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ON SMOOTHNESS OF SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR SECOND-ORDER DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS

ПРО ГЛАДКІСТЬ РОЗВ'ЯЗКУ ПЕРШОЇ КРАЙОВОЇ ЗАДАЧІ ДЛЯ ВИРОДЖЕНИХ ЕЛІПТИЧНО-ПАРАБОЛІЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

In this work, the first boundary-value problem is considered for second-order degenerate elliptic-parabolic equation with, generally speaking, discontinuous coefficients. The matrix of senior coefficients satisfies the parabolic Cordes condition with respect to space variables. We prove that the generalized solution to the problem belongs to the Hölder space $C^{1+\lambda}$ if the right-hand side f belongs to L_p, $p > n$.

Розглянуто першу крайову задачу для виродженого еліптично-параболічного рівняння другого порядку із, взагалі кажучи, розривними коефіцієнтами. Матриця старших коефіцієнтів задовольняє параболічну умову Кордеса за просторовими змінними. Доведено, що узагальнений розв'язок задачі належить до простору Гельдера $C^{1+\lambda}$, якщо права частина f належить L_p, $p > n$.

Introduction. Investigations of boundary-value problems for second-order degenerate elliptic-parabolic equations ascend to the work by Keldysh [1], where correct statements for boundary-value problems were considered for the case of one space variable as well as existence and uniqueness of solutions. In the work by Fichera [2], boundary-value problems were given for multidimensional case. He proved existence of generalized solutions to these boundary-value problems. In the work by Oleynik [3], existence and uniqueness of generalized solution to these problems were proved for smooth and piecewise smooth domains. In the case of smooth coefficients and some weighted functions, the generalized solvability was studied in [4] and [5]. Moreover, the smoothness of the solution was studied and the condition (15) and the example (22) were apparently given for the first time in the paper [5].

Let R_{n+1} be an $(n+1)$-dimensional Euclidian space of points $(x,t) = (x_1, x_2, \ldots, x_n, t)$, Ω be a bounded n-dimensional domain in R_n with the boundary $\partial\Omega$, $Q_T = \Omega \times (0, T)$ be a cylinder in R_{n+1}, $T \in (0, \infty)$, $Q_0 = \{(x,t)\colon x \in \Omega,\ t = 0\}$ and let $\Gamma(Q_T) = Q_0 \bigcup (\partial\Omega \times [0, T])$ be a parabolic boundary of the cylinder Q_T.

Let us consider in Q_T the first boundary-value problem for second-order degenerate elliptic-parabolic operator

$$Zu = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \psi(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t), \quad (1)$$

$$u|_{\Gamma(Q_T)} = 0. \quad (2)$$

Assume that the coefficients of the operator Z satisfy the following conditions: $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with elements measurable in Q_T and, for any $(x,t) \in Q_T$ and $\xi \in R_n$, the following inequalities are true:

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \gamma^{-1}|\xi|^2, \quad (3)$$

where $\gamma \in (0, 1]$ is a constant,

$$\sigma = \frac{\sup\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}^2(x,t)}{\inf\limits_{Q_T} \left[\sum_{i=1}^{n} a_{ii}(x,t)\right]^2} < \frac{1}{n - 1/2}, \quad (4)$$

$$c(x,t) \le 0, \qquad c(x,t) \in L_{n+1}(Q_T), \quad (5)$$

$$|b(x,t)| \in L_{n+2}(Q_T), \qquad |b(x,t)|^2 + Kc(x,t) \le 0. \quad (6)$$

Assume that the following conditions are true for the weighted function:

$$\psi(x,t) = \lambda(\rho)w(t)\varphi(T-t),$$

where

$$\rho = \rho(x) = \operatorname{dist}(x, \partial\Omega), \quad \lambda(\rho) \ge 0, \quad \lambda(\rho) \in C^1[0, \operatorname{diam}\Omega],$$

$$|\lambda'(\rho)| \le p\sqrt{|\lambda(\rho)|}, \quad w(t) \ge 0, \quad w(t) \in C^1[0,T],$$

$$\varphi(z) \ge 0, \quad \varphi'(z) \ge 0, \quad \varphi(z) \in C^1[0,T], \quad \varphi(0) = \varphi'(0) = 0, \quad \varphi(z) \ge \beta z\varphi'(z), \quad (7)$$

where p, β are positive constants and $\psi(x,t)$ has bounded derivatives of the second order.

The condition (4) is called the condition of Cordes type and is taken within the accuracy of a linear nonsingular transformation. This means that the Cordes condition is taken within the accuracy of nondegenerate linear transformation, that is the domain Q_T can be covered by a finite number of domains $Q^1, \ldots, Q^M$ so that in each Q^i there exists such a nondegenerate linear transformation that a matrix of sinior coefficients of the image of the operator Z satisfies the condition (4) in the image of Q^i, $i = \overline{1, M}$.

Before we move to the proof of the basic result, let us consider some auxiliary problems. Let

$$L'u = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \psi(x,t)\frac{\partial^2 u}{\partial t^2} +$$

$$+ \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + b_0(x,t)\frac{\partial u}{\partial t} + c(x,t)u = f(x,t). \quad (8)$$

Without loss of generality, we may assume that the coefficients are smooth in $\overline{Q_T}$ and their derivatives are bounded. To speak more exactly, let us say that the coefficients and the right-hand side have the first Hölder derivatives. Let

$$L'_\varepsilon u = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \psi_\varepsilon(x,t)\frac{\partial^2 u}{\partial t^2} +$$

$$+ \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + b_0(x,t)\frac{\partial u}{\partial t} + c(x,t)u = f(x,t), \tag{9}$$

where $\psi_\varepsilon(x,t)$ is defined so: for any fixed $\varepsilon \in (0,T)$

$$\varphi_\varepsilon(z) = \varphi(\varepsilon) - \frac{\varphi'(\varepsilon)\varepsilon}{m} + \frac{\varphi'(\varepsilon)}{m\varepsilon^{m-1}} z^m \quad \text{for} \quad z \in (0,\varepsilon], \quad \varphi_\varepsilon(z) = \varphi(z)$$

for $z \in [\varepsilon, T]$, $m = \dfrac{2}{\beta}$ (we denoted by z the argument of $\varphi(T-t)$). Similarly, for any fixed $\varepsilon \in (0,T)$

$$w_\varepsilon(z) = w(\varepsilon) - \frac{w'(\varepsilon)\varepsilon}{m} - \frac{w'(\varepsilon)}{m\varepsilon^{m-1}} z^m \quad \text{for} \quad z \in (0,\varepsilon],$$

$w_\varepsilon(z) = w(z)$ for $z \in [\varepsilon, T]$, $m = \dfrac{2}{\beta}$ (we denoted by z the argument of $w(t)$).

Analogously the new value of $\lambda_\varepsilon(z) = \lambda(\varepsilon) + \varepsilon$ on the correspondent segment. We multiply all the approximated functions to obtain $\psi_\varepsilon(x,t)$.

Everywhere further, we consider the case where $\psi(z) > 0$ for $z > 0$. If $\psi(z) \equiv 0$, then the equation (1) is parabolic, and the corresponding result on smoothness of the solution ensues from [6]. But if $\psi(z) = 0$ for $z \in [0, z^0]$, then the solution to the problem (1), (2) can be obtained by composition of the solution $u(x,t)$ to the first boundary-value problem in the cylinder Q_{z^0} and the solution $v(x,t)$ to the first boundary-value problem for the parabolic equation in the sylinder $\Omega \times (z^0, T)$ with the boundary conditions

$$v(x, z^0) = u(x, z^0), \qquad v|_{\partial\Omega \times [z^0, T]} = 0.$$

Note that under the conditions (3) – (6) for the coefficients, the smoothness of the solution results from [7]. Denote by Σ^0 the part of Q_T, where $\psi(x,t) = 0$, i.e., where the equation (8) degenerates: denote by Γ^0 the part of intersection of Σ^0 and the boundary Γ, where a tangent plane to the surface Γ is orthogonal to the axis t, i.e., has a characteristical direction.

By maximum principle, the solutions $u_\varepsilon(x,t)$ of the equation (9) in the domain satisfy the following estimate:

$$|u_\varepsilon(x,t)| \le \left|\frac{f(x,t)}{c(x,t)}\right|,$$

that is $u_\varepsilon(x,t)$ are uniformly bounded with respect to ε.

Lemma 1. *The derivatives of the solution $u_\varepsilon(x,t)$ are uniformly bounded on a closed subset of the boundary Γ, that belongs to $\Gamma \setminus \Gamma^0$.*

Proof. Let us take a point $(x', t) \in \Gamma \setminus \Gamma^0$ such that at the point a tangent plane to Γ is not orthogonal to the axis t, i.e., the surface Γ near the point has an equation of the kind $x_1 = \theta(x_2, \ldots, x_n, t)$, where θ has derivatives up to second order. Let $\chi(x_2, \ldots, x_n, t)$ be twice continuously differentiable function equal to a positive constant β in some neighborhood of a projection (x', t) onto the plane $(x_2, \ldots, x_n, t)$ and equal to zero in a little greater neighborhood $0 \le \chi(x_2, \ldots, x_n, t) \le \beta$. We denote by Q_T^1 the part of Q_T being between the surfaces Γ and $\sigma\{x_1 = \theta + \chi\}$. Let Γ^1 denote that part of Γ, where $\chi = \beta$. Consider a function $v = e^{\alpha(-x_1+\theta+\chi)}$. It is obvious, that on the surface σ $v = 1$. Then in Q_T^1, for sufficiently great α, we have

$$L'_\varepsilon(v) \ge \alpha^2\gamma - \alpha\mu - \mu_1 > \frac{\alpha^2\gamma}{2}, \qquad L'_\varepsilon(v \pm u_\varepsilon) > \frac{\alpha^2\gamma}{2} - \max_{Q_T}|f(x,t)| > 0, \quad (10)$$

where μ, μ_1 are maximums of the modules of the solution itself and its first derivatives within Q_T. Then we choose α independent of ε so, that (10) is true and, moreover, $e^{\alpha\beta} > 1 + \max_{Q_T}|u_\varepsilon(x,t)|$. This means that on Γ^1 the values of functions $v \pm u_\varepsilon$ equal to $e^{\alpha\beta}$ are greater than their values on σ, where $v = 1$ (taking into account that $u_\varepsilon(x,t)|_\Gamma = 0$). By maximum principle, it results from the following estimate (10) that functions $v \pm u_\varepsilon$ within the domain Q_T^1 cannot take maximal positive value. Hence, they reach maximum on the boundary Γ, i.e., on the part Γ^1 too, while on the other part of Γ $v \pm u_\varepsilon = e^{\alpha\chi} \le e^{\alpha\beta}$. So, at points that belong to Γ^1, we have

$$\frac{\partial(v \pm u_\varepsilon)}{\partial x_1} \le 0 \quad \text{or} \quad \left|\frac{\partial u_\varepsilon(x,t)}{\partial x_1}\right|_{\Gamma^1} \le -\left.\frac{\partial v}{\partial x_1}\right|_{\Gamma^1} = \alpha e^{\alpha\beta}.$$

In other words on Γ^1, the derivatives $\dfrac{\partial u_\varepsilon(x,t)}{\partial x_1}$ are uniformly bounded. Moreover, derivatives of $u_\varepsilon(x,t)$ with respect to directions lying in a tangent plane are equal to zero as $u_\varepsilon(x,t)|_\Gamma = 0$. Thus, the derivatives $\dfrac{\partial u_\varepsilon(x,t)}{\partial x_i}$, $i = \overline{1,n}$, are uniformly bounded with respect to ε on Γ^1.

Let us take a point $(x', t) \in \Gamma \setminus \Gamma^0$. Let a tangent plane to Γ at this point be orthogonal to the axis t. This case can be proved similarly.

The lemma is proved.

***Remark* 1.** If the boundary does not contain points of Γ^0, then $\dfrac{\partial u_\varepsilon(x,t)}{\partial t}$ are uniformly bounded on the entire boundary.

Lemma 2. *Suppose that on* Σ^0 *the condition*

$$c(x,t) + \frac{\partial b_0(x,t)}{\partial t} < 0 \qquad (11)$$

is true and $\overline{\Sigma^1}$ *is any closed domain with a boundary* σ_1, *which belongs to* $\overline{Q_T}$. *Then at* $(x,t) \in \overline{\Sigma^1}$

$$\sum_{i=1}^{n}\left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i}\right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 \le$$

$$\leq C \max_{(x,t)\in\sigma_1} \left[\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i} \right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 \right] + C_1, \tag{12}$$

where C, C_1 *are constants depending on a structure of the equation.*

Proof. Introduce the notation $\left(\overline{\Sigma^1} \cap \Sigma^0\right) \cap \Sigma^1 = \Sigma^2$. Let us prove the inequality in some neighborhood of closed domain Σ^2. The boundary of Σ^2 consists of the part σ_1 of the boundary Σ^1 and the surface σ_2 being in the part, where $\psi(x,t) > 0$. At points $(x,t) \in \sigma_2$ the inequality

$$\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i} \right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 \leq$$

$$\leq C \max_{(x,t)\in\sigma_1} \left[\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i} \right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 \right] + C_1 \tag{13}$$

is true.

The following estimate (13) is obtained from the fact, that derivatives of the solution are bounded in any closed subdomain for the case of bounded derivatives up to the boundary of a domain. Now if we show that the following estimate (13) is also true for the domain Σ^2, the from (13) and this following estimate we will get (12) for the domain Σ^1. Assume that (11) is also satisfied in Σ^2. For simplicity of calculations, we will find following estimates for one space variable and in the end show the changes in calculations in the case of many space variables. Without loss of generality, we take the coefficient at second derivative with respect to a space variable x equal to unit, as it can be easily obtained by division by terms by the coefficient. Denote

$$z = \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^m + \alpha_1 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^{m-2} u_\varepsilon^2(x,t).$$

First, we show that for corresponding n, α_1, we have $L'_\varepsilon z > 0$ in Σ^2, if $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 > \mu_1$. Let n be a positive even number. We get

$$L'_\varepsilon \left[\left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^m + \alpha_1 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^{m-2} u_\varepsilon^2(x,t) \right] = L'_\varepsilon z > 0,$$

if $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 > \mu_1$. Now if z takes its maximum within Σ^2, then at this point $L_\varepsilon z \leq 0$. So, either $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 \leq \mu_1$ or the value of z within Σ^2 is not greater than the maximum on the boundary Σ^2. Since

$$\left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 \leq z^{2/m} \leq C_2 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t} \right)^2 + C_3$$

and

$$\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2\Bigg|_{\Sigma^2} \le z^{2/m}\Big|_{\Sigma^2} \le C_2 \max_{\sigma_2\cup\sigma_1} z^{2/m} + C_3 <$$

$$< C_4 \max_{\sigma_2\cup\sigma_1}\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 + C_5, \tag{14}$$

$\dfrac{\partial u_\varepsilon(x,t)}{\partial x}$ can be estimated similarly.

The lemma is proved.

Lemma 3. *Assume that on the set* Σ^0, *the following condition is satisfied*:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} + 2\frac{\partial b_0(x,t)}{\partial t} + c(x,t) < 0 \tag{15}$$

and first derivatives of $u_\varepsilon(x,t)$ *are uniformly bounded in a closed domain* $\overline{\Sigma^1}\subset\overline{Q_T}$ *with the boundary* σ. *Then*

$$\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 + \sum_{i,j=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 + \left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^2 \le$$

$$\le C\max_{(x,t)\in\sigma}\left[\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 + \sum_{i,j=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 + \left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^2\right] + C_1, \tag{16}$$

where C, C_1 *do not depend on* ε.

Proof. As $c(x,t) < 0$, $\dfrac{\partial^2\psi}{\partial t^2} \ge 0$ on Σ^0, the statement of the lemma for first derivatives results form Lemma 2. To prove the lemma, as in the proof of Lemma 2, we have to show that in some neighborhood of $\Sigma^0\cap\Sigma^1$: $L'_\varepsilon z_1 > 0$ at the corresponding m (an even number) and α_i. Here, z_1 is the same as in Lemma 2, but it contains additional terms. An element $\left(\dfrac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m$ is the main in it, so we have to estimate

$$L'_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right] = m\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-1} L'_\varepsilon\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right) +$$

$$+ m(m-1)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\sum_{i,j=1}^n a_{ij}(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_j}\right) +$$

$$+ m(m-1)\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^{m-2}\psi_\varepsilon(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 - (m-1)c(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m.$$

Taking into account (15) on Σ^0, for sufficiently great m, we have

$$-m\left(\frac{\partial^2\psi_\varepsilon}{\partial t^2} + 2\frac{\partial b_0}{\partial t} + c - \frac{c}{m}\right) > \mu_1 m$$

in some neighborhood of Σ^0. Now we choose $\beta < \mu_1 - \mu_2$, where $\mu_2 > 0$, and fix β. Then

$$L_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right] >$$

$$> m(m-1)\mu\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\sum_{i=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)^2 + m\mu_2\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m +$$

$$+ m(m-1)\mu\psi_\varepsilon(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 -$$

$$- m\mu_3\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\left[\psi_\varepsilon(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 + \sum_{i+j>0;\, i\neq j}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 +\right.$$

$$\left.+ \sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 + \sum_{i,j=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial x_i\partial x_j\partial t}\right)^2 + 1\right].$$

Let us choose sufficiently great m, so that $-m\mu_3 + m(m-1)\mu > \mu_3 > 0$ and fix m. Under this condition,

$$L_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right] \geq \mu_4\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\sum_{i=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)^2 +$$

$$+ \mu_5\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m + \mu_3\psi_\varepsilon(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 -$$

$$- \mu_4\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2}\left[\sum_{i+j>0;\, i\neq j}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 + \sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 +\right.$$

$$\left.+ \sum_{i,j=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 + 1\right].$$

Having obtained the other estimates similarly to Lemma 2, we get the statement of the lemma.

The lemma is proved.

Lemma 4. *Let the condition* (15) *be satisfied on the set* Σ^0 *and the boundary of* Q_T *have no points of* Σ^0*. Then in the closed domain* $\overline{Q_T}$*, derivatives of* $u_\varepsilon(x,t)$ *with respect to space up to the second-order variables are uniformly bounded.*

Proof. Let us take a point $(x^*, t^*) \in \Gamma$ and let in its neighborhood the boundary Γ be presented in the form $x_1 = \varphi(x_2, \ldots, x_n, t)$. By means of change of variables $t = t^*$, $\xi_1 = x_1 - \varphi(x_2, \ldots, x_n, t)$, $\xi_2 = x_2, \ldots, \xi_n = x_n$ in the neighborhood of (x^*, t^*), the equation (9) is reduced to the form

$$L_\varepsilon^* u_\varepsilon = \sum_{i,j=1}^n a_{ij}^*(\xi, t^*)\frac{\partial^2 u_\varepsilon}{\partial\xi_i\partial\xi_j} + \psi_\varepsilon^*(\xi, t^*)\frac{\partial^2 u_\varepsilon}{\partial(t^*)^2} + \sum_{i=1}^n b_i^*(\xi, t^*)\frac{\partial u_\varepsilon}{\partial\xi_i} +$$

$$+ b_0^*(\xi, t^*)\frac{\partial u_\varepsilon}{\partial t^*} + c^*(\xi, t^*)u_\varepsilon = f^*(\xi, t^*), \qquad (17)$$

where $a_{11}^*(\xi, t^*) \geq \mu > 0$, $c^*(\xi, t^*) < 0$, and due to assumptions on smoothness of the coefficients and boundary, the coefficients of (17) have uniformly bounded derivatives. The boundary Γ will have the equation $\xi_1 = 0$ in the neighborhood of (x^*, t^*). For clarity, we take the axis ξ_1 to be pointed into Q_T. As in Lemma 1, we denote by $\chi(\xi_2, \ldots, \xi_n, t^*)$ a nonnegative twice continuously differentiable function equal to the constant β inside some neighborhood Γ^1 of the point (x^*, t^*) on the boundary Γ and equal to zero outside a little greater neighborhood $0 \le \chi \le \beta$. The part of the domain Q_T lying between the boundary $\Gamma\{\xi_1 = 0\}$ and $\sigma\{\xi_1 = \chi(\xi_2, \ldots$ $\ldots, \xi_n, t^*)\gamma/\alpha\}$, will be denoted by Q_T^ε. Further, α will be chosen as depending on ε, and γ as not depending on ε. In $\overline{Q_T}$, the uniform boundedness results from Lemma 2 for first derivatives of $u_\varepsilon(x, t)$ with respect to x_i and t, and hence, with respect to ξ_i, t^* in a neighborhood of (x^*, t^*). By Lemma 3, second derivatives of $u_\varepsilon(x, t)$ are estimated via their values on the boundary, and as second derivatives with respect to x_i and t, as well as with respect to ξ_i, t^*, are mutually expressed by each other and by first derivatives in a neighborhood of (x^*, t^*) in a uniformly bounded way, so

$$\left|\frac{\partial^2 u_\varepsilon(\xi, t^*)}{\partial \xi_i \partial \xi_j}\right| + \left|\frac{\partial^2 u_\varepsilon(\xi, t^*)}{\partial \xi_i \partial t}\right| + \left|\frac{\partial^2 u_\varepsilon(\xi, t^*)}{\partial t^2}\right| < \mu H(\varepsilon) + \mu_1 \tag{18}$$

at $(\xi, t^*) \in \overline{Q_T}$, $i, j = \overline{1, n}$. Here, a maximum of second derivatives on the boundary Γ is denoted by $H(\varepsilon)$.

If at the point (x^*, t^*) a tangent plane to Γ is orthogonal to the axis t, then by definition of Σ^0, at the point, and that's in some its neighborhood, $\psi_\varepsilon(x, t) > \mu_1 > 0$. Thus, for each point $(x^*, t^*) \in \Gamma$, a neighborhood exists on the boundary such that

$$\left|\frac{\partial^2 u_\varepsilon}{\partial \xi_i \partial \xi_j}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial \xi_i \partial t}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial t^2}\right| < \mu_6 \sqrt{H(\varepsilon)} + \mu_7, \qquad i, j = \overline{1, n}.$$

Taking a finite number of such neighborhoods covering Γ, and taking into account the smoothness of change of coordinates in each of these neighborhoods, we get

$$\left|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial t}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial t^2}\right| < \mu_8 \sqrt{H(\varepsilon)} + \mu_9$$

on entire boundary Γ or, due to definition of $H(\varepsilon)$, $H(\varepsilon) \le \mu_{10}\sqrt{H(\varepsilon)} + \mu_{11}$. Hence, $H(\varepsilon) < \mu_{12}$, i.e., we have boundedness of second derivatives on the boundary, and by Lemma 3, in the whole domain $\overline{Q_T}$. Here, we used only boundedness of first derivatives of coefficients of the equation (17).

The lemma is proved.

Now we can move to the proof of existence and the uniqueness theorem for the first boundary-value problem for the equation (8).

Theorem 1. *Let the equation* (8) *defined in a cylindrical domain* Q_T *with the boundary* Γ, *degenerate on the set* $\Sigma^0 \subset \overline{Q_T}$ *into a parabolic one, let the condition* (3) *be satisfied and let all the coefficients and the right-hand side of the equation* (8) *have bounded derivatives up to the first order, satisfying the Hölder condition. Assume that, in a cylindrical domain* $Q_T' \supset \overline{Q_T}$, $\psi(x, t) \ge 0$ *and the conditions* (7) *are satisfied. If the boundary* Γ *has no points of* Γ^0 *and the condition* (15) *is satisfied on* Σ^0, *then in* Q_T *there exists a unique solution of the equation* (8) *that satisfy the condition* (2) *and have in* $\overline{Q_T}$ *derivatives of the first* *order satisfying the Hölder condition; and the following estimate is true*:

$$\|u\|_{C^{1+\lambda}(Q_T)} \le K_1\left(\|f\|_{C^{\lambda}(Q_T)} + \sup_{Q_T}|u|\right). \tag{19}$$

Proof. From Lemma 4 it results that solutions of the equation

$$L'_\varepsilon u_\varepsilon(x,t) = f(x,t), \tag{20}$$

vanishing on Γ, are uniformly bounded in the closed domain $\overline{Q_T}$ along with their derivatives up to the second order. In other words, it is possible to find a sequence $u_\varepsilon(x,t)$ such that as $\varepsilon \to 0$, it uniformly converges to some function $u(x,t)$ along with its derivatives up to the first order in the closed domain $\overline{Q_T}$. And it is clear that these derivatives of $u(x,t)$ will be Hölder derivatives and the function $u(x,t)$ equals zero on the boundary Γ. Moreover, for such solutions, the estimate (19) is true (see [6, p. 235], Chapter 3). Passing to the limit in the equation (20) as $\varepsilon \to 0$, we obtain that $u(x,t)$ satisfies the equation (8) and the estimate (19) is true. Uniqueness of the solution follows directly from maximum principle.

***Remark* 2.** From the proof of Theorem 1, the convergence of the solutions of the equation (20) to the solution of the equation (1) as $\varepsilon \to 0$ also follows.

***Remark* 3.** The condition (15) cannot be omitted. There exists an essential difference from existence theorems proved for a smooth solution of the Dirichlet problem for elliptic equation. Let us give an example.

***Example* 1.** Let us consider the equation

$$t^2\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} + \beta t\frac{\partial u(x,t)}{\partial t} + cu = 0 \tag{21}$$

with sufficiently smooth coefficients, where β, c are constants, $c \le 0$. It is easy to check that the equation has a solution

$$u(x,t) = t^\gamma \sin px, \tag{22}$$

$$\gamma(\gamma-1) + \beta\gamma + c = p^2. \tag{23}$$

The equation degenerates on the axis x. The condition (15) for the equation means that $2+2\beta+c < 0$. Let the condition be not satisfied, e.g., $2+2\beta+c > 0$. Then such p, $\gamma < 2$ exist that they satisfy (23). Let us consider the domain Q_T containing a segment of the axis x, whose boundary near the axis x consists of straight lines $x = 0$ and $x = \pi/p$ and everywhere is sufficiently smooth. Then the solution (22) will be sufficiently smooth on the boundary (near the axis $x = 0$ it is zero), but, nevertheless, its first order derivatives will not satisfy the Hölder condition for $t = 0$, $0 < x < \pi/p$.

Let us give the scheme of proof of the solvability when passing from smooth coefficients to coefficients satisfying (3) – (6), (8).

First, let $f(x,t)$ be sufficiently smooth in $\overline{Q_T}$. Denote by $v(x,t)$ a classical solution of the first boundary-value problem

$$\begin{gathered} \Delta v - v_t = f(x,t), \qquad (x,t)\in Q_T, \\ v|_{\Gamma(Q_T)} = 0. \end{gathered} \tag{24}$$

It is known that the solution $v(x,t)$ to the problem exists and $v(x,t)\in C^{2,1}(\overline{Q_T})$. Now we take an operator L_ε. Let $u_\varepsilon(x,t)$ be a classical solution of the Dirichlet problem

$$L_\varepsilon u_\varepsilon(x,t) = f(x,t), \qquad (x,t) \in Q_T,$$
$$u_\varepsilon|_{\Gamma(Q_T)} = 0, \qquad u_\varepsilon|_{t=T} = v|_{t=T}.$$

Such a solution $u_\varepsilon(x,t)$ exists due to smoothness of $\psi_\varepsilon(x,t)$ and $f(x,t)$. As we have shown, $\{u_\varepsilon(x,t)\}$ are uniformly bounded with respect to ε in $C_0^{2,1}(Q_T)$. Therefore, it is compact in this space, i.e., there exist such a function $u(x,t) \in C_0^{2,1}(Q_T)$ and a sequence $\varepsilon_k \to 0$, $k \to \infty$, that the corresponding sequence $\{u_{\varepsilon_k}(x,t)\}$ converges to the function $u(x,t) \in C_0^{2,1}(Q_T)$ as $k \to \infty$. Further, we can obtainthat $L_0 u = f$ in Q_T. Now let $f(x,t) \in L_p(Q_T)$, $p > n+2$. Then a sequence $\{f_m(x,t)\}$, $f_m(x,t) \in$ $\in C^\infty(\overline{Q_T})$ exists such that

$$\lim_{m\to\infty} \| f_m - f \|_{L_p(Q_T)} = 0.$$

For natural m, denote by $u_m(x,t)$ the sequence of solutions of the first boundary-value problem for

$$u_m(x,t) \in C_0^{2,1}(Q_T),$$
$$L_0 u_m(x,t) = f_m(x,t), \qquad (x,t) \in Q_T.$$

It is proved that the limit $u(x,t)$ of the sequence $\{u_m(x,t)\}$ in $C_0^{2,1}(Q_T)$, $m \to \infty$, satisfies in Q_T the equation $L_0 u(x,t) = f(x,t)$.

Note that as we said above, $\psi(x,t) > 0$. If $\psi(x,t) \equiv 0$, then the equation (1) is parabolic and that is why under the conditions (3) – (6) and $f(x,t) \in L_p(Q_T)$, $p > n +$ $+ 2$, for the bounded solution of the equation (1) the following estimate is true:

$$\|u\|_{C^{1+\lambda}(Q_T^\rho)} \le K_1 \left(\|f\|_{L_p(Q_T)} + \sup_{Q_T} |u| \right). \tag{25}$$

If $\psi(x,t) > 0$ and the condition of Theorem 1 is satisfied for the coefficients, then for the bounded solution of the equation (1) the estimate (25) is true. The estimate (25) can be obtained by composition of the solution $u(x,t)$ to the problem in the cylinder Q_{z^0}, where $\psi(z) = 0$ for $z \in [0, z^0]$, and the solution $v(x,t)$ to the first boundary-value problem for parabolic equation in the cylinder $\Omega \times (z^0, T)$ with boundary conditions $v(x,z^0) = u(x,z^0)$, $v|_{\partial\Omega\times[z^0,T]} = 0$. It must be noted that the theorem has been obtained for smooth coefficients, but we can pass to $f(x,t) \in L_p(Q_T)$ by means of the above mentioned scheme. Further, to prove the estimate (25) under the conditions (3) – (7), we apply the method of continuation by parameter.

Theorem 2. *Suppose that the equation* (1) *defined in* Q_T *degenerates on the set* $\Sigma^0 \subset \overline{Q_T}$ *into a parabolic one, the conditions* (3) – (7) *are satisfied for the coefficients, and the right-hand side of the equation* $f(x,t) \in L_p(Q_T)$, $p > n+2$. *If the boundary* Γ *has no points of* Γ^0 *and on* Σ^0 *the condition* (15) *is satisfied, then for the bounded solution* $u(x,t)$ *of the equation* (1) *the following estimate is true*:

$$\|u\|_{C^{1+\lambda}(Q_T^\rho)} \le K_1 \left(\|f\|_{L_p(Q_T)} + \sup_{Q_T} |u| \right),$$

where $\lambda > 0$ *depends only on coefficients of the operator* L *and* n; *and* K_1, *moreover, depends on* p, ρ, $\operatorname{diam} Q_T$.

***Remark* 4.** Theorem 2 in this formulation is also true for the equation (1), if in the condition (15) instead of $b_0(x,t)$ will be taken $b_1(x,t)$.

***Proof of Theorem* 2.** To prove it, we consider a family of operators $Z^{(\tau)} = (1-\tau)L' + \tau Z$ for $\tau \in [0,1]$, where L' is a model operator defined from the equation (8) with Laplacian main part and smooth coefficients, and the operator Z is defined from the equation (1). Let us show that the set E of points τ, at which for solutions of the problem

$$Z^{(\tau)}u = f(x,t), \qquad (x,t)\in Q_T, \tag{26}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{27}$$

the estimate (25) is true if $f(x,t)\in L_p(Q_T)$, $p > n+2$, is nonempty, and open and closed simultaneously with respect to the segment $[0,1]$. Hence, $E = [0,1]$ and, in particular, for the solution of the problem (26), (27) the estimate (26) is true for $\tau = 1$, i.e., when $Z^{(1)} = Z$. The set E is nonempty by Theorem 1. Let us show that it is open. For this purpose, we prove that for solutions of the problem (26), (27) the estimate (25) is true for all $\tau\in[0,1]$ such that $|\tau-\tau_0| < \varepsilon$ (here, $\tau_0 \in E$ and $\varepsilon > 0$ will be chosen later). Rewrite the problem (26), (27) in the equivalent form

$$Z^{(\tau_0)}u = f(x,t) - (Z^{(\tau)} - Z^{(\tau_0)})u, \qquad (x,t)\in Q_T, \tag{28}$$

$$u(x,t)\in C_0^{2,1}(Q_T).$$

We introduce an arbitrary function $\upsilon(x,t)\in C_0^{2,1,\lambda}(Q_T)$ and consider the first boundary-value problem

$$Z^{(\tau_0)}u = f(x,t) - (Z^{(\tau)} - Z^{(\tau_0)})\upsilon, \qquad (x,t)\in Q_T, \tag{29}$$

$$u(x,t)\in C_0^{2,1}(Q_T).$$

It is clear that $(Z^{(\tau)} - Z^{(\tau_0)})\upsilon \in C^{2,1,\lambda}(Q_T)$. Indeed, note that for all operators $Z^{(\tau)}$ the conditions (3) and (4) are satisfied with constants $\gamma^0_{(\tau)} \ge \min\{\gamma, n\}$ and $\sigma_{(\tau)} \le \sigma$, respectively. Let us show this. Denote by $a_{ij}^{(\tau)}(x,t)$, $i = \overline{1,n}$, the coefficients of the operator $Z^{(\tau)}$ at higher derivatives with respect to space variables. Let

$$\bar\iota = \sup_{Q_T}\frac{\sum_{i,j=1}^n a_{ij}^2(x,t)}{g^2(x,t)}, \qquad \iota^{(\tau)} = \sup_{Q_T}\frac{\sum_{i,j=1}^n\left[a_{ij}^{(\tau)}(x,t)\right]^2}{\left[\sum_{i=1}^n a_{ii}^{(\tau)}(x,t)\right]^2}, \qquad \overline{\iota^{(\tau)}} = \sup_{Q_T}\iota^{(\tau)}(x,t),$$

where

$$g(x,t) = \sum_{i=1}^n a_{ii}(x,t).$$

Taking into account (4) and the fact that for any operator of Z-type the inequality $\bar\iota \ge 1$ is true, we conclude that

$$\iota^{(\tau)}(x,t) = \frac{n(1-\tau)^2 + 2\sigma(1-\tau)g(x,t) + \tau^2\sum_{i,j=1}^n a_{ij}^2(x,t)}{n^2(1-\tau)^2 + 2\tau(1-\tau)ng(x,t) + \tau^2 g^2(x,t)} \le$$

$$\le \frac{1}{n} + \frac{\tau^2(\bar\iota - 1/n)g^2(x,t)}{\tau^2 g^2(x,t)} = \bar\iota. \tag{30}$$

Let now

$$\lambda^- = \inf_{Q_T} g(x,t), \qquad \lambda^+ = \sup_{Q_T} g(x,t), \qquad \bar{\lambda}(\tau) = \frac{\inf\limits_{Q_T} \sum_{i=1}^n a_{ii}^{(\tau)}(x,t)}{\sup\limits_{Q_T} \sum_{i=1}^n a_{ii}^{(\tau)}(x,t)}.$$

Calculations we made before show that $\bar{\lambda}(\tau) = \dfrac{(1-\tau)n + \tau\lambda^-}{(1-\tau)n+\tau\lambda^+}$. But on the other hand, $\bar{\lambda}'(\tau) = \dfrac{\lambda^- - \lambda^+}{[(1-\tau)n+\tau\lambda^+]^2} \le 0$. That is why,

$$\bar{\lambda}(\iota) \ge \bar{\lambda}(1) = \lambda, \tag{31}$$

(30) and (31) imply that $\sigma_{(\tau)} = \bar{\iota}^{(\tau)} - \dfrac{1}{n - \bar{\lambda}^2(\tau)} \le \bar{\iota} - \dfrac{1}{n-\lambda^2} = \sigma$, and the needed statement is obtained.

Note that all above mentioned reasonings and Lemma 4 imply that if $T \le T^0$, the following estimate is true for any $\tau \in [0,1]$ and any function $u(x,t) \in C^{2,1,\lambda}(Q_T)$:

$$\|u\|_{C^{2,1,\lambda}(Q_T)} \le K_2\left(\left\|Z^{(0)}u\right\|_{C^{0,\lambda}(Q_T)}\right). \tag{32}$$

For the solution $u(x,t)$ of the boundary-value problem (29), due to the assumption made, the estimate (25) is true for any $v(x,t) \in C_0^{2,1,\lambda}(Q_T)$. Thus, an operator Φ is defined from $C_0^{2,1,\lambda}(Q_T)$ to $C_0^{2,1,\lambda}(Q_T)$ and $u = \Phi v$. This operator is compressing at ε chosen in an appropriate way. Indeed, let $v^{(i)}(x,t) \in C_0^{2,1,\lambda}(Q_T)$, $u^{(i)} = \Phi v^{(i)}$, $i = 1, 2$. Then, takling into account that $(Z^{(\tau)} - Z^{(\tau_0)}) = (\tau - \tau_0)(Z - L')$, we conclude that $u^{(1)}(x,t) - u^{(2)}(x,t)$ is a classical solution of the first boundary-value problem

$$Z^{(\tau_0)}(u^{(1)}(x,t) - u^{(2)}(x,t)) = (\tau-\tau_0)(Z-L')(v^{(1)}(x,t) - v^{(2)}(x,t)),$$

$$u^{(1)}(x,t) - u^{(2)}(x,t) \in C_0^{2,1,\lambda}(Q_T).$$

Using (32), we get

$$\left\|u^{(1)}(x,t) - u^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)} \le$$

$$\le K_2|\tau-\tau_0|\left\|(Z-L')v^{(1)}(x,t) - v^{(2)}(x,t)\right\|_{C^{0,\lambda}(Q_T)}. \tag{33}$$

On the other hand,

$$\left\|(Z-L')v^{(1)}(x,t) - v^{(2)}(x,t)\right\|_{C^{0,\lambda}(Q_T)} \le K_3(Z,n,\Omega,T)\left\|v^{(1)}(x,t) - v^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)}.$$

So,

$$\left\|u^{(1)}(x,t) - u^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)} \le K_2K_3\varepsilon\left\|v^{(1)}(x,t) - v^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)}.$$

Now taking $\varepsilon = 1/2K_2K_3$, we prove that the operator Φ is compressing. Hence, it has a stationary point $u = \Phi u$, that is a classical solution of the boundary-value problem (28), and of (26), (27) as well, and for the solution the estimate (25) is true. So, we have proved that the set E is open.

Let us show that the set E is closed. Let $\tau_k \in E$, $k = 1, 2, \ldots$, $\lim\limits_{k\to\infty} \tau_k = \tau$. For natural k, we denote by $u_{[k]}(x,t)$ the solution of the first boundary-value problem $Z^{(\tau_k)}u_{[k]}(x,t) = f(x,t)$, $(x,t) \in Q_T$, $u_{[k]}(x,t)\big|_{\Gamma(Q_T)} = 0$, for which the following estimate takes place:

$$\left\| u_{[k_l]}(x,t) \right\|_{C^{2,1}(Q_T)} \;\le\; K_3 \| f \|_{L_p(Q_T)}. \tag{34}$$

So, from (34) we obtain that the family of functions $\{u_{[k]}(x,t)\}$ is compact in $C_0^{2,1}(Q_T)$, i.e., there exists such a subsequence of natural numbers $\{k_l\}$, $\lim\limits_{l\to\infty} k_l = \infty$ and a function $u(x,t) \in C_0^{2,1}(Q_T)$ that, for any $\varphi(x,t) \in C_0^{\infty}(\overline{Q_T})$,

$$\lim_{l\to\infty} \left(Z^{(\tau_{k_l})} u_{[k_l]}, \varphi\right) \;=\; \left(Z^{(\tau)} u, \varphi\right). \tag{35}$$

However,

$$\left(Z^{(\tau)} u_{[k_l]}, \varphi\right) \;=\; \left(\left(Z^{(\tau)} - Z^{(\tau_{k_l})}\right) u_{[k_l]}, \varphi\right) + (f, \varphi) \;=\; J_1(l) + (f, \varphi). \tag{36}$$

Moreover, taking into account (33) and (34), we have

$$\begin{aligned} |J_1(l)| \;&\le\; |\tau - \tau_{kl}| \left| (Z - L') u_{[k_l]}, \varphi \right| \;\le\; |\tau - \tau_{kl}| K_4 \left\| u_{[k_l]} \right\|_{C^{2,1}(Q_T)} \|\varphi\|_{C^{0,\lambda}(Q_T)} \;\le \\ &\le\; K_3 K_4 |\tau - \tau_{kl}| \, \| f \|_{L_p(Q_T)} \|\varphi\|_{C^{0,\lambda}(Q_T)}. \end{aligned} \tag{37}$$

It follows from (37) that $\lim\limits_{l\to\infty} J_1(l) = 0$. From (36) and (37) we get that $\left(Z^{(\tau)} u, \varphi\right) = (f, \varphi)$, i.e., $Z^{(\tau)} u = f(x,t)$, everywhere in Q_T. Thus, we showe that $\tau \in E$, i.e., the set E is closed.

The theorem is proved.

Now we prove some estimate for the solution, which can also be taken as an independent result.

Theorem 3. *Let the conditions* (3) – (7) *be satisfied for the coefficients of the operator* (1). *Then for any function* $u(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$, *the following estimate is true*:

$$\| u(x,t) \|_{C(Q_T)} \;\le\; k \| f \|_{L_{n+1}(Q_T)}, \tag{38}$$

where $k = k(\gamma, n)$.

Proof. Suppose that $(x^0, t^0) \in Q_T$ and at this point $\sup\limits_{Q_T} u = u(x^0, t^0) = \mu > 0$. Let us take an auxiliary function $z = u^m$, where $m \ge 2$ is a natural number, which will be chosen later. Denote by A_z the set

$$\Big\{ (x,t) : (x,t) \in Q_T,\; u(x,t) \ge 0,\; z_t(x,t) \ge 0,\; z_{tt}(x,t) \le 0,$$
$$\left\| z_{ij}(x,t) \right\| \text{ is a positively defined matrix} \Big\}.$$

We have

$$\begin{aligned} \mu^{m(n+1)} \;&\le\; K_1 \int\limits_{A_z} \left(z_t - \sum_{i,j=1}^{n} a_{ij} z_{ij} \right)^{n+1} dx\,dt \;\le\; K \int\limits_{A_z} \left(z_t - \sum_{i,j=1}^{n} a_{ij} z_{ij} - \psi(x,t) z_{tt} \right)^{n+1} dx\,dt \;\le \\ &\le\; K_2 \int\limits_{A_z} \left[m u^{m-1}(-Zu) + m u^{m-2} \left(u(x,t) \left(\sum_{i=1}^{n} b_i(x,t) \right)^2 \right)^{1/2} |\nabla_x u(x,t)| \;+ \right. \\ &\qquad \left. +\; c(x,t) u^2 - (m-1)\gamma |\nabla_x u(x,t)|^2 \right]^{n+1} dx\,dt. \end{aligned} \tag{39}$$

If $(x,t) \in A_z$ is such that

$$|\nabla_x u(x,t)| \geq \frac{|b(x,t)|}{(m-1)\gamma} u(x,t),$$

then

$$u|b||\nabla_x u(x,t)| + cu^2 - (m-1)\gamma|\nabla_x u(x,t)|^2 \leq 0.$$

However, if

$$|\nabla_x u(x,t)| \leq \frac{|b(x,t)|}{(m-1)\gamma} u(x,t) \quad \text{for} \quad (x,t) \in A_z,$$

then

$$u|b||\nabla_x u(x,t)| + cu^2 - (m-1)\gamma|\nabla_x u(x,t)|^2 \leq \frac{u^2}{(m-1)\gamma}\left(|b|^2 + (m-1)\gamma c\right).$$

Now we take $\max\left\{2,\ 1+\frac{m}{\gamma}\right\}$ as $\psi_\varepsilon(x,t)m$. Then from (14) we get that

$$\mu^{m(n+1)} \leq K_2 m^{n+1} \mu^{(m-1)(n+1)} \int\limits_{Q_{T_z}} |f|^{n+1} dx dt.$$

Hence, the estimate (38) with $K = K_2^{1/(n+1)} m$ is obtained in a standard way. The case where $(x^0, t^0) = (x^0, T)$, $x^0 \in \Omega$ is considered similarly.

Theorem 4. *The conditions of Theorem* 2 *be satisfied and in the cylinder* Q_T *the solution to the first boundary-value problem* (1), (2) *be defined,* $f \in L_p(Q_T)$, $p > n+2$. *Then the following estimate is true*:

$$\|u(x,t)\|_{C^{1+\lambda}(Q_T)} \leq K_4 \|f\|_{L_p(Q_T)}. \tag{40}$$

Proof. To prove this, we should use the estimate (25) from Theorem 2 and the estimate (38) from Theorem 3, which implies the estimate (40).

As a consequence of the estimate (40), we get the theorem on classical solvability of the first boundary-value problem for the operator Z, which can be proved by the standard Lere – Schauder method [6].

Theorem 5. *Let the conditions of Theorem* 2 *be satisfied. Then the problem* (1), (2) *has a classical solution* $u(x,t) \in C^{2,1,\lambda}(Q_T)$ *and* $\lambda > 0$ *depends only on* σ, n.

Note that classical solvability can be proved analogously to Theorem 2.

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