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ENERGY INTERACTION BETWEEN LINEAR AND NONLINEAR OSCILLATORS (ENERGY TRANSIENT THROUGH THE SUBSYSTEMS IN THE HYBRID SYSTEM)*

ЕНЕРГЕТИЧНА ВЗАЄМОДІЯ МІЖ ЛІНІЙНИМИ ТА НЕЛІНІЙНИМИ ОСЦИЛЯТОРАМИ (ПРОЦЕС ПЕРЕХОДУ ЕНЕРГІЇ ЧЕРЕЗ ПІДСИСТЕМИ У ГІБРИДНІЙ СИСТЕМІ)

The study of the transfer of energy between subsystems coupled in hybrid system is very important for different applications. This paper presents an analytical analysis of the transfer of energy between linear and nonlinear oscillators for free vibrations when oscillators are statically, as well as dynamically, connected into double-oscillator system, as the two new hybrid systems, every with two degrees of freedom. The analytical analysis showed that the elastic connection between oscillators caused the appearance of a like two-frequency regime of time function, and that the energy transfer between subsystems appears. Also, the dynamical linear constraint between oscillators, each with one degree of freedom, coupled in hybrid system changes dynamics from single frequency regimes into like two-frequency regimes. The dynamical constraint as a connection between subsystems is realized by rolling element with inertia properties. In this case, an analytical analysis of the transfer energy between linear and nonlinear oscillators for free vibrations is also performed.

The two Lyapunov exponents corresponding to each of two eigen modes are expressed by using energy of the corresponding eigen time component.

Вивчення переносу енергії між підсистемами, що поєднані у гібридну систему, є дуже важливим для різних застосувань. У даній статті проведено аналітичне дослідження переносу енергії між лінійним та нелінійним осциляторами при вільних коливаннях для випадків як статичного, так і динамічного поєднання осциляторів у подвійно-осциляторну систему у вигляді двох нових гібридних систем із двома ступенями вільності кожна. Аналітичне дослідження показало, що пружне поєднання осциляторів зумовлює встановлення двочастотно-подібного режиму функції часу і спричиняє перенос енергії між підсистемами. Динамічний лінійний зв'язок між осциляторами, що поєднані у гібридну систему і мають один ступінь вільності кожен, змінює динаміку з одночастотних режимів до двочастотно-подібних режимів. Динамічний зв'язок як поєднання підсистем реалізовано елементом, що котиться і має інерційні властивості. Також проведено аналітичне дослідження переносу енергії між лінійним і нелінійним осциляторами у цьому випадку.

Для двох експонент Ляпунова, що відповідають кожній з двох власних мод, побудовано вирази з використанням енергії відповідних власних компонент часу.

1. Introduction. *1.1.* The study of the transfer of energy between subsystems coupled in hybrid system (see [1 – 16]) is very important for different applications. The new manuscripts by author (see [17 – 19, 41 – 44]) presents analytical analysis of the transfer of energy between plates for free and forced transversal vibrations of an elastically connected double-plate system. The analytical analysis showed that the elastic connection between plates caused the appearance of a two-frequency regime of time functions, which corresponds to one eigenamplitude function of one mode, and also that time functions of different vibration modes are uncoupled, but energy transfer between plates in one eigen mode appears. It was shown for each shape of vibrations. Series of two Lyapunov exponents corresponding to one eigen amplitude mode are expressed by using energy of the corresponding eigen amplitude time component. Then it is very important to investigate transfer of energy between subsystems in the hybrid system, and also between different subprocesses in the system with hybrid process containing these subprocesses coupled with different type of connections. The

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elements of connections between subsystems are possible to realize by static, dynamical and rheological elements and also by hybrid connection containing elements with static, dynamic and rheological properties. This is the reason to investigate basic types of elementary connections between subsystem coupled in the hybrid system beginning with subsystems, every of both, with one degree of freedom and compare kinetic parameters, properties of the simple dynamical processes in the separate decoupled subsystems into separate their integrity and compare processes in coupled subsystem taking into account interaction and changes in the dynamics of the components in system present in their subsystems. It is especially important to analyze energy in the decoupled subsystem as a separate system with proper integrity of dynamical processes.

1.2. When, at an international conference ICNO in Kiev in 1969, my professor of mechanics and mathematics, D. Rašković (see [16, 20]) introduced me to academician Yurii Alekseevich Mitropolskii (see [21 – 24]) and when I started really to understand the differences between linear and nonlinear dynamics, I knew I was on the right path of research which enchanted me ever more by understanding new phenomena and their variety in nonlinear dynamics of realistic engineering and other dynamical systems. (First my knowledge about properties of nonlinearity and the nonlinear function I obtained in high school from my excellent professor of mathematics Draginja Nikolić and during my research Matura work on the subject of elementary functions and their graphics as a final high school examination.)

For beginning of this paper, it is necessary to present a survey of original results of the author and of researchers from Faculty of Mechanical Engineering University of Niš (see [1 – 3, 10, 15, 20, 25 – 32]), inspired and/or obtained by the Krylov – Bogolyubov – Mitropolskii asymptotic method (see [21 – 24]), and as a direct influence of professor Rašković [16] scientific instruction and published Mitropolskii's papers and monographs. These results have been published in scientific journals, and were presented on the scientific conferences and in numerous degree works and theses supervised by Mitropolskii (in 1972 – 1975), Rašković (in 1964 – 1974), and Hedrih (in 1976 – 2006).

The original results obtained by Hedrih (Stevanović), Rašković, Kozić, Pavlović, Mitić, Filipovski, Janevski, and Simonović contain asymptotic analysis of the nonlinear oscillatory motions of elastic bodies: beams, plates, shells and shafts. The multifrequency oscillatory motion of elastic bodies was studied. Corresponding differential equations systems of first approximations for amplitudes and phases multifrequency regime of elastic bodies nonlinear oscillations were composed. The characteristic properties of nonlinear systems passing through coupled multifrequency resonant state and mutual influences excited modes were discovered.

In the same cited papers amplitude-frequency and phase frequency curves for stationary and nonstationary coupled multifrequency resonant states, based on the numerical experiment on the differential equations systems are presented.

Using ideas of averaging and asymptotic methods of Krylov, Bogolyubov, and Mitropolskii. K. Hedrih gives the asymptotic approximations of the solutions for one-, two-, three- and four-frequency vibrations of elastic beams, shaft and thin elastic plates, as well as of the thin elastic shells with positive constant Gauss' curvatures and finite deformations, and differential equations system for amplitude and phase corresponding vibration regimes.

Some results of an investigation of multifrequency vibrations in single frequency regime in nonlinear systems with many degrees of freedom and with slowchanging parameters are presented by Stevanović and Rašković article from 1974 (see [20]). Application of the Krylov – Bogolyubov – Mitropolskii asymptotic method for study of elastic bodies nonlinear oscillations and energetic analysis of the elastic bodies oscillatory motions give new results in theses by Stevanović in 1975. One-frequency transversal oscillations of thin rectangular plate with nonlinear constitutive material stressstrain relations and nonlinear transversal vibrations of a plate with special analysis of influence of weak nonlinear boundary conditions are contents of the articles by Hedrih (1979, 1981) and than asymptotic solution of the nonlinear equations of the thin elastic shell with positive Gauss' curvature in two-frequency regime is pointed out in the article by Hedrih (1983). Two-frequency oscillations of the thin elastic shells with finite deformations and interactions between harmonics have been studied by Hedrih and Mitić (see [27]) and multifrequency forced vibrations of thin elastic shells with a positive Gauss's curvature and finite displacements by Hedrih in 1984, and also on the mutual influence between modes in nonlinear systems with small parameter applied to the multifrequencies plate oscillations are studied by Hedrih, Kozić, Pavlović, and Mitić (see [28]), as well as Herdih and Simonović (see [42, 43]), and Janevski (see [44]).

Multifrequency forced vibrations of thin elastic shells with a positive Gauss' curvature and finite deformations and initial deformations influence of the shell middle surface to the phase-frequency characteristics of the nonlinear stationary forced shell's vibrations and numerical analysis of the four-frequency vibrations of thin elastic shells with Gauss' positive curvature and finite deformations by Hedrih and Mitić [27] and also initial displacement deformation influence of the thin elastic shell middle surface to the resonant jumps appearance by Hedrih and Mitić (1987). By means of the graphical presentations from the cited references, analysis was made and some conclusions about nonlinear phenomenon in multifrequency vibrations regimes were pointed out. Some of these conclusions we quote here: nonlinearities are the reason for the appearance of interaction between modes in multifrequency regimes; in the coupled resonant state one or several resonant jumps appear on the amplitude-frequency and phase frequency curves; these resonant jumps are from smaller to greater amplitudes and vice versa.

Unique trigger of singularities (see [13, 33 – 38]) with one unstable homoclinic saddle type point, and with two singular stable knot type points appear in one frequency stationary resonant state. It is visible on the phase-frequency as well as on the amplitude-frequency for stationary resonant state.

In the case of the multifrequency coupled resonant state and in the appearance of the more resonant coupled modes in resonant range of corresponding frequencies, unique trigger, and multiplied triggers appear (see [33]). Maximum number of triggers coupled is adequate to number of coupled modes and resonant frequencies of external excitations. Multiplied triggers contain as many unstable saddle homoclinic points in the mapped plane as the number of resonant frequencies of external excitations. For example, if a four-frequency coupled resonant process in u-v plane is in question, four homoclinic saddle type points appear. The appearance of these unstable homoclinic points requires further study, since it induces instability elements in a stationary nonlinear multifrequency process.

Using differential equations systems of the first approximation of multifrequency regime of stationary and nonstationary resonant states, we analyzed the energy of excited modes and transfer of energy from one to other modes (see [26, 28]). On the basis of this analysis, the question of excitation of lower frequency modes by higher frequency mode in the nonlinear multifrequency vibration regimes was opened.

1.3. In many engineering systems with nonlinearity, high frequency excitations are sources of the appearance of multifrequency resonant regimes with high frequency modes as well as low frequency modes. It is visible from many experimental research results and also theoretical results (see [14]). The interaction between amplitudes and phases of the different modes in the nonlinear systems with many degrees of the freedom as well in the deformable body infinite numbers frequency vibration free and forced regimes is observed theoretically by averaging asymptotic methods of Krylov, Bogolyubov, and Mitropolskii. This knowledge has great practical importance.

Application of the Krylov – Bogolyubov – Mitropolskii asymptotic method as well as energy approach given in monographs by Mitropolskii (see [21, 22]) for study of the elastic bodies nonlinear oscillations and energetic analysis of the elastic bodies oscillatory motions give new results in master and doctoral theses by Stevanović in 1972 and 1975.

In numerous papers (see [1, 2, 9, 15, 27, 28, 41]) author presented transfer of energy between modes in nonlinear deformable body vibrations by using averaging and asymptotic methods of Krylov, Bogolyubov, and Mitropolskii for obtaining system of the differential equations of amplitudes and phases in first approximations and expression for energy of the excited modes depending of amplitudes, phases and frequencies of different nonlinear modes. By means of these obtained asymptotic approximations of the solutions, the energy analysis of the interaction of the modes in the cases of then multifrequency vibration regimes in the nonlinear elastic systems (beams, plates and shells) excited by initial conditions for free and forced vibrations were made and transfer energy between modes is identified. Also, for the case of the forced frequency of the external excitation in the resonant frequency range near to one of the natural eigen frequency of the basic linear system two or more resonant energy jumps are present and it is possible to identified by the obtained system of the differential equations of amplitudes and phases in first approximations. Trigger of coupled singularities, as well as coupled triggers of the energy values is also present in the nonlinear system multifrequency resonant stationary and nonstationary regimes during to increasing and decreasing values of the external excitation frequencies though corresponding mode resonant ranges.

In conclusion of tills part we can summarize the following: oscillatory processes in dynamical systems depend on systems character. In such systems energy is also transformed from one form to another and has different flows inside a dynamical system. Transformation of kinetic energy into potential energy and vice versa occurs in conservative systems, but when lineal systems are in question, the energy carried by a considered harmonic (mode) of adequate frequency remains constant during a dynamical process, as does the total systems energy. There is no mutual influence between harmonics and the system may be presented by partial oscillators, the number of which is equal to the number of oscillations freedom degrees, or to the number of free vibrations own circular frequencies. During that the total energy of a single partial oscillator remains constant and the transformation of kinetic energy into potential occurs. In sash linear system transfer energy between modes no occurs (see [16]).

When nonlinear conservative systems are in question such conclusion for linear systems would be incorrect. The theoretical and experimental studies reveal that the interactions between widely separated nonlinear modes result in various bifurcations, the coexistence of multiple attractors, and chaotic attractors. The theoretical results show also that damping may be destabilizing. The different types of nonlinear phenomena in single degree of freedom nonlinear system dynamics are investigated between other researchers, also by Hedrih (see [39, 40]).

1.4. For introducing to the problem of the energy transfer or transient in the hybrid nonlinear systems, it is useful to take, for simple analysis, into consideration the change energy between parts of the energy carrying on the generalized coordinates ϕ and ρ in the very known system, known under name *spring pendulum system* with two degree of freedom. For the analysis of the energy in the spring pendulum we can write the kinetic and potential energies in the forms

$$E_k = \frac{1}{2} m\left[\dot{\rho}^2 + (\rho + l)^2 \dot{\phi}^2\right] \quad \text{and} \quad E_p = \frac{1}{2} c\rho^2 + mg(\rho + l)(1 - \cos\phi),$$

where m is mass of the pendulum, l length of pendulum string-neglected mass spring in the static equilibrium state of the pendulum, and c spring axial rigidity and ϕ and ρ are respectfully, angle and extension part of length of the string-spring of the pendulum with comparison of the spring length in static equilibrium state of the pendulum, take as the generalized coordinates of the system. For the linearized case for kinetic energy, after neglecting small member — part of kinetic energy on the generalized coordinate ϕ we can taking into account following expression:

expression $E_{k2} = \frac{1}{2} m(\rho + l)^2 \dot{\phi}^2$ changes into approximation $E_{k2} \approx \frac{1}{2} m(l\dot{\phi})^2$.

Only for small oscillations — perturbations from equilibrium position it is possible to use approximation of the expression for kinetic and potential energy in the form

$$E_k \approx \frac{1}{2} m\left[\dot{\rho}^2 + (l\dot{\phi})^2\right] \quad \text{and} \quad E_p \approx \frac{1}{2} c\rho^2 + \frac{1}{2} mgl\phi^2$$

in for that linearized case the generalized coordinates are normal coordinates of the small oscillations of the spring pendulum around equilibrium position $\rho = 0$, $\phi = 0$ and coordinates are decoupled. In this linearized case of the spring pendulum model, the energy carried on the these normal coordinates are uncoupled and transfer or transient of the total energy don't appeared between proper parts of the separate normal coordinate and on the separate processes defined by normal coordinates are conservative systems every with one degree of the freedom. In this case on the every of the coordinate there are conversion of the energy from kinetic to potential, but some of the both on one normal coordinates is constant:

$$E_{k\rho} \approx \frac{1}{2} m\dot{\rho}^2 \quad \text{and} \quad E_{p\rho} \approx \frac{1}{2} c\rho^2,$$

$$E_{k\phi} \approx \frac{1}{2} m(l\dot{\phi})^2 \quad \text{and} \quad E_{p\phi} \approx \frac{1}{2} mgl\phi^2.$$

This is visible from system differential equations of the linearized system

$$m\ddot{\rho} + c\rho = 0,$$

$$ml^2\ddot{\phi} + mgl\phi = 0$$

or in the form

$$\ddot{\rho} + \omega_2^2\rho = 0, \qquad \omega_2^2 = \frac{c}{m},$$

$$\ddot{\phi} + \omega_1^2\phi = 0, \qquad \omega_1^2 = \frac{g}{l}.$$

But for the nonlinear case the interaction between coordinates is present and then energy transient appears:

$$E_k = \frac{1}{2} m\left[\dot{\rho}^2 + l^2\dot{\phi}^2 + \rho^2\dot{\phi}^2 + 2\rho l\dot{\phi}^2\right]$$

and

$$E_p = \frac{1}{2} c\rho^2 + mgl(1 - \cos\phi) + mg\rho(1 - \cos\phi).$$

We can separate the following parts:

I. Kinetic and potential energy carrying on the coordinate ρ are

$$E_{k\rho} = \frac{1}{2} m\dot{\rho}^2 \quad \text{and} \quad E_{p\rho} = \frac{1}{2} c\rho^2 + mg\rho.$$

By analyzing these previous expressions we can see that with these expressions for decoupled oscillator with coordinate ρ, we have pure linear oscillator or harmonic oscillator with coordinate ρ and frequency $\omega_2^2 = \frac{c}{m}$, and separated process is isochronous.

II. Kinetic and potential energy carrying on the coordinate ϕ are

$$E_{k\phi} = \frac{1}{2} m l^2 \dot{\phi}^2 \quad \text{and} \quad E_{p\phi} = mgl(1-\cos\phi).$$

By analyzing these previous expressions we can see that with these expression for decoupled oscillator with coordinate ϕ, we have pure nonlinear oscillator with coordinate ϕ, and separated process is no isochronous. For linearized case this oscillator have frequency $\omega_1^2 = \frac{g}{l}$.

III. Then we can conclude that formally we have coupled two oscillators, one pure linear with one degree of freedom, and second nonlinear, also with one degree of freedom. In the hybrid system these oscillators are coupled and energy of the coupling containing two parts: one kinetic and second potential energy. Then, in the coupling, hybrid connections with static and dynamic properties are introduced.

Kinetic and potential energies of the coordinate ϕ and ρ interaction in the nonlinear hybrid model are:

$$E_{k(\phi,\rho)} = \frac{1}{2} m[\rho + 2l]\rho\dot{\phi}^2 \quad \text{and} \quad E_{p(\phi,\rho)} = -mg\rho\cos\phi. \tag{1}$$

For nonlinear case differential equations are

$$m\ddot{\rho} + c\rho + mg(1-\cos\phi) = 0,$$

$$ml^2\ddot{\phi} + \frac{d}{dt}\left[m(\rho+2l)\rho\dot{\phi}\right] + mgl\sin\phi = 0$$

or in the form

$$\ddot{\rho} + \omega_2^2\rho + g(1-\cos\phi) = 0,$$

$$l^2\ddot{\phi} + 2(\rho+l)\dot{\rho}\dot{\phi} + (\rho+2l)\rho\ddot{\phi} + \omega_1^2 l^2\sin\phi = 0,$$

or

$$\ddot{\rho} + \omega_2^2\rho = -g(1-\cos\phi), \tag{2}$$

$$\ddot{\phi} + \omega_1^2\phi = \omega_1^2(\phi - \sin\phi) - \frac{2}{l^2}\dot{\rho}\dot{\phi}(\rho+l) - \frac{1}{l^2}\rho(\rho+2l)\ddot{\phi} \tag{3}$$

or in nonlinear approximation forms for small oscillations around zero coordinates $\rho =$ $= 0$, $\phi = 0$ or of the around stable equilibrium position of the spring pendulum

$$\ddot{\rho} + \omega_2^2\rho \approx -g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \ldots\right), \tag{4}$$

$$\ddot{\phi} + \omega_1^2\phi \approx -\omega_1^2\left(\frac{\phi^3}{3} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \ldots\right) - \frac{2}{l^2}\dot{\rho}\dot{\phi}(\rho+l) - \frac{1}{l^2}\rho(\rho+2l)\ddot{\phi}. \tag{5}$$

If we introduce phase coordinate, then we can write

$$v = \dot{\rho},$$

$$\dot{v} = -\omega_2^2\rho - g(1-\cos\phi),$$

$$u = \dot{\phi},$$

$$\dot{u} = -\omega_1^2\phi + \omega_1^2(\phi - \sin\phi) - \frac{2}{l^2}\dot{\rho}\dot{\phi}(\rho+l) - \frac{1}{l^2}\rho(\rho+2l)\ddot{\phi}$$

or in the approximation

$$v = \dot{\rho},$$

$$\dot{v} \approx -\omega_2^2\rho - g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \ldots\right),$$

$$u = \dot{\phi},$$

$$\dot{u} \approx -\omega_1^2\phi - \omega_1^2\left(\frac{\phi^3}{3} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \ldots\right) - \frac{2}{l^2}\dot{\rho}\dot{\phi}(\rho + l) - \frac{1}{l^2}\rho(\rho + 2l)\ddot{\phi}.$$

From system equations (2), (3), as well from their approximations (4), (5) we can see that their right-hand parts are nonlinear and are functions of generalized coordinates, as well as of the generalized coordinates first and second derivatives. Also we can see that generalized coordinates ϕ and ρ around their zero values, when $\rho = 0$, $\phi = 0$ at the stable equilibrium position of the spring pendulum are also main coordinates of the linearized model. It is reason that the asymptotic averaged method is applicable for obtaining first asymptotic approximation of the solutions and it is possible to use for energy analysis of the transfer energy between energies carried by generalized coordinates ϕ and ρ in this nonlinear system with two degree of freedom, but formally we can take into account that we have two oscillators, one non linear and one linear every with one degree of freedom as two subsystems coupled in the hybrid system with two degree of freedom, by hybrid connection realized by statical and dynamical connections. This interconnection have two part of energy interaction between subsystems expressed by kinetic and potential energy in the form (1).

Taking into consideration some conclusion from considered system of the spring pendulum we can conclude also that it is important to consider more simple case of the coupling between linear and nonlinear systems with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the coupled subsystems.

2. Energy analysis of the oscillatory processes and of the modes in nonlinear system. When nonlinear conservative systems are in question such conclusion for linear systems would be incorrect. The theoretical and experimental studies reveal that the interactions between widely separated modes result in various bifurcations, the coexistence of multiple attractors, and chaotic attractors. The theoretical results show also that damping may be destabilizing.

The interaction between high-frequency and low-frequency modes observed experimentally and demonstrated theoretically is of great practical importance. In many engineering systems, high-frequency excitations can be caused by rotating machinery and some debalances.

Kinetic energy and potential energy in first asymptotic approximation for nonlinear conservative system nonlinear modes using normal coordinates of unperturbed corresponding linear system are (see [1 – 3]):

$$\mathbf{E}_k = \sum_{s=1}^{s=n} \mathbf{E}_{ks} = \sum_{s=1}^{s=n} \left(\dot{\xi}_s^2\right) +$$

$$+ \varepsilon g\left(\xi_1, \xi_2, \ldots, \xi_s, \xi_r, \ldots, \xi_{n-1}, \xi_n, \dot{\xi}_1, \dot{\xi}_2, \ldots, \dot{\xi}_s, \dot{\xi}_r, \ldots, \dot{\xi}_{n-1}, \dot{\xi}_n\right),$$

$$\mathbf{E}_p = \sum_{s=1}^{s=n} \left(\omega_s^2 \xi_s^2\right) + f\left(\xi_1, \xi_2, \ldots, \xi_s, \xi_r, \ldots, \xi_{n-1}, \xi_n\right),$$

where

$$\xi_s = a_s \cos(\theta_s + \psi_s), \quad s = 1, 2, \ldots, n,$$

are first asymptotic approximations of normal coordinates, and a_s are amplitudes, and $\theta_s + \psi_s$ are phases as a functions of time and which are calculate from differential equations first approximations (see [21, 22]).

2.1. Nonlinear oscillator. Kinetic and potential energies and Rayleigh dissipative function of nonlinear oscillator with one degree of freedom and generalized coordinate x_1 are:

$$E_{k(1)} = \frac{1}{2}m_1\dot{x}_1^2, \qquad E_{p(1)} = \frac{1}{2}c_1x_1^2 + \frac{1}{4}\tilde{c}_1x_1^4 \quad \text{and} \quad \Phi_{(1)} = \frac{1}{2}b_1\dot{x}_1^2,$$

where m_1 is masses, c_1 is the spring rigidity coefficient of the linear elasticity low, and $\tilde{c}_1$ the spring rigidity coefficient of the nonlinear elasticity low, b_1 coefficient of the system linear dumping force. For this nonlinear oscillator it is right: $\frac{d}{dt}\big(E_{k(1)} + E_{p(1)}\big) = -2\Phi_{(1)}$ and for the case of the free vibrations.

For this case differential equation is in the following form:

$$\ddot{x}_1 + 2\delta_1\dot{x}_2 + \omega_1^2x_1 = -\tilde{\omega}_{N1}^2x_1^3,$$

where

$$\omega_1^2 = \frac{c_1}{m_1}, \qquad 2\delta_1 = \frac{b_1}{m_1}, \qquad \tilde{\omega}_{N1}^2 = \frac{\tilde{c}_1}{m_1}$$

and characteristic equation of the basic liner equation corresponding to previous has the following characteristic numbers: $\lambda_{1,2} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^2 - \delta_1^2} = -\delta_1 \mp \mathbf{i}p_1$ for the small damping coefficient $\delta_1 < \omega_1$, and solution for free vibrations is: $x_1(t) = R_{01}e^{-\delta_1 t}\cos(p_1t + \alpha_{01})$. To obtain approximation by using averaged method, we propose solution in the following form:

$$x_1(t) = R_1(t)e^{-\delta_1 t}\cos\Phi_1(t),$$

where $R_1(t)$ and $\Phi_1(t)$ are unknown functions. Also we can write: $\Phi_1(t) = p_1t + \phi_1$. After averaging with respect to the full phase $\Phi_1(t)$ we obtain the following system of the averaged equations:

$$\dot{R}_1(t) = 0,$$

$$\dot{\phi}_1(t) = -\frac{3}{8p_1}\tilde{\omega}_{N1}^2R_1^2(t)e^{-2\delta_1 t}$$

and after integration we obtain for amplitude and phase the following first approximation:

$$R_1(t) = R_{01} = \text{const},$$

$$\phi_1(t) = \frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t} + \alpha_{01}$$

and for full phase:

$$\Phi_1(t) = p_1t + \frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t} + \alpha_{01}$$

and solution in the first averaged approximation form is:

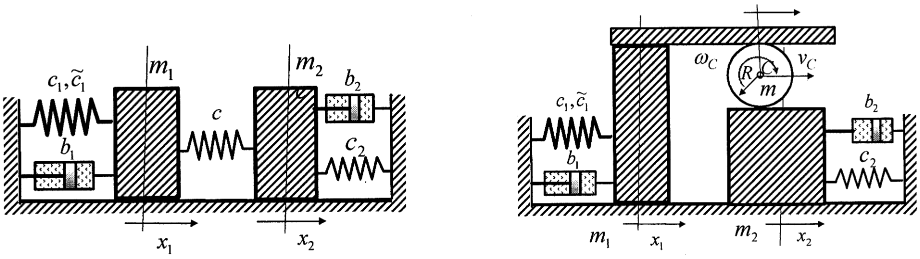
$$x_1(t) = R_{01}e^{-\delta_1 t}\cos\Phi_1(t) = R_{01}e^{-\delta_1 t}\cos\left(p_1t + \frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t} + \alpha_{01}\right)$$

we can see that amplitude of the solution in the first averaged approximation form is in the form $R_{01}e^{-\delta_1 t}$ and that phase are also function of the time, and also frequency $\tilde{p}_1(t) = p_1 + \frac{3}{8p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t}$ is changeable with time in the first approximation obtained by averaged method.

By using previous averaged solution we obtain Lyapunov exponent in the form

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t)+\frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta_1 < 0$$

or in the form



a *b*

Fig. 1. Two hybrid systems containing coupled subsystems by (*a*) static constraint, coupled by linear spring rigidity c and (*b*) dynamical constraint, coupled by rolling element of the mass m — dynamic coupling: one nonlinear (left) and second linear (right).

$$\tilde{\lambda}_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t)+\frac{\omega_{N1}^2}{\omega_1^2}x_1^4(t)+\frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = \lim_{t\to\infty}\frac{1}{2t}\ln\left[\frac{E_{\text{sist}}}{2m_1\omega_1^2}\right] = -\delta_1 < 0.$$

In our research, we can investigate system with small nonlinearity and small vibrations around periodic vibrations.

2.2. Linear oscillator. Kinetic and potential energies and Rayleigh dissipative function of linear oscillator with one degree of freedom and generalized coordinate x_2 are:

$$E_{k(2)} = \frac{1}{2}m_2\dot{x}_2^2, \quad E_{p(2)} = \frac{1}{2}c_2x_2^2 \quad \text{and} \quad \Phi_2 = \frac{1}{2}b_2\dot{x}_2^2,$$

where m_2 is mass, c_2 is the spring rigidity coefficient of the linear elasticity low, b_2 coefficient of the system linear dumping force.

For this system it is possible to show that is $\frac{d}{dt}\left(E_{k(2)}+E_{p(2)}\right) = -2\Phi_2$. For this case differential equation is in the following form: $\ddot{x}_2 + 2\delta_2\dot{x}_2 + \omega_2^2x_2 = 0$, where $\omega_2^2 = \frac{c_2}{m_2}$, $2\delta_2 = \frac{b_2}{m_2}$, and with characteristic numbers $\lambda_{1,2} = -\delta_2 \mp \mathbf{i}\sqrt{\omega_2^2-\delta_2^2}$ for the small damping coefficient $\delta_2 < \omega_2$. Solution for free vibrations is $x_2(t) = R_0e^{-\delta_2 t}\cos(p_2t+\alpha_2)$.

2.3. Hybrid system — coupled nonlinear and linear oscillators by statical constraint. Kinetic and potential energies and Rayleigh dissipative function of the hybrid system, containing two subsystems — one linear oscillator and one nonlinear oscillator, with two degree of freedom expressed by generalized coordinates x_1 and x_2 (see Fig. 1 (*a*)) are:

$$E_k = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2,$$

$$E_p = \frac{1}{2}c_1x_1^2 + \frac{1}{4}\tilde{c}_1x_1^4 + \frac{1}{2}c(x_1-x_2)^2 + \frac{1}{2}c_2x_2^2,$$

$$\Phi = \frac{1}{2}b_1\dot{x}_1^2 + \frac{1}{2}b_2\dot{x}_2^2,$$

where m_1 and m_2 are masses, c_1, c and c_2 are the spring rigidity coefficients of the linear elasticity low, and $\tilde{c}_1$ the spring rigidity coefficient of the nonlinear elasticity low, b_1 and b_2 coefficient of the system linear dumping forces. For this system it is possible to show that is $\frac{d}{dt}(E_k + E_p) = -2\Phi$.

Energy interaction in this hybrid system, containing two coupled subsystems by statical constraint, is potential energy of the spring for coupling nonlinear and linear subsystem and is expressed in the form

$$E_{p(1,2)} = \frac{1}{2}c(x_2 - x_1)^2.$$

Coupled system of differential equations of the hybrid system containing two subsystems, one nonlinear and one linear are in the form

$$\ddot{x}_1 + 2\delta_1\dot{x}_2 + \left(\omega_1^2 + a_1^2\right)x_1 - a_1^2 x_2 = -\tilde{\omega}_{N1}^2 x_1^3,$$

$$\ddot{x}_2 + 2\delta_2\dot{x}_2 + \left(\omega_2^2 + a_2^2\right)x_2 - a_2^2 x_1 = 0,$$

where are

$$\omega_i^2 = \frac{c_i}{m_i}, \quad 2\delta_i = \frac{b_i}{m_i}, \quad a_i^2 = \frac{c}{m_i}, \quad \tilde{\omega}_{N1}^2 = \frac{\tilde{c}_1}{m_i}, \quad i = 1, 2.$$

Taking into account that consideration of the homogeneous system does not lose generality of the phenomena, next our considerations are applied to this homogeneous hybrid system.

For the basic linear equations of the coupled system of the differential equations of the hybrid system containing two subsystems, one linearized and one linear are in the form:

$$\ddot{x}_1 + 2\delta_1\dot{x}_1 + \left(\omega_1^2 + a_1^2\right)x_1 - a_1^2 x_2 = 0,$$

$$\ddot{x}_2 + 2\delta_2\dot{x}_2 + \left(\omega_2^2 + a_2^2\right)x_2 - a_2^2 x_1 = 0$$

and for case that linearized and linear systems are equal $\omega_1^2 = \omega_2^2$ and $\delta_1 = \delta_2$ and $a_1^2 = a_2^2$, we can define:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{2\delta} & \\ & \mathbf{2\delta} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \omega_1^2 + a_1^2 & -a_1^2 \\ -a_1^2 & \omega_1^2 + a_1^2 \end{pmatrix}$$

and frequency equation is in the form:

$$\left|\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}\right| = \begin{vmatrix} \lambda^2 + 2\delta\lambda + \omega_1^2 + a_1^2 & -a_1^2 \\ -a_1^2 & \lambda^2 + 2\delta\lambda + \omega_1^2 + a_1^2 \end{vmatrix} = 0$$

with characteristic numbers:

$$\lambda_{1,2} = -\delta \mp \mathbf{i}\,p_1, \quad \lambda_{1,2} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^2 - \delta_1^2} = -\delta_1 \mp \mathbf{i}p_1$$

for the small damping coefficient $\delta_1 < \omega_1$,

$$\lambda_{3,4} = -\delta \mp \mathbf{i}\,\tilde{p}_1, \quad \lambda_{3,4} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^2 + 2a_1^2 - \delta_1^2} = -\delta_1 \mp \mathbf{i}\tilde{p}_1,$$

where

$$p_1 = \sqrt{\omega_1^2 - \delta_1^2}$$

for the small damping coefficient $\delta_1 < \omega_1$,

$$\tilde{p}_2 = \tilde{p}_1 = \sqrt{\omega_1^2 + 2a_1^2 - \delta_1^2}$$

for the small damping coefficient $\delta_1 < \omega_1$, and solution of the linear coupled system we can write in the following two-frequency form:

$$x_1(t) = e^{-\delta t}\left[R_{01}\cos(p_1 t + \alpha_{01}) + R_{02}\cos(\tilde{p}_2 t + \alpha_{02})\right],$$

$$x_2(t) = e^{-\delta t}\left[R_{01}\cos(p_1 t + \alpha_{01}) - R_{02}\cos(\tilde{p}_2 t + \alpha_{02})\right],$$

where amplitudes and phases R_{0i} and α_{0i} are constants.

By using averaged method, a first approximation of the solution of the hybrid system, containing coupled nonlinear and linear system, we propose solutions in the following forms:

$$x_1(t) = e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) + R_{21}(t)\cos\Phi_2(t)\right],$$

$$x_2(t) = e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) - R_{21}(t)\cos\Phi_2(t)\right],$$

where $R_i(t)$ and $\Phi_i(t)$ are unknown functions. Also we can write: $\Phi_i(t) = p_1 t + \phi_i$. Then after averaging with respect to the full phase $\Phi_1(t)$ we obtain the following system of the first asymptotic approximation of the system differential equations for amplitude $R_i(t)$ and phase $\Phi_i(t)$:

$$\dot{R}_1(t) = 0,$$

$$\dot{\phi}_1(t) = -\frac{3}{16 p_1}\tilde{\omega}_{N1}^2\left[R_1^2(t) + 2R_2^2(t)\right]e^{-2\delta_1 t},$$

$$\dot{R}_2(t) = 0,$$

$$\dot{\phi}_2(t) = -\frac{3}{16 \tilde{p}_2}\tilde{\omega}_{N1}^2\left[R_2^2(t) + 2R_1^2(t)\right]e^{-2\delta_1 t}$$

and after integration we obtain

$$R_1(t) = R_{01} = \text{const},$$

$$\phi_1(t) = \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta_1 t} + \alpha_{01},$$

$$R_2(t) = R_{02} = \text{const},$$

$$\phi_2(t) = \frac{3}{32\delta \tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta_1 t} + \alpha_{02}.$$

Solutions in the first asymptotic approximation in averaged form of the hybrid system are

$$x_1(t) = e^{-\delta t}\left\{R_{01}\cos\left(p_1 t + \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t} + \alpha_{01}\right) + \right.$$

$$\left. + R_{02}\cos\left(\tilde{p}_2 t + \frac{3}{32\delta\tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t} + \alpha_{02}\right)\right\},$$

$$x_2(t) = e^{-\delta t}\left\{R_{01}\cos\left(p_1 t + \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t} + \alpha_{01}\right) - \right.$$

$$\left. - R_{02}\cos\left(\tilde{p}_2 t + \frac{3}{32\delta\tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t} + \alpha_{02}\right)\right\}$$

we can see that amplitudes of the solution in the first approximation is in the form $R_{0i}e^{-\delta t}$ and that phases are also functions of the time, and also frequencies

$$p_1(t) = p_1 + \frac{3}{16\tilde{p}_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t}$$

and

$$\tilde{p}_2(t) = \tilde{p}_2 + \frac{3}{16\tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t}$$

are changeable with time in the first approximation obtained by averaged method.

By using previous averaged solution we can obtain Lyapunov exponents in the forms

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta < 0,$$

$$\lambda_2 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_2^2(t) + \frac{1}{\omega_2^2}\dot{x}_2^2(t)\right] = -\delta < 0.$$

Also, taking into account that system is nonlinear

$$\tilde{\lambda}_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t) + \frac{\tilde{\omega}_{N1}^2}{\omega_1^2}x_1^4(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] =$$

$$= \lim_{t\to\infty}\frac{1}{2t}\ln\left[\frac{E_{\text{subsist}(1)}}{2m_1\omega_1^2}\right] = -\delta < 0.$$

For the nonhomogeneous case we can define

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1}\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{2}\delta_1 & \\ & \mathbf{2}\delta_2\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \omega_1^2 + a_1^2 & -a_1^2 \\ -a_2^2 & \omega_2^2 + a_2^2\end{pmatrix}$$

and frequency equation is in the form:

$$\left|\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}\right| = \begin{vmatrix} \lambda^2 + 2\delta_1\lambda + \omega_1^2 + a_1^2 & -a_1^2 \\ -a_2^2 & \lambda^2 + 2\delta_2\lambda + \omega_2^2 + a_2^2\end{vmatrix} = 0$$

or in the form

$$\left(\lambda^2 + 2\delta_1\lambda + \omega_1^2 + a_1^2\right)\left(\lambda^2 + 2\delta_2\lambda + \omega_2^2 + a_2^2\right) - a_1^2a_2^2 = 0$$

with four roots: $\lambda_{1,2} = -\hat{\delta}_1 \mp \mathbf{i}\,\hat{p}_1$ and $\lambda_{3,4} = -\hat{\delta}_2 \mp \mathbf{i}\,\hat{p}_2$. Own amplitude numbers we obtain from

$$\frac{A_1^{(s)}}{a_1^2} = \frac{A_2^{(s)}}{\lambda_s^2 + 2\delta_1\lambda_s + \omega_1^2 + a_1^2} = \tilde{C}_s \quad \text{or} \quad \frac{A_1^{(s)}}{K_{21}^{(s)}} = \frac{A_2^{(s)}}{K_{22}^{(s)}} = C_s$$

and solution of the linear coupled system we can write in the following form:

$$x_1(t) = K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left(\hat{p}_1t + \alpha_{01}\right) + K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left(\hat{p}_2t + \alpha_{02}\right),$$

$$x_2(t) = K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left(\hat{p}_1t + \alpha_{01}\right) + K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left(\hat{p}_2t + \alpha_{02}\right),$$

where amplitudes and phases R_{0i} and α_{0i} are constants.

By using asymptotic averaged method, a first asymptotic approximation of the solution of the hybrid system, containing coupled nonlinear and linear system as subsystems, we propose solutions in the following forms:

$$x_1(t) = K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_1(t)\cos\Phi_1(t) + K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\Phi_2(t),$$

$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_{01} \cos\Phi_1(t) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos\Phi_2(t),$$

where $R_i(t)$ and $\Phi_i(t)$ are unknown functions. Also we can write: $\Phi_i(t) = \hat{p}_i t + \phi_i$. And all next is similar as in previous considered part.

2.4. Hybrid system — coupled nonlinear and linear oscillators by dynamical constraint. In Fig. 1(b) we can see hybrid system containing two subsystems, one linear and one nonlinear coupled by dynamical constraint. Dynamical constraint consists of the one disk with mass m and mass inertia axial moment $\mathbf{J}_C$ with possibility of rolling between two masses m_1 and m_2 of the subsystems. In our research, we can investigate small nonlinearity in the subsystem, and also in the hybrid system and also small vibrations around periodic.

Angular velocity of the disk rotation, with mass m and mass inertia axial moment $\mathbf{J}_C$, between two masses m_1 and m_2 is: $\omega_C = \dfrac{\dot{x}_2 - \dot{x}_1}{2R}$, and velocity of the mass center is: $v_C = \dfrac{\dot{x}_1 + \dot{x}_2}{2}$. Kinetic energy of the coupling nonlinear and linear subsystems is

$$\mathbf{E}_{k(1,2)} = \frac{1}{2}\left[m v_C^2 + \mathbf{J}_C \omega_C^2\right] = \frac{1}{2}\left[m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \mathbf{J}_C\left(\frac{\dot{x}_2 - \dot{x}_1}{2R}\right)^2\right]$$

or

$$\mathbf{E}_{k(1,2)} = \frac{1}{2}\left(\hat{a}_{11}\dot{x}_1^2 + \hat{a}_{22}\dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\hat{a}_{12}\right),$$

where

$$\hat{a}_{11} = \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}, \quad \hat{a}_{22} = \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2} \quad \text{and} \quad \hat{a}_{12} = \frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}.$$

Then we have hybrid system with dynamic, but also linear, constraint between subsystems as a dynamic coupled subsystems.

Kinetic and potential energies and Rayleigh dissipative function of the hybrid system, containing two subsystems — one linear oscillator and one nonlinear oscillator, with two degree of freedom expressed by generalized coordinates x_1 and x_2 (see Fig. 1(a)) are

$$\mathbf{E}_k = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\left[m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \mathbf{J}_C\left(\frac{\dot{x}_2 - \dot{x}_1}{2R}\right)^2\right],$$

$$E_p = \frac{1}{2}c_1 x_1^2 + \frac{1}{4}\tilde{c}_1\dot{x}_1^4 + \frac{1}{2}c_2 x_2^2$$

and

$$\Phi = \frac{1}{2}b_1\dot{x}_1^2 + \frac{1}{2}b_2\dot{x}_2^2,$$

where m_1 and m_2 are masses, c_1, c and c_2 are the spring rigidity coefficients of the linear elasticity low, and $\tilde{c}_1$ the spring rigidity coefficient of the nonlinear elasticity low, b_1 and b_2 coefficient of the system linear dumping forces. For this system it is possible to show that is $\dfrac{d}{dt}(E_k + E_p) = -2\Phi$.

Energy interaction in this system is potential energy of the spring for coupling nonlinear and linear system and is expressed in the form:

$$E_k = \frac{1}{2}\left[m_1 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_1^2 + \frac{1}{2}\left[m_2 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_2^2 + \frac{1}{2}2\dot{x}_1\dot{x}_2\left[\frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}\right]$$

or in the form

$$E_k = \frac{1}{2}\left(\tilde{a}_{11}\dot{x}_1^2 + \tilde{a}_{22}\dot{x}_2^2 + 2\tilde{a}_{12}\dot{x}_1\dot{x}_2\right),$$

where

$$\tilde{a}_{11} = m_1 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2} = a_{11} + \hat{a}_{11},$$

$$\tilde{a}_{22} = m_2 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2} = a_{22} + \hat{a}_{22}, \qquad \tilde{a}_{12} = \frac{m}{4} - \frac{\mathbf{J}_C}{4R^2} = \hat{a}_{12}.$$

Coefficient $\tilde{a}_{12} = \frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}$ is coefficient of the subsystems coupling and as the constraint is dynamical then this coefficient is coefficient of inertia. When this coefficient is equal to zero, then the system coordinate x_1 and x_2 are decoupled and there are not energy of the coupling, but there are energy of the influence of the dynamic constraint by additional members.

Kinetic energy of the first subsystem as a one part of the hybrid system is

$$E_{k(1)} = \frac{1}{2}\left[m_1 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_1^2 \quad \text{or} \quad E_k = \frac{1}{2}\tilde{a}_{11}\dot{x}_1^2.$$

Kinetic energy of the second subsystem as a one part of the hybrid system is

$$E_{k(2)} = \frac{1}{2}\left[m_2 + \frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_2^2 \quad \text{or} \quad E_k = \frac{1}{2}\tilde{a}_{22}\dot{x}_2^2.$$

Kinetic energy of the coupling of the subsystems as a two parts of the hybrid system is

$$E_{k(1,2)} = \frac{1}{2}2\dot{x}_1\dot{x}_2\left[\frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}\right] \quad \text{or} \quad E_k = \tilde{a}_{12}\dot{x}_1\dot{x}_2.$$

Additional part of the kinetic energy of the first subsystem — reduction of the dynamic constraint to the first subsystem

$$E_{k(1)d} = \frac{1}{2}\left[\frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_1^2 = \frac{1}{2}\hat{a}_{11}\dot{x}_1^2.$$

Additional part of the kinetic energy of the first subsystem — reduction of the dynamic constraint to the second subsystem

$$E_{k(2)d} = \frac{1}{2}\left[\frac{m}{4} + \frac{\mathbf{J}_C}{4R^2}\right]\dot{x}_2^2 = \frac{1}{2}\hat{a}_{22}\dot{x}_2^2.$$

When the coefficient of subsystems coupling equals zero: $\tilde{a}_{12} = \left[\frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}\right] = 0$, then subsystems do not have kinetic energy interaction, but have additional part of kinetic energy of the first subsystem — reduction of the dynamic constraint to the first subsystem and additional part of the kinetic energy of the second subsystem — reduction of the dynamic constraint to the second subsystem.

System of differential equations is based on the kinetic and potential energy and Rayleigh dissipative function

$$E_k = \frac{1}{2}\left(\tilde{a}_{11}\dot{x}_1^2 + \tilde{a}_{22}\dot{x}_2^2 + 2\tilde{a}_{12}\dot{x}_1\dot{x}_2\right),$$

$$E_p = \frac{1}{2}c_1 x_1^2 + \frac{1}{4}\tilde{c}_1 x_1^4 + \frac{1}{2}c_2 x_2^2,$$

$$\Phi = \frac{1}{2}b_1\dot{x}_1^2 + \frac{1}{2}b_2\dot{x}_2^2.$$

System of differential equations based on the kinetic and potential energy and Rayleigh dissipative function are in the following form:

$$\tilde{a}_{11}\ddot{x}_1 + \tilde{a}_{12}\ddot{x}_2 + c_1x_1 + \tilde{c}_1x_1^3 + b_1\dot{x}_1 = 0,$$
$$\tilde{a}_{22}\ddot{x}_2 + \tilde{a}_{12}\ddot{x}_1 + c_2x_2 + b_2\dot{x}_2 = 0.$$

After multiplying first and second equation or the previous system by $\frac{1}{\hat{a}_{11}}$ and $\frac{1}{\hat{a}_{22}}$ respective and after introducing the following notations:

$$\kappa_1 = \frac{\tilde{a}_{12}}{\tilde{a}_{11}}, \quad \kappa_2 = \frac{\tilde{a}_{12}}{\tilde{a}_{22}}, \quad \tilde{\omega}_1^2 = \frac{c_1}{\tilde{a}_{11}},$$

$$\tilde{\omega}_2^2 = \frac{c_1}{\tilde{a}_{22}}, \quad \tilde{\tilde{\omega}}_{N1}^2 = \frac{\tilde{c}_1}{\tilde{a}_{11}} = \tilde{\omega}_{N1}^2\frac{m_1}{\tilde{a}_{11}}, \quad 2\tilde{\delta}_i = \frac{b_i}{\tilde{a}_{ii}}, \quad i = 1, 2,$$

we can write the following system of differential equations of the hybrid system:

$$\ddot{x}_1 + \kappa_1\ddot{x}_2 + \tilde{\omega}_1^2\dot{x}_1 + 2\tilde{\delta}_1\dot{x}_1 = -\tilde{\tilde{\omega}}_{N1}^2x_1^3,$$

$$\ddot{x}_2 + \kappa_2\ddot{x}_1 + \tilde{\omega}_2^2x_2 + 2\tilde{\delta}_2\dot{x}_2 = 0.$$

For the basic linear equations of the linear dynamically coupled system of the differential equations of the hybrid system containing two subsystems, one linearized and one linear are in the form

$$\ddot{x}_1 + \kappa_1\ddot{x}_2 + \tilde{\omega}_1^2x_1 + 2\tilde{\delta}_1\dot{x}_1 = 0,$$

$$\ddot{x}_2 + \kappa_2\ddot{x}_1 + \tilde{\omega}_2^2x_2 + 2\tilde{\delta}_2\dot{x}_2 = 0$$

we can define the following matrices:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \kappa_1 \\ \kappa_2 & \mathbf{1} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{2}\delta_1 & \\ & \mathbf{2}\delta_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \tilde{\omega}_1^2 & 0 \\ 0 & \tilde{\omega}_2^2 \end{pmatrix}$$

and frequency equation is in the form

$$\left|\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}\right| = \begin{vmatrix} \lambda^2 + 2\delta_1\lambda + \tilde{\omega}_1^2 & \kappa_1\lambda^2 \\ \kappa_2\lambda^2 & \lambda^2 + 2\delta_2\lambda + \tilde{\omega}_2^2 \end{vmatrix} = 0$$

or in the form

$$\left(\lambda^2 + 2\delta_1\lambda + \tilde{\omega}_1^2\right)\left(\lambda^2 + 2\delta_2\lambda + \tilde{\omega}_2^2\right) - \kappa_1\kappa_2\lambda^4 = 0$$

with four roots $\lambda_{1,2} = -\hat{\delta}_1 \mp \mathbf{i}\hat{p}_1$ and $\lambda_{3,4} = -\hat{\delta}_2 \mp \mathbf{i}\hat{p}_2$. Own amplitude numbers we obtain from

$$\frac{A_1^{(s)}}{-\kappa_1\lambda_s^2} = \frac{A_2^{(s)}}{\lambda_s^2 + 2\delta_1\lambda_s + \omega_1^2} = \tilde{C}_s \quad \text{or} \quad \frac{A_1^{(s)}}{K_{21}^{(s)}} = \frac{A_2^{(s)}}{K_{22}^{(s)}} = C_s$$

and the solution of the basic linear coupled system we can write in the following form:

$$x_1(t) = K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left(\hat{p}_1t + \alpha_{01}\right) + K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left(\hat{p}_2t + \alpha_{02}\right),$$

$$x_2(t) = K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left(\hat{p}_1t + \alpha_{01}\right) + K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left(\hat{p}_2t + \alpha_{02}\right)$$

where amplitudes and phases R_{0i} and α_{0i} are constants, depending of initial conditions.

By using asymptotic averaged method, a first asymptotic approximation of the solution of the hybrid system, containing dynamical coupled nonlinear and linear system, we propose solutions in the following forms:

$$x_1(t) = K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t) \cos\Phi_1(t) + K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos\Phi_2(t),$$

$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_{01} \cos\Phi_1(t) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos\Phi_2(t),$$

where $R_i(t)$ and $\Phi_i(t)$ are unknown functions. Also we can write $\Phi_i(t) = \hat{p}_1 t + \phi_i$. After averaging with respect to the full phase $\Phi_1(t)$ we obtain the following system of the first asymptotic averaged approximation of the equations for amplitudes $R_i(t)$ and phases $\Phi_i(t)$:

$$\dot{R}_1(t) = 0,$$

$$\dot{\phi}_1(t) = \frac{3}{16 p_1 \left[K_{21}^{(1)} K_{22}^{(2)} - K_{22}^{(1)} K_{21}^{(2)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times \left\{ e^{-2\hat{\delta}_1 t} \left(K_{21}^{(1)}\right)^3 [R_1(t)]^2 + 2 e^{-2\hat{\delta}_2 t} K_{21}^{(1)} \left[K_{21}^{(2)}\right]^2 [R_2(t)]^2 \right\},$$

$$\dot{R}_2(t) = 0,$$

$$\dot{\phi}_2(t) = \frac{3}{16 \hat{p}_2 \left[K_{21}^{(2)} K_{22}^{(1)} - K_{22}^{(2)} K_{21}^{(1)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times \left\{ 2 e^{-2\hat{\delta}_1 t} \left(K_{21}^{(1)}\right)^3 [R_1(t)]^2 + e^{-2\hat{\delta}_2 t} K_{21}^{(1)} \left[K_{21}^{(2)}\right]^2 [R_2(t)]^2 \right\}.$$

After integrating the system of averaged equations we obtain first approximation of the amplitudes and phases of the solution

$$\dot{R}_1(t) = R_{01} = \text{const},$$

$$\phi_1(t) = -\frac{3}{16 p_1 \left[K_{21}^{(1)} K_{22}^{(2)} - K_{22}^{(1)} K_{21}^{(2)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times \left\{ \frac{e^{-2\hat{\delta}_1 t}}{2\hat{\delta}_1} \left(K_{21}^{(1)}\right)^3 \left[R_{01}\right]^2 + \frac{e^{-2\hat{\delta}_2 t}}{\hat{\delta}_2} K_{21}^{(1)} \left[K_{21}^{(2)}\right]^2 \left[R_{02}\right]^2 \right\} + \alpha_{01},$$

$$R_2(t) = R_{02} = \text{const},$$

$$\phi_2(t) = -\frac{3}{16 \hat{p}_2 \left[K_{21}^{(2)} K_{22}^{(1)} - K_{22}^{(2)} K_{21}^{(1)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times \left\{ \frac{e^{-2\hat{\delta}_1 t}}{\hat{\delta}_1} \left(K_{21}^{(1)}\right)^3 \left[R_{01}\right]^2 + \frac{e^{-2\hat{\delta}_2 t}}{2\hat{\delta}_2} K_{21}^{(1)} \left[K_{21}^{(2)}\right]^2 \left[R_{02}\right]^2 \right\} + \alpha_{02}.$$

Full phases $\Phi_i(t)$ are

$$\Phi_1(t) = \hat{p}_1 t - \frac{3}{16 p_1 \left[K_{21}^{(1)} K_{22}^{(2)} - K_{22}^{(1)} K_{21}^{(2)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times \left\{ \frac{e^{-2\hat{\delta}_1 t}}{2\hat{\delta}_1} \left(K_{21}^{(1)}\right)^3 \left[R_{01}\right]^2 + \frac{e^{-2\hat{\delta}_2 t}}{\hat{\delta}_2} K_{21}^{(1)} \left[K_{21}^{(2)}\right]^2 \left[R_{02}\right]^2 \right\} + \alpha_{01},$$

$$\Phi_2(t) = \hat{p}_2 t - \frac{3}{16 \hat{p}_2 \left[K_{21}^{(2)} K_{22}^{(1)} - K_{22}^{(2)} K_{21}^{(1)}\right]} \tilde{\omega}_{N1}^2 \times$$

$$\times\left\{\frac{e^{-2\hat{\delta}_1 t}}{\hat{\delta}_1}\left(K_{21}^{(1)}\right)^3\left[R_{01}\right]^2+\frac{e^{-2\hat{\delta}_2 t}}{2\hat{\delta}_2}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_{02}\right]^2\right\}+\alpha_{02}.$$

Solution in the first averaged asymptotic approximation

$$x_1(t) = K_{21}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left\langle \hat{p}_1 t-\frac{3}{16p_1\left[K_{21}^{(1)}K_{22}^{(2)}-K_{22}^{(1)}K_{21}^{(2)}\right]}\tilde{\omega}_{N1}^2\times\right.$$
$$\left.\times\left\{\frac{e^{-2\hat{\delta}_1 t}}{2\hat{\delta}_1}\left(K_{21}^{(1)}\right)^3\left[R_{01}\right]^2+\frac{e^{-2\hat{\delta}_2 t}}{\hat{\delta}_2}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_{02}\right]^2\right\}+\alpha_{01}\right\rangle+$$
$$+K_{21}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left\langle \hat{p}_2 t-\frac{3}{16\hat{p}_2\left[K_{21}^{(2)}K_{22}^{(1)}-K_{22}^{(2)}K_{21}^{(1)}\right]}\tilde{\omega}_{N1}^2\times\right.$$
$$\left.\times\left\{\frac{e^{-2\hat{\delta}_1 t}}{\hat{\delta}_1}\left(K_{21}^{(1)}\right)^3\left[R_{01}\right]^2+\frac{e^{-2\hat{\delta}_2 t}}{2\hat{\delta}_2}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_{02}\right]^2\right\}+\alpha_{02}\right\rangle,$$

$$x_2(t) = K_{22}^{(1)}e^{-\hat{\delta}_1 t}R_{01}\cos\left\langle \hat{p}_1 t-\frac{3}{16p_1\left[K_{21}^{(1)}K_{22}^{(2)}-K_{22}^{(1)}K_{21}^{(2)}\right]}\tilde{\omega}_{N1}^2\times\right.$$
$$\left.\times\left\{\frac{e^{-2\hat{\delta}_1 t}}{2\hat{\delta}_1}\left(K_{21}^{(1)}\right)^3\left[R_{01}\right]^2+\frac{e^{-2\hat{\delta}_2 t}}{\hat{\delta}_2}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_{02}\right]^2\right\}+\alpha_{01}\right\rangle+$$
$$+K_{22}^{(2)}e^{-\hat{\delta}_2 t}R_{02}\cos\left\langle \hat{p}_2 t-\frac{3}{16\hat{p}_2\left[K_{21}^{(2)}K_{22}^{(1)}-K_{22}^{(2)}K_{21}^{(1)}\right]}\tilde{\omega}_{N1}^2\times\right.$$
$$\left.\times\left\{\frac{e^{-2\hat{\delta}_1 t}}{\hat{\delta}_1}\left(K_{21}^{(1)}\right)^3\left[R_{01}\right]^2+\frac{e^{-2\hat{\delta}_2 t}}{2\hat{\delta}_2}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_{02}\right]^2\right\}+\alpha_{02}\right\rangle.$$

By using previous averaged solution we can obtain Lyapunov exponents in the forms

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t)+\frac{1}{\tilde{\omega}_1^2}\dot{x}_1^2(t)\right] = -\hat{\delta}_1 < 0,$$
$$\lambda_2 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_2^2(t)+\frac{1}{\tilde{\omega}_2^2}\dot{x}_2^2(t)\right] = -\hat{\delta}_2 < 0.$$

Also, taking into account that system is nonlinear we can introduce first Lyapunov exponent in the forms

$$\tilde{\lambda}_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t)+\frac{\tilde{\omega}_{N1}^2}{\tilde{\omega}_1^2}\dot{x}_1^4(t)+\frac{1}{\tilde{\omega}_1^2}\dot{x}_1^2(t)\right] =$$
$$= \lim_{t\to\infty}\frac{1}{2t}\ln\left[\frac{E_{\text{subsist}(1)}}{2m_1\tilde{\omega}_1^2}\right] = -\hat{\delta}_1 < 0.$$

3. Concluding remarks. As the energy (kinetic, potential and the power of energy dissipation caused by dissipative forces), “carried” by a harmonic of a corresponding oscillations “stroll” frequency depends both on the amplitude square and on the square of its time derivatives, or frequency, the harmonic amplitudes, phases or frequencies change during the oscillatory process and regime itself as well as the interaction between them causes the energy change. The appearance of energy transfer from one harmonic onto other or others of higher or lower frequencies can also be noticed here.

In nonlinear systems we can observe the idea of equivalent systems exchange by the use of elementary linear simple oscillators which would be uncoupled and would make an equivalent replacement for a linearized system. And after that we may, using the asymptotic methods of nonlinear mechanics. for instance the method of Krylov – Bogolyubov – Mitropolskii, compose a system of necessary approximation of the differential equations for nonlinear oscillation harmonic amplitudes and phases that are close to an unperturbed oscillations. From such a system of adequate approximation differential equations for amplitudes and phases that are mutually coupled by nonlinear members, we may, using either quantitative or qualitative analysis, derive certain conclusions about the flows and the transfer of energy by following the phase and harmonics trajectories through the phase space of dynamical system local states around singularities.

A generalization of a analytical analysis of the transfer energy between linear and nonlinear oscillators for free vibrations with different type constraints as a couple between two subsystems every of them with one degree of freedom is also important but it is new task.

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