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REGULARIZATION INERTIAL PROXIMAL POINT ALGORITHM FOR UNCONSTRAINED VECTOR CONVEX OPTIMIZATION PROBLEMS*

РЕГУЛЯРИЗАЦІЙНИЙ ІНЕРЦІАЛЬНИЙ АЛГОРИТМ ТИПУ ПРОКСИМАЛЬНОЇ ТОЧКИ ДЛЯ ВЕКТОРНИХ ОПУКЛИХ ЗАДАЧ ОПТИМІЗАЦІЇ БЕЗ ОБМЕЖЕНЬ

The purpose of the paper is to investigate an iterative regularization method of proximal point type for solving ill-posed vector convex optimization problems in Hilbert spaces. The application to the convex feasibility problems and the common fixed points for nonexpansive potential mappings is also given.

Досліджено ітеративний метод регуляризації типу проксимальної точки для розв'язку некоректних векторних опуклих задач оптимізації у гільбертових просторах. Наведено також застосування методу до задач опуклої припустимості та до задачі про спільні нерухомі точки для нерозширних відображень потенціала.

1. Introduction. Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively.

Consider the problem of unconstrained vector convex optimization: find an element $x_0 \in H$ sych that

$$\varphi_j(x_0) = \inf_{x\in H} \varphi_j(x) \quad \forall j = 0, 1, \ldots, N, \tag{1.1}$$

where φ_j are the weakly lower semi-continuous and proper convex functionals on H.

Set

$$S_j = \left\{\bar{x} \in H : \varphi_j(\bar{x}) = \inf_{x\in H} \varphi_j(x)\right\}, \quad j = 0, 1, \ldots, N, \quad S = \bigcap_{j=0}^{N} S_j.$$

Here, we suppose that $S \neq \varnothing$, and $\theta \notin S$, where θ is the zero element of H.

It is well known that S_j coincides with the set of solutions of the following inclusion:

$$\partial\varphi_j(x) \ni \theta, \tag{1.2}$$

and is a closed convex subset in H, where $\partial\varphi_j(x)$ is the subdifferential of φ_j at the point $x \in H$ and assummed to be bounded in the sense

$$\|y\| \leq d_1 \quad \forall y \in \bigcup_{x\in B} \partial\varphi_j(x), \quad B = \{x \in H : \|x\| \leq d_0\}$$

in this paper, where d_0, d_1 are some positive constants.

Without additional conditions on $\partial\varphi_j$ such as the strongly or uniformly monotone property each inclusion (1.2) is ill-posed. By this we mean that the solution set S_j does not depend continuously on the data $\partial\varphi_j$. Therefore, problem (1.1) is also ill-posed. To solve (1.1), in [1] when φ_j are Gateau differentiable with the derivative A_j $(= \partial\varphi_j)$, we have proposed an operator method of regularization describing by the

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operator equation

$$\sum_{j=0}^{N} \alpha^{\mu_j} A_j^n(x) + \alpha U(x) = \theta, \tag{1.3}$$

$$\mu_0 = 0 < \mu_j < \mu_{j+1} < 1, \quad j = 1, 2, \ldots, N-1,$$

depending on the regularization parameter α, where A_j^n are the monotone hemi-continuous approximations for A_j in the sense

$$\left\| A_j(x) - A_j^n(x) \right\| \le h_n g(\|x\|),$$

with $h_n \to 0$, as $n \to +\infty$, $g(t)$ is a positive bounded (image of bounded set is bounded) function, and U is the normalized duality mapping of Banach space which is the identity operator I in H. Equation (1.3) has a unique solution x_α^n for each fixed $\alpha > 0$, and the sequence $\{x_\alpha^n\}$ converges strongly to the solution x_0 with

$$\|x_0\| = \min_{x \in S} \|x\|,$$

as h_n/α, $\alpha \to 0$, $n \to +\infty$.

In this paper, we consider the regularization inertial proximal point algorithm, where z_{n+1} is defined by

$$c_n \left(\sum_{j=0}^{N} \alpha_n^j A_j^n(z_{n+1}) + \alpha_n^{N+1} z_{n+1} \right) + z_{n+1} - z_n \ni \gamma_n (z_n - z_{n-1}), \quad z_0, z_1 \in H, \tag{1.4}$$

where $\{c_n\}$ and $\{\gamma_n\}$ are the sequences of positive numbers, and A_j^n are the maximal monotone approximations for $\partial\varphi_j$ in the sense

$$\rho\left(A_j^n(x), \partial\varphi_j(x)\right) \le h_n g(\|x\|), \tag{1.5}$$

where $\rho(P, Q)$ is the Hausdorff metric for the set P and Q.

Since A_j^n are maximal monotone, then the operators in (1.4) are maximal monotone (see [2]) and coercive. Hence, (1.4) has a unique solution denoted by z_{n+1} for $n \ge 1$.

To solve the inclusion $A(x) \ni f$ involving the maximal monotone operator A in H, in [3] the proximal point algorithm

$$c_n\left(A_0(z_{n+1}) - f\right) + z_{n+1} \ni z_n, \quad z_0 \in H, \tag{1.6}$$

where $c_n > c_0 > 0$, is studied. Under some conditions $\{z_n\}$ converges weakly to a solution of (1.1), if this solution is unique. R. T. Rockafellar in [3] posed an open question whether (or not) the proximal algorithm (1.6) always converges strongly. This question was resolved in the negative by O. Güler [4] and after by H. H. Bauschke et al. in [5]. To obtain the strong convergence M. V. Solodov and B. F. Svaiter in [6] have combined the proximal algorithm with simple projection step onto intersection of two halfspaces containing solution set. Recently, to obtain the strong convergence I. P. Ryazantseva in [7] has combined the proximal point algorithm with Tikhonov regularization in the form

$$c_n\left(A^n(z_{n+1}) + \alpha_n U(z_{n+1}) - f_n\right) + U(z_{n+1}) \ni U(z_n), \quad z_0 \in X,$$

for the case of reflexive Banach space X, where $\|f_n - f\| \le \delta_n \to 0$, as $n \to +\infty$.

The strong convergence of $\{z_n\}$ of this algorithm is guaranteed by it boundedness which is followed from the same property of the solution set of $A(x) \ni f$ (see [7]). There is an open question for the case where the solution set is not bounded. For example, the system of linear algebraic equations $Ax = b$ with a nonnegative matrix A, $\det A = 0$, and $r(A) = r([A, b])$ has the unbounded solution set.

Notice that for the simple case $N = 0$, the algorithm (1.4) without the regularization term was proposed to solve the monotone inclusions in [8] when $A_0^n \equiv \partial\varphi_0$. Further, this algorithm was generalized for the case $A_0^n = A_0^{\varepsilon_n}$, the enlargement of the operator $\partial\varphi_0$ in [9, 10].

In this paper, for the more general case $N \geq 0$, in Sectoin 2 we shall show that the boundedness of the sequence $\{z_n\}$ is automatically confirmed by combiniting the inertial proximal algorithm with regularization in form (1.4). An application for the convex feasibility problems and the problem of common fixed points for nonexpansive potential operators is given in Section 3.

Above and below, the symbols $\rightharpoonup$ and $\rightarrow$ denote the weak convergence and convergence in the norm, respectively.

2. Main result. First, consider the inclusion

$$\sum_{j=0}^{N} \alpha_n^j A_j^n(x) + \alpha_n^{N+1} x \ni \theta. \tag{2.1}$$

Since A_j^n are the maximal monotone operators defined on H, then the operator $\sum_{j=0}^{N} \alpha_n^j A_j^n + \alpha_n^{N+1} I$ is maximal monotone (see [3] and coercive. Hence, (2.1) has a unique solution denoted by x_n.

We have a result.

Theorem 2.1. *If* $0 < \alpha_n \leq 1$, h_n / α_n^{N+1}, $\alpha_n \to 0$, *as* $n \to +\infty$, *then* $\lim_{n\to+\infty} x_n = x_0 \in S$ *with*

$$\| x_{n+1} - x_n \| = O\left(\frac{h_{n+1} + h_n}{\alpha_{n+1}^{N+1}} + \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}^{N+1}} \right).$$

Proof. When $\partial\varphi_j = A_j$ and A_j^n are hemi-continuous monotone operators, the proof of the first part is given in [1]. For convenience, we do it here again.

From (2.1) it follows

$$\sum_{j=0}^{N} \alpha_n^j \left\langle A_j^n(x_n), x_n - x \right\rangle + \alpha_n^{N+1} \langle x_n, x_n - x \rangle = 0 \quad \forall x \in S. \tag{2.2}$$

On the base of (1.2), (1.5) and the monotone property of A_j^n we obtain

$$\| x_n \| \leq \| x \| + \frac{h_n}{\alpha_n^{N+1}} g(\| x \|)(N+1). \tag{2.3}$$

Hence, $\{x_n\}$ is bounded. Let $x_{n_k} \rightharpoonup \bar{x} \in H$ as $k \to +\infty$. First, we prove that $\bar{x} \in S_0$. Indeed, by virtue of the monotone property of $A_0^{n_k}$ and (2.1) we can write

$$\left\langle A_0^{n_k}(x), x - x_{n_k} \right\rangle \geq \left\langle A_0^{n_k}(x_{n_k}), x - x_{n_k} \right\rangle \geq$$

$$\geq \sum_{j=1}^{N} \alpha_{n_k}^j \left\langle A_j^{n_k}(x_{n_k}), x_{n_k} - x \right\rangle + \alpha_{n_k}^{N+1} \left\langle x_{n_k}, x_{n_k} - x \right\rangle \geq$$

$$\geq \sum_{j=1}^{N} \alpha_{n_k}^{j} \left\langle A_j^{n_k}(x),\, x_{n_k} - x \right\rangle + \alpha_{n_k}^{N+1} \left\langle x,\, x_{n_k} - x \right\rangle \quad \forall\, x \in H.$$

By tending $k \to +\infty$ in the last inequality we have

$$\langle \partial\varphi_0(x),\, x - \bar{x} \rangle \geq 0 \quad \forall\, x \in H.$$

Since $\partial\varphi_0$ is maximal monotone, then $\bar{x} \in S_0$. Now, we shall prove that $\bar{x} \in S_j$, $j = 1, 2, \ldots, N$. Indeed, from (1.2), (2.1) and the monotone property of $A_0^{n_k}$ it implies that

$$\left\langle A_1^{n_k}(x_{n_k}),\, x_{n_k} - x \right\rangle + \sum_{j=2}^{N} \alpha_{n_k}^{j-1} \left\langle A_j^{n_k}(x_{n_k}),\, x_{n_k} - x \right\rangle +$$

$$+ \alpha_{n_k}^{N} \left\langle x_{n_k},\, x_{n_k} - x \right\rangle \leq 0 \quad \forall\, x \in S_0$$

or

$$\left\langle A_1^{n_k}(x),\, x_{n_k} - x \right\rangle + \sum_{j=2}^{N} \alpha_{n_k}^{j-1} \left\langle A_j^{n_k}(x),\, x_{n_k} - x \right\rangle + \alpha_{n_k}^{N} \left\langle x,\, x_{n_k} - x \right\rangle \leq 0.$$

After passing $k \to +\infty$, it gives

$$\langle \partial\varphi_1(x),\, \bar{x} - x \rangle \leq 0 \quad \forall\, x \in S_0.$$

Thus, $\bar{x}$ is a local minimizer for φ_1 on S_0. Since $S_0 \cap S_1 \neq \varnothing$, then $\bar{x}$ is also a global minimizer for φ_1, i.e., $\bar{x} \in S_1$.

Set $\tilde{S}_i = \bigcap_{l=0}^{i} S_l$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \varnothing$.

Now, suppose that we have proved $\bar{x} \in \tilde{S}_i$, and need to show that $\bar{x}$ belongs to S_{i+1}. Again, by virtue of (2.1) for $x \in \tilde{S}_i$ we can write

$$\left\langle A_{i+1}^{n_k}(x_{n_k}),\, x_{n_k} - x \right\rangle + \sum_{j=i+2}^{N} \alpha_{n_k}^{j-(i+1)} \left\langle A_j^{n_k}(x_{n_k}),\, x_{n_k} - x \right\rangle + \alpha_{n_k}^{N-i} \left\langle x_{n_k},\, x_{n_k} - x \right\rangle \leq 0,$$

or

$$\left\langle A_{i+1}^{n_k}(x),\, x_{n_k} - x \right\rangle + \sum_{j=i+2}^{N} \alpha_{n_k}^{j-(i+1)} \left\langle A_j^{n_k}(x),\, x_{n_k} - x \right\rangle + \alpha_{n_k}^{N-i} \left\langle x,\, x_{n_k} - x \right\rangle \leq 0.$$

After passing $k \to +\infty$, it is clear that

$$\langle \partial\varphi_{i+1}(x),\, \bar{x} - x \rangle \leq 0 \quad \forall\, x \in \tilde{S}_i.$$

So, $\bar{x} \in S_{i+1}$. It means that $\bar{x} \in S$. S is a closed convex subset in H, because each S_j is closed convex. Hence, from (2.3) and $x_{n_k} \rightharpoonup \bar{x}$ it deduces that $\bar{x}$ is the minimal norm element of S. This element is unique. Consequently, all sequence $\{x_n\}$ converges weakly to $\bar{x}$. Again, from (2.2), $\bar{x} \in S$, and the weakly convergent property of $\{x_n\}$, we have $\|\bar{x}\| \leq \|x\| \quad \forall x \in S$. Therefore, $\lim_{n\to+\infty} x_n = \bar{x}$, and $\bar{x} = x_0$.

Now, because of (2.1),

$$\left\langle \alpha_{n+1}^{N+1} y - \alpha_n^{N+1} x,\, y - x \right\rangle = \alpha_{n+1}^{N+1} \| y - x \|^2 + (\alpha_{n+1}^{N+1} - \alpha_n^{N+1}) \langle x,\, y - x \rangle,$$

$$\left\langle \alpha_{n+1}^{j} A_j^{n+1}(y) - \alpha_n^{j} A_j^{n}(x),\, y - x \right\rangle = \alpha_{n+1}^{j} \left\langle A_j^{n+1}(y) - A_j^{n+1}(x),\, y - x \right\rangle +$$

$$+\ \alpha_{n+1}^{j}\left\langle A_j^{n+1}(x)-A_j^{n}(x),\ y-x\right\rangle + (\alpha_{n+1}^{j}-\alpha_n^{j})\left\langle A_j^{n}(x),\ y-x\right\rangle,$$

$$\left\langle A_j^{n+1}(x)-A_j^{n}(x),\ y-x\right\rangle \le (h_{n+1}+h_n)\,g(\|x\|)\|y-x\|$$

for $\|x\| \le d_0$, and

$$a^j-b^j = (a-b)(a^{j-1}+a^{j-2}b+\ldots+ab^{j-2}+b^{j-1}),$$

we obtain the estimation

$$\|x_{n+1}-x_n\| \le \frac{d_1}{\alpha_{n+1}^{N+1}}\sum_{j=1}^{N}\left|\alpha_{n+1}^{j}-\alpha_n^{j}\right| +$$

$$+\ d_0\frac{\left|\alpha_{n+1}^{N+1}-\alpha_n^{N+1}\right|}{\alpha_{n+1}^{N+1}} + \frac{h_{n+1}+h_n}{\alpha_{n+1}^{N+1}}g(\|x\|)\sum_{j=0}^{N}\alpha_{n+1}^{j} \le$$

$$\le \tilde{M}\left(\frac{h_{n+1}+h_n}{\alpha_{n+1}^{N+1}}+\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^{N+1}}\right) := \tilde{D}_n,$$

and $\tilde{M}$ is some positive constant.

The theorem is proved.

Theorem 2.2. *Assume that the parameters* c_k, γ_k *and* α_k *are chosen such that*

(i) $0<c_0<c_n<C_0$, $0\le\gamma_n<\gamma_0<1$, $\alpha_n \searrow 0$,

(ii) $\sum_{n=1}^{\infty}\tilde{\alpha}_n = +\infty$, $\tilde{\alpha}_n = c_n\alpha_n^{N+1}/(1+c_n\alpha_n^{N+1})$,

(iii) $\sum_{n=1}^{\infty}\gamma_n\|z_n-z_{n-1}\| < +\infty$,

(iv) $\lim_{n\to+\infty}D_n/\tilde{\alpha}_n = \lim_{n\to+\infty}\gamma_n\|z_n-z_{n-1}\|/\tilde{\alpha}_n = 0$ *where*

$$D_n = \frac{h_{n+1}+h_n}{\alpha_{n+1}^{N+1}}+\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^{N+1}}.$$

Then, the sequence $\{z_n\}$ *defined by* (1.5) *converges strongly to the element* x_0, *as* $n\to+\infty$.

Proof. From (1.5) and (2.1), it follows

$$\mu_n\sum_{j=0}^{N}\alpha_n^{j}A_j^{n}(z_{n+1}) + z_{n+1} \ni \beta_n z_n + \beta_n\gamma_n(z_n-z_{n-1}),$$

$$\mu_n\sum_{j=0}^{N}\alpha_n^{j}A_j^{n}(x_n) + x_n \ni \beta_n x_n,$$

$$\mu_n = c_n\beta_n, \qquad \beta_n = 1/(1+c_n\alpha_n^{N+1}).$$

Hence,

$$\mu_n\sum_{j=0}^{N}\alpha_n^{j}\left\langle A_j^{n}(z_{n+1})-A_j^{n}(x_n),\ z_{n+1}-x_n\right\rangle + \langle z_{n+1}-x_n,\ z_{n+1}-x_n\rangle =$$

$$= \beta_n\langle z_n-x_n,\ z_{n+1}-x_n\rangle + \beta_n\gamma_n\langle z_n-z_{n-1},\ z_{n+1}-x_n\rangle.$$

Therefore,

$$\|z_{n+1}-x_n\| \le \beta_n\|z_n-x_n\| + \beta_n\gamma_n\|z_n-z_{n-1}\|.$$

Consequently,

$$\|z_{n+1} - x_{n+1}\| \le \|z_{n+1} - x_n\| + \|x_{n+1} - x_n\| \le$$

$$\le \beta_n \|z_n - x_n\| + \gamma_n \|z_n - z_{n-1}\| + \tilde{D}_n \le (1 - \tilde{\alpha}_n)\|z_n - x_n\| + \tilde{d}_n,$$

since $\beta_n < 1$, where $\tilde{d}_n = \gamma_n \|z_n - z_{n-1}\| + \tilde{D}_n$. Therefore, $\|z_{n+1} - x_{n+1}\| \to 0$, as $n \to +\infty$ is followed from the lemma.

Lemma. *Let* $\{u_n\}$, $\{a_n\}$, $\{b_n\}$ *be the sequences of positive numbers satisfying the conditions*

(i) $u_{n+1} \le (1 - a_n)u_n + b_n, \quad 0 \le a_n \le 1,$

(ii) $\sum_{n=0}^{\infty} a_n = +\infty, \quad \lim\limits_{n \to +\infty} \dfrac{b_n}{a_n} = 0.$

Then, $\lim\limits_{n \to +\infty} u_n = 0.$

On the other hand, $\|x_n - x_0\| \to 0$, as $n \to +\infty$. In final, we have $x_n \to x_0$, as $n \to +\infty$.

The theorem is proved.

Remark. The sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ which are defined by

$$h_n = (1+n)^{-h}, \qquad \alpha_n = (1+n)^{-p}, \qquad 0 < 2p(N+1) < h < 1,$$

$$\gamma_n = \begin{cases} (1+n)^{-\tau} \dfrac{\|z_n - z_{n-1}\|}{1 + \|z_n - z_{n-1}\|^2}, & \text{if } \|z_n - z_{n-1}\| \ne 0, \\ 0, & \text{if } \|z_n - z_{n-1}\| = 0, \end{cases}$$

with $\tau > 1 + p(N+1)$ satisfy all conditions in Theorem 2.2.

3. Application. Given a finite family of weakly lower semi-continuous convex functionals f_j, $j = 0, 1, \ldots, N$, find an $x_0 \in H$ such that

$$f_j(x_0) \le 0, \quad j = 0, 1, \ldots, N.$$

Denote by $C_j = \{x : f_j(x) \le 0\}$, $j = 0, 1, \ldots, N$. Then, C_j are closed convex. The problem of finding $x_0 \in \bigcap_{j=0}^{N} C_j$ is the convex feasibility one. It is intensively studied for the last time (see [11 – 13] and references therein), and can be rewritten in the form of unconstrained vector convex optimization as follows. Define

$$\varphi_j(x) = \max\{0, f_j(x)\}.$$

Then C_j is coincided with the set S_j.

It is easy to see that every convex program with the objective function f and the constrain described by the functions f_j, $j = 0, \ldots, N-1$, can be also rewritten in the form of unconstrained vector optimization with $f_N = f$.

The problem of common fixed point is formulated as follows. Find $x_0 \in C = \bigcap_{j=0}^{N} C_j$, where $C_j = F(T_j)$, $j = 0, \ldots, N$, where $F(T_j)$ is the fixed point set of the nonexpansive operator T_j. It is intensively studied in recent under condition

$$C = F(T_N T_{N-1} \ldots T_0) = F(T_{N-1} \ldots T_0 T_N) = \ldots = F(T_0 T_1 \ldots T_{N-1} T_N)$$

(see [14 – 16]). After, this results are generalized to Banach spaces in [17 – 19]. Evidently, this condition can be replaced by the potential property of T_j, i.e., there exists a functional $f_j(x)$ such that $f_j'(x) = T_j(x)$ for each j. Then, $\varphi_j(x) = \|x\|^2/2 - f_j(x)$ is convex, since its derivative $I - T_j$ are monotone. Moreover, $S_j = C_j$, and the presented method in this paper can be applied to solve the problems.

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