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THE CONSTANT OF RENORMALIZATION FOR SELF-INTERSECTION LOCAL TIME OF DIFFUSION PROCESS IN THE PLANE*

КОНСТАНТА ПЕРЕНОРМУВАННЯ ЛОКАЛЬНОГО ЧАСУ САМОПЕРЕТИНІВ ДИФУЗІЙНОГО ПРОЦЕСУ НА ПЛОЩИНІ

The self-intersection local times of a diffusion process in the plane are studied. The main result consists in investigating asymptotic behavior of renormalizing constant for this local time.

Розглянуто локальний час самоперетинів для дифузійного процесу на площині. Головним результатом є дослідження асимптотичної поведінки константи перенормування цього локального часу.

Introduction and the main result. The local time of self-intersections of Brownian motion arose in the study of Euclidian field theory [1]. Since then, many papers devoted to renormalized self-intersection local time have appeared. For instance Dynkin in [2] have studied the local time of self-intersections for planar Brownian motion.

Le Gall in [3] defined the Wiener sausage and got some results for it in terms of the local time of self-intersections of Wiener process.

Rosen in [4, 5, 6] investigated the local time of self-intersections for the stable process in $\mathbb{R}^2$.

Bass and Khoshnevisan in[7] gave a new method of constructing intersection local times for Brownian motion in $\mathbb{R}^2$ and $\mathbb{R}^3$ by using stochastic calculus and additive functionals of Markov process and obtained Tanaka formula for self-intersection local times of planar Brownian motion.

Chen, Xia in [8] have studied large deviation and law of the iterated logarithm for self-intersection local times of additive process.

More references can be found in papers mentioned above.

The goal of this paper is to consider the self-intersection local times for diffusion process in the plane and to study the asymptotic behaviour of renormalizing constant for it.

In studying double self-intersections of planar Wiener process $\{w(t),\ t \geq 0\}$ one naturally tries to use the formal expression

$$\int\limits_{\Delta_2(0,1)} \delta_0(w(s_2) - w(s_1))\, d\vec{s}\,, \tag{1}$$

where $\Delta_2(0,1) = \{0 \leq s_1 \leq s_2 \leq 1\}$, δ_0 is the δ-function concentrated at the point zero.

Let $\{f_\varepsilon(x),\ \varepsilon > 0\}$ be an approximate sequence for δ_0 given by the following formula:

$$f_\varepsilon(x) = \frac{1}{2\pi\varepsilon} e^{-\frac{\|x\|^2}{2\varepsilon}}. \tag{2}$$

Consider

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$$\int\limits_{\Delta_2(0,1)} f_\varepsilon(w(s_2) - w(s_1))d\vec{s}\,. \tag{3}$$

The expectation of (3) blows up as $\varepsilon \to 0+$. Therefore, instead of (3) consider

$$\mathcal{T}_2^\varepsilon := \int\limits_{\Delta_2(0,1)} f_\varepsilon(w(s_2) - w(s_1))d\vec{s} - E\int\limits_{\Delta_2(0,1)} f_\varepsilon(w(s_2) - w(s_1))d\vec{s}\,. \tag{4}$$

Dynkin in [2] proved that there exists $L_2\text{-}\lim\limits_{\varepsilon\to 0+} \mathcal{T}_2^\varepsilon$, where this limit is called the renormalized self-intersection local time for planar Wiener process.

Let us call the following expression as the renormalizing constant for self-intersection local time of Wiener process in the plane:

$$C_\varepsilon^w := E\int\limits_{\Delta_2(0,1)} f_\varepsilon(w(s_2) - w(s_1))d\vec{s}\,. \tag{5}$$

In [2] it was showed that

$$C_\varepsilon^w \sim \frac{1}{2\pi}\ln\frac{1}{\varepsilon}, \quad \varepsilon \to 0+.$$

Let Y be a diffusion process in $\mathbb{R}^2$ described by the stochastic differential equation

$$\begin{aligned} dY(s) &= a(Y(s))ds + B(Y(s))dw(s), \\ Y(0) &= x_0. \end{aligned} \tag{6}$$

Here the coefficients a and B are Lipschitz functions. We suppose that

$$m_1 I < B^* B < m_2 I, \tag{7}$$

where m_1, m_2 are some positive constants, and I is identity matrix in $\mathbb{R}^2$.

Note that by using the fact that B is Lipschitz function and taking into account (7), one can show that $\det B$ is Lipschitz function too.

The purpose of the article is to find the asymptotic behavior of renormalizing constant for self-intersection local times of diffusion process given in (6), namely, to find asymptotic behavior of

$$C_\varepsilon^Y := E\int\limits_{\Delta_2(0,1)} f_\varepsilon(Y(s_2) - Y(s_1))d\vec{s}\,. \tag{8}$$

The main result of this paper is the following statement.

Theorem. *Suppose that diffusion process Y satisfies* (6), f_ε *is given by the formula* (2). *Then*

$$C_\varepsilon^Y \sim E\int\limits_0^1 \frac{1}{|\det B(Y(s))|}ds\,\frac{1}{2\pi}\ln\frac{1}{\varepsilon}, \quad \varepsilon \to 0+. \tag{9}$$

Note that there exists another case, when the renormalization can be described precisely.

By using this statement the following conclusion can be made.

Let $X(t)$ be Ito process in terms of [9, Sec. 2] with the following representation:

$$X(t) = w(t) + \int\limits_0^t \alpha(s)ds, \tag{10}$$

where α is measurable bounded random function adapted to the flow of w.

By using Girsanov theorem and Dynkin result for Brownian motion, it can be checked that there exists both

$$\lim_{\varepsilon\to 0+}\left(C_\varepsilon^X - C_\varepsilon^w\right) \tag{11}$$

and

$$L_2\text{-}\lim_{\varepsilon\to 0+}\left[\int_{\Delta_2(0,1)} f_\varepsilon(X(s_2)-X(s_1))\,d\vec{s} - C_\varepsilon^X\right]. \tag{12}$$

As a result we have the following equivalence:

$$C_\varepsilon^X \sim \frac{1}{2\pi}\ln\frac{1}{\varepsilon}, \qquad \varepsilon \to 0+. \tag{13}$$

Let us rewrite (6) as follows:

$$Y(s) = x_0 + \int_0^s a(Y(r))\,dr + \int_0^s B(Y(r))\,dw(r). \tag{14}$$

In case of $B = I$, by using (9) we get

$$C_\varepsilon^Y \sim \frac{1}{2\pi}\ln\frac{1}{\varepsilon}, \quad \varepsilon \to 0+. \tag{15}$$

Therefore, the hypothesis can be put forward that the renormalizing constant for the class of Ito processes of the form

$$X(s) = X_0 + \int_0^s B(r)\,dw(r) + \int_0^s \alpha(r)\,dr$$

is equivalent to

$$E\int_0^1 \frac{1}{|\det B(s)|}\,ds\,\frac{1}{2\pi}\ln\frac{1}{\varepsilon}, \quad \varepsilon \to 0+.$$

We will need the following estimations for transition density of diffusion process [10, Sec. 6].

Denote by $\mathcal{E}_Y(s_1, x_1; s_2, x_2)$ transition density of the process Y, then there exist $C_1 > 0$, $C_2 > 0$ such that

$$\mathcal{E}_Y(s_1, x_1; s_2, x_2) \geq \frac{C_1}{s_2 - s_1} e^{-\frac{\|x_1 - x_2\|^2}{C_1(s_2 - s_1)}}, \tag{16}$$

$$\mathcal{E}_Y(s_1, x_1; s_2, x_2) \leq \frac{C_2}{s_2 - s_1} e^{-\frac{\|x_1 - x_2\|^2}{C_2(s_2 - s_1)}}. \tag{17}$$

By using the estimations (16), (17), one can show that

$$\overline{\lim_{\varepsilon\to 0+}} \frac{C_\varepsilon^Y}{\ln\frac{1}{\varepsilon}} \leq C^*, \tag{18}$$

$$\underline{\lim_{\varepsilon\to 0+}} \frac{C_\varepsilon^Y}{\ln\frac{1}{\varepsilon}} \geq C_*, \tag{19}$$

where C^*, C_* are some positive constants.

It should not be made any conclusion about the precise asymptotic behaviour of C_ε^Y as $\varepsilon \to 0+$. That is why to prove the theorem we use another approach related not only to estimations (16), (17) but also to the parametrix method consisting of the following procedure.

Asume that Y satisfies the equation

$$Y(s) = Y(s_0) + \int_{s_0}^{s} a(Y(r))\,dr + \int_{s_0}^{s} B(Y(r))\,dw(r), \tag{20}$$

where s_0 is some fixed point under $Y(s_0) = x_0$ being fixed. According to the parametrix method, the transition density of process Y on the interval $[s_0, 1]$ can be expressed as follows:

$$\mathcal{E}_Y(s_1, x_1; s_2, x_2) = \mathcal{E}_X(s_1, x_1; s_2, x_2) + \int_{s_1}^{s_2}\int_{\mathbb{R}^2} \Phi(s_1, x_1; s_3, x_3)\mathcal{E}_X(s_3, x_3; s_2, x_2)\,dx_3\,ds_3,$$

where the process X has the representation

$$X(s) = x_0 + \int_{s_0}^{s} a(x_0)\,dr + \int_{s_0}^{s} B(x_0)\,dw(r) \tag{22}$$

and Φ is some function satisfying the estimation [10, Sec. 4].

There exists $C_3 > 0$ such that

$$\left|\Phi(s_1, x_1; s_3, x_3)\right| \le \frac{C_3}{(s_3 - s_1)^{3/2}} e^{-\frac{\|x_1 - x_3\|^2}{C_3(s_3 - s_1)}}. \tag{23}$$

So, transition density of diffusion process Y on the small time intervals is close to the transition density of Wiener process.

Proof of theorem. Let n be fixed. Consider the partition of $\Delta_2(0,1)$ such that

$$\Delta_2(0,1) = \bigcup_{k=0}^{n-1} \Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right) \cup \bigcup_{k=0}^{n-2} R_2^k, \tag{24}$$

where

$$\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right) = \left\{\frac{k}{n} \le s_1 \le s_2 \le \frac{k+1}{n}\right\}, \tag{25}$$

$$R_2^k = \left\{\frac{k}{n} \le s_1 \le \frac{k+1}{n},\ \frac{k+1}{n} \le s_2 \le 1\right\}. \tag{26}$$

Then

$$C_\varepsilon^Y = I_{1,\varepsilon}^n + I_{2,\varepsilon}^n,$$

where

$$I_{1,\varepsilon}^n = E\sum_{k=0}^{n-1} \int_{\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right)} f_\varepsilon(Y(s_2) - Y(s_1))\,d\vec{s}, \tag{27}$$

$$I_{2,\varepsilon}^n = E\sum_{k=0}^{n-2} \int_{R_2^k} f_\varepsilon(Y(s_2) - Y(s_1))\,d\vec{s}. \tag{28}$$

Let us consider $I^n_{2,\varepsilon}$. By using (17) and usual calculations, $I^n_{2,\varepsilon}$ can be estimated as follows:

$$I^n_{2,\varepsilon} \le \sum_{k=0}^{n-2} \int_{R_2^k} \frac{1}{c_2(s_2 - s_1)} d\vec{s} \le c(n), \tag{29}$$

where $c(n)$ is some positive constant which does not depend on ε.

$I^n_{1,\varepsilon}$ can be rewritten as

$$I^n_{1,\varepsilon} = \sum_{k=0}^{n-1} \int_{\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right)} EE\left(f_\varepsilon(Y(s_2) - Y(s_1)) \,|\, F_{\frac{k}{n}} \right) d\vec{s} =$$

$$= E\sum_{k=0}^{n-1} \int_{\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right)} \int_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1 - x_2)\mathcal{E}_Y\left(\frac{k}{n}, Y\left(\frac{k}{n}\right); s_1, x_1\right)\mathcal{E}_Y(s_1, x_1; s_2, x_2)\, d\vec{x}\, d\vec{s}, \tag{30}$$

where as usual $F_t = \sigma(Y(s),\ s \le t)$.

According to the parametrix method, we get the following representation for $I^n_{1,\varepsilon}$:

$$I^n_{1,\varepsilon} = E\sum_{k=0}^{n-1} \int_{\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right)} \int_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1 - x_2)\Bigg(\mathbb{E}_X\left(\frac{k}{n}, Y\left(\frac{k}{n}\right); s_1, x_1\right) +$$

$$+ \int_{\frac{k}{n}}^{s_1} \int_{\mathbb{R}^2} \Phi\left(\frac{k}{n}, Y\left(\frac{k}{n}\right); x_3, s_3\right)\mathbb{E}_X(x_3, s_3; x_1, s_1)\, dx_3\, ds_3 \Bigg) \times$$

$$\times \left(\mathcal{E}_{\tilde{X}}(s_1, x_1; s_2, x_2) + \int_{s_1}^{s_2} \int_{\mathbb{R}^2} \Phi(s_1, x_1; s_3, x_3)\mathcal{E}_{\tilde{X}}(s_3, x_3; s_2, x_2)\, dx_3\, ds_3 \right) d\vec{x}\, d\vec{s}, \tag{31}$$

The process X on the interval $[k/n, s_1]$ and $\tilde{X}$ on the interval $[s_1, s_2]$ have the following representations:

$$\begin{aligned} dX(s) &= a\left(Y\left(\frac{k}{n}\right)\right)ds + B\left(Y\left(\frac{k}{n}\right)\right)dw(s), \\ X\left(\frac{k}{n}\right) &= Y\left(\frac{k}{n}\right), \end{aligned} \tag{32}$$

$$\begin{aligned} d\tilde{X}(s) &= a(X(s_1))ds + B(X(s_1))dw(s), \\ \tilde{X}(s_1) &= X(s_1). \end{aligned} \tag{33}$$

After some transformation, we obtain

$$I^n_{1,\varepsilon} = E\sum_{k=0}^{n-1} \int_{\Delta_2\left(\frac{k}{n}, \frac{k+1}{n}\right)} \int_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1 - x_2) \times$$

$$\times\ \mathcal{E}_X\left(\frac{k}{n}, Y\left(\frac{k}{n}\right); s_1, x_1\right)\mathcal{E}_{\tilde{X}}(s_1, x_1; s_2, x_2)\, d\vec{x}\, d\vec{s}\ +$$

$$+\ \mathcal{T}^n_{2,\varepsilon} + \mathcal{T}^n_{3,\varepsilon} + \mathcal{T}^n_{4,\varepsilon}, \tag{34}$$

where

$$\mathcal{T}^n_{2,\varepsilon} = E\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1-x_2)\mathcal{E}_{\tilde X}(s_1,x_1;s_2,x_2)\ \times$$

$$\times \int\limits_{\frac{k}{n}}^{s_1}\int\limits_{\mathbb{R}^2}\Phi\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ x_3,s_3\right)\mathcal{E}_X(s_3,x_3;s_1,x_1)\,dx_3\,ds_3\,d\vec{x}\,d\vec{s}, \qquad (35)$$

$$\mathcal{T}^n_{3,\varepsilon} = E\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1-x_2)\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\ \times$$

$$\times \int\limits_{s_1}^{s_2}\int\limits_{\mathbb{R}^2}\Phi(s_1,x_1;s_3,x_3)\mathcal{E}_{\tilde X}(s_3,x_3;s_2,x_2)\,dx_3\,ds_3\,d\vec{x}\,d\vec{s}, \qquad (36)$$

$$\mathcal{T}^n_{4,\varepsilon} = E\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1-x_2)\int\limits_{s_1}^{s_2}\int\limits_{\frac{k}{n}}^{s_1}\int\limits_{\mathbb{R}^{2\times 2}}\Phi\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ x_3,s_3\right)\ \times$$

$$\times\ \Phi(s_1,x_1;s_4,x_4)\mathcal{E}_X(s_3,x_3;s_1,x_1)\mathcal{E}_{\tilde X}(s_4,x_4;s_2,x_2)\,dx_3\,dx_4\,ds_3\,ds_4\,d\vec{x}\,d\vec{s}. \qquad (37)$$

A few remarks about notation should be given. From now, we will denote

$$B(Y(s)) = \begin{pmatrix} b_{11}(Y(s)) & b_{12}(Y(s)) \\ b_{21}(Y(s)) & b_{22}(Y(s)) \end{pmatrix}, \qquad (38)$$

$$\vec{a}(Y(s)) = \begin{pmatrix} a_1(Y(s)) \\ a_2(Y(s)) \end{pmatrix}. \qquad (39)$$

Let us consider

$$E\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^{2\times 2}} f_\varepsilon(x_1-x_2)\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\mathcal{E}_{\tilde X}(s_1,x_1;s_2,x_2)\,d\vec{x}\,d\vec{s}. \qquad (40)$$

After some calculations (40) can be rewritten as

$$E\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^2}\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\frac{1}{2\pi}\ \times$$

$$\times\ \frac{1}{\sqrt{(\det B(x_1))^2(s_2-s_1)^2+\varepsilon\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2}}\, e^{-\frac{(\vec{a},B,\vec{s},\varepsilon)}{2}}\,d\vec{x}\,d\vec{s}, \qquad (41)$$

where

$$e^{-\frac{(\vec{a},B,\vec{s},\varepsilon)}{2}} =$$

$$= \exp\left\{-\frac{1}{2}\frac{(b_{21}(x_1)a_1(x_1)-b_{11}(x_1)a_2(x_1))^2(s_2-s_1)^3}{(\det B(x_1))^2(s_2-s_1)^2+\varepsilon\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2}\right\}\ \times$$

$$\times\ \exp\left\{-\frac{\varepsilon}{2}\frac{\left(a_1^2(x_1)+a_2^2(x_1)\right)(s_2-s_1)^2}{(\det B(x_1))^2(s_2-s_1)^2+\varepsilon\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2}\right\}\times$$

$$\times\ \exp\left\{-\frac{1}{2}\frac{(b_{22}(x_1)a_1(x_1)-b_{12}(x_1)a_2(x_1))^2(s_2-s_3)^3}{(\det B(x_1))^2(s_2-s_1)^2+\varepsilon\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2}\right\}.$$

For small $\varepsilon > 0$ the following inequality holds:

$$e^{-\frac{(\vec{a},B,\vec{s},\varepsilon)}{2}} \le 1. \tag{42}$$

By using (42) we can write that (41) is less or equal to

$$E\sum_{k=0}^{n-1}\int_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int_{\mathbb{R}^2}\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\frac{1}{2\pi}\ \times$$

$$\times\ \frac{1}{\sqrt{(\det B(x_1))^2(s_2-s_1)^2+\varepsilon\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2}}\,d\vec{x}_1\,d\vec{s}\ =$$

$$=\ E\sum_{k=0}^{n-1}\int_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int_{\mathbb{R}^2}\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\frac{1}{2\pi}\,\frac{1}{|\det B(x_1)|}\ \times$$

$$\times\ \frac{1}{\sqrt{(s_2-s_1)^2+\varepsilon(\det B(x_1))^{-2}\sum_{i,j=1}^2 b_{ij}^2(x_1)(s_2-s_1)+\varepsilon^2(\det B(x_1))^{-2}}}\,dx_1\,d\vec{s}\,. \tag{43}$$

Let us note that

$$\frac{1}{|\det B(x_1)|}\ \le\ \frac{1}{|\det B(Y(k/n))|}\ +\ L\left\|x_1-Y\left(\frac{k}{n}\right)\right\| \tag{44}$$

with some positive constant L.

By using (7) and (44) we can write that (43) is less or equal to

$$E\sum_{k=0}^{n-1}\int_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int_{\mathbb{R}^2}\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\frac{1}{2\pi}\frac{1}{|\det B(Y(k/n))|}\ \times$$

$$\times\ \frac{1}{\sqrt{(s_2-s_1)^2+\varepsilon d_1(s_2-s_1)+\varepsilon^2 d_2}}\,dx_1\,d\vec{s}\ +$$

$$+\ LE\sum_{k=0}^{n-1}\int_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int_{\mathbb{R}^2}\mathcal{E}_X\left(\frac{k}{n},Y\left(\frac{k}{n}\right);\ s_1,x_1\right)\frac{1}{2\pi}\left\|x_1-Y\left(\frac{k}{n}\right)\right\|\ \times$$

$$\times\ \frac{1}{\sqrt{(s_2-s_1)^2+\varepsilon d_1(s_2-s_1)+\varepsilon^2 d_2}}\,dx_1\,d\vec{s}\,, \tag{45}$$

where $d_1 = d_1(m_1,m_2)$, $d_2 = d_2(m_2)$ are some positive constants.

Denote by $S_{1,\varepsilon}^n$ and $S_{2,\varepsilon}^n$ the first and the second terms of (45). After a usual calculations we get

$$S^n_{1,\varepsilon} \;\le\; \frac{1}{2}\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right) E\sum_{k=0}^{n-1} \frac{1}{|\det B(Y(k/n))|}\frac{1}{n} \tag{46}$$

with some positive constant $c(d_1, d_2)$.

For arbitrary $r > 0$, $S^n_{2,\varepsilon}$ can be rewritten as

$$LE\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{B\left(Y\left(\frac{k}{n}\right),r\right)} \mathcal{E}_X\left(\frac{k}{n}, Y\left(\frac{k}{n}\right);\, s_1, x_1\right)\frac{1}{2\pi}\left\|x_1 - Y\left(\frac{k}{n}\right)\right\| \times$$

$$\times\;\frac{1}{\sqrt{(s_2-s_1)^2 + \varepsilon d_1(s_2-s_1) + \varepsilon^2 d_2}}\,dx_1\,d\vec{s} \;+$$

$$+\;LE\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^2\setminus\overline{B}\left(Y\left(\frac{k}{n}\right),r\right)} \mathcal{E}_X\left(\frac{k}{n}, Y\left(\frac{k}{n}\right);\, s_1, x_1\right)\frac{1}{2\pi}\left\|x_1 - Y\left(\frac{k}{n}\right)\right\| \times$$

$$\times\;\frac{1}{\sqrt{(s_2-s_1)^2 + \varepsilon d_1(s_2-s_1) + \varepsilon^2 d_2}}\,dx_1\,d\vec{s}\,. \tag{47}$$

where $\overline{B}\left(Y\left(\frac{k}{n}\right), r\right) \;=\; \left\{x_1 : \left\|x_1 - Y\left(\frac{k}{n}\right)\right\| \le r\right\}.$

The first and the second terms of (47) will be denoted by $S^{n,r}_{2,\varepsilon}$ and $S^{n,\bar{r}}_{2,\varepsilon}$.

Let us consider $S^{n,r}_{2,\varepsilon}$. It can be estimated as follows:

$$S^{n,r}_{2,\varepsilon} \;\le\; LrE\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\frac{1}{2\pi}\frac{1}{\sqrt{(s_2-s_1)^2 + \varepsilon d_1(s_2-s_1) + \varepsilon^2 d_2}}\,d\vec{s} \;=$$

$$=\;\frac{1}{2\pi}Lr\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right)\frac{n-1}{n}.$$

Consider $S^{n,\bar{r}}_{2,\varepsilon}$. By using (32) we can write

$$\mathcal{E}_X\left(\frac{k}{n}, Y\left(\frac{k}{n}\right);\, s_1, x_1\right) \;\le\; \frac{1}{2\pi(s_1 - k/n)}\,e^{-\frac{\|x_1 - Y(k/n)\|^2}{2(s_1 - k/n)}}. \tag{48}$$

By using (48) and polar coordinates transformation we get

$$S^{n,\bar{r}}_{2,\varepsilon} \;\le\; LE\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\int\limits_{\mathbb{R}^2\setminus\overline{B}\left(Y\left(\frac{k}{n}\right),r\right)}\frac{1}{2\pi(s_1 - k/n)}\,e^{-\frac{\|x_1 - Y(k/n)\|^2}{2(s_1 - k/n)}}\left\|x_1 - Y\left(\frac{k}{n}\right)\right\| \times$$

$$\times\;\frac{1}{\sqrt{(s_2-s_1)^2 + \varepsilon d_1(s_2-s_1) + \varepsilon^2 d_2}}\,dx_1\,d\vec{s} \;=$$

$$=\;LE\sum_{k=0}^{n-1}\int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)}\frac{1}{s_1 - k/n}\int\limits_{r}^{+\infty} e^{-\rho^2}\, 2\left(s_1 - \frac{k}{n}\right)\rho^2\, d\rho\, d\vec{s}\,.$$

Put $\dfrac{\rho}{\sqrt{s_1 - k/n}} \;=\; \upsilon$, then

$$S_{2,\varepsilon}^{n,\bar r} \le LE\sum_{k=0}^{n-1} \int\limits_{\Delta_2\left(\frac{k}{n},\frac{k+1}{n}\right)} \frac{1}{s_1 - k/n}\left(s_1 - \frac{k}{n}\right)\sqrt{s_1 - \frac{k}{n}} \times$$

$$\times \int\limits_{r/\sqrt{s_1 - k/n}}^{+\infty} e^{-\frac{v^2}{2}} v^2\, dv \frac{1}{\sqrt{(s_2 - s_1)^2 + \varepsilon d_1 (s_2 - s_1) + \varepsilon^2 d_2}}\, d\vec{s} \le$$

$$\le \frac{1}{\sqrt{n}} L^* \frac{n-1}{n}\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right)$$

with some positive constant L^*. By using (17), (23), one can check that

$$\mathcal{T}_{2,\varepsilon}^n \le c_2(n) \ln\frac{1}{\varepsilon},$$

$$\mathcal{T}_{3,\varepsilon}^n \le c_3(n) \ln\frac{1}{\varepsilon},$$

$$\mathcal{T}_{4,\varepsilon}^n \le c_4(n),$$

where $c_2(n)$, $c_3(n)$ $c_4(n)$ are some positive constants such that

$$c_2(n) \to 0 \quad \text{as} \quad n \to +\infty,$$

$$c_3(n) \to 0 \quad \text{as} \quad n \to +\infty.$$

Therefore, the following inequality holds:

$$C_3^Y \le \frac{1}{2\pi}\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right) E \sum_{k=0}^{n-1} \frac{1}{|\det B(Y(k/n))|}\frac{1}{n} +$$

$$+ \frac{1}{2\pi} Lr\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right)\frac{n-1}{n} + \frac{1}{\sqrt{n}} L^* \frac{n-1}{n}\left(\ln\frac{1}{\varepsilon} + c(d_1, d_2)\right) +$$

$$+ (c_2(n) + c_3(n)) \ln\frac{1}{\varepsilon} + c_4(n) + c(n).$$

This inequality implies that

$$\overline{\lim_{\varepsilon\to 0+}} \frac{C_\varepsilon^Y}{\ln\frac{1}{\varepsilon}} \le \overline{\lim_{r\to 0+}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0+}} \left[\frac{1}{2\pi}\left(1 + \frac{c(d_1, d_2)}{\ln(1/\varepsilon)}\right) E \sum_{k=0}^{n-1} \frac{1}{|\det B(Y(k/n))|}\frac{1}{n} +\right.$$

$$+ \frac{1}{2\pi} Lr\left(1 + \frac{c(d_1, d_2)}{\ln(1/\varepsilon)}\right)\frac{n-1}{n} + \frac{1}{n} L^* \frac{n-1}{n}\left(1 + \frac{c(d_1, d_2)}{\ln(1/\varepsilon)}\right) +$$

$$\left. + (c_2(n) + c_3(n)) + \frac{c_4(n)}{\ln(1/\varepsilon)} + \frac{c(n)}{\ln(1/\varepsilon)}\right] = E\int_0^1 \frac{1}{|\det B(Y(s))|} ds \frac{1}{2\pi}. \tag{49}$$

By using the same arguments, one can show that

$$\underline{\lim_{\varepsilon\to 0+}} \frac{C_\varepsilon^Y}{\ln\frac{1}{\varepsilon}} \ge \frac{1}{2\pi} E\int_0^1 \frac{1}{|\det B(Y(s))|} ds. \tag{50}$$

The inequalities (49), (50) imply that

$$C_\varepsilon^Y \sim E\int_0^1 \frac{1}{|\det B(Y(s))|} ds \frac{1}{2\pi} \ln\frac{1}{\varepsilon}, \quad \varepsilon \to 0+.$$

Theorem is proved.

Remark. In this paper, we have investigated the asymptotic behavior of the constant of renormalization for self-intersection local times of planar diffusion process. The questions devoted to the existence of double self-intersections and self-intersections of order k, the asymptotical behavior of the constant of renormalization for self-intersection local times of diffusion process in k dimensions will be considered in further papers.

1. *Wolpert R. L.* Local time and a particle for Euclidian field theory // J. Func. Anal. – 1978. – **30**. – P. 341 – 357.
2. *Dynkin E. B.* Regularized self intersection local times of planar Brownian motion // Ann. Probab. – 1988. – **16**, № 1. – P. 58 – 74.
3. *Le Gall J.-F.* Fluctuation results for the Wiener sausage // Ann. Probab. – 1988. – **16**. – P. 991 – 1018.
4. *Rosen J.* Dirichlet processes and an intrinsic characterization for renormalizaed intersection local times // Ann. Inst. H. Poincare. – 2001. – **37**. – P. 403 – 420.
5. *Rosen J.* Continuity and singularity of the intersection local time of stable processes in $\mathbb{R}^2$ // Ann. Probab. – 1988. – **16**. – P. 75 – 79.
6. *Rosen J.* Limit laws for the intersection local time of stable processes in $\mathbb{R}^2$ // Stochastics. – 1988. – **23**. – P. 219 – 240.
7. *Bass R. F., Khoshnevisan D.* Intersection local times and Tanaka formulas // Ann. Inst. H. Poincare Prob. Stat. – 1993. – **29**. – P. 419 – 452.
8. *Chen, Xia.* Self-intersection local times of additive processes: large deviation and law of the iterated logarithm // Stochastic Process. Appl. – 2006. – **9**. – P. 1236 – 1253.
9. *Liptser R., Shiryaev A.* Statistics of random processes. – Berlin: Springer, 2001. – 427 p.
10. *Friedman A.* Partial differential equations of parabolic type. – Englewood Cliffs, New York: Prentice-Hall, Inc., 1964. – 347 p.

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