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MUCKENHOUPT – WHEEDEN THEOREM FOR GENERALIZED f-RIESZ TYPE POTENTIALS

ТЕОРЕМА МАКЕНХАУПТА – ВІДЕНА ДЛЯ УЗАГАЛЬНЕНИХ ПОТЕНЦІАЛІВ f-РІСІВСЬКОГО ТИПУ

We obtain the Muckenhoupt-Wheeden theorem for some class of potentials. As a consequence, we describe the equivalent norm in the generalized Bessel potential space of negative order.

Одержано теорему Макенхаупта – Відена для одного класу потенціалів. Як наслідок, описано еквівалентну норму в просторі узагальнених потенціалів Бесселя від'ємного порядку.

Introduction. The paper is devoted to the generalization of Muckenhoupt – Wheeden theorem (see [1], Theorem 3.6.1, also [2]) to the case of potentials

$$I^f\mu(x) = \int_{\mathbb{R}^n} \frac{\mu(dy)}{|x-y|^n f\left(|x-y|^{-2}\right)}, \tag{1}$$

where μ is any positive measure on $\mathbb{R}^n$, and f is a Bernstein function, which means that f is a real-valued function defined on $(0, \infty)$, satisfying the following conditions:

1) $f \in C^\infty(0,\infty)$,
2) $f(x) \geq 0$,
3) $(-1)^k f^{(k)}(x) \leq 0$ for all $k \geq 1$.

For a positive measure μ, we define the f-maximal function $M^f\mu$

$$M^f\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{f(r^{-2/n})\omega_n r^n}, \tag{2}$$

where $\omega_n = \int_{S^{n-1}} dx$ is the volume of a unit ball in $\mathbb{R}^n$. For $f(x) = x^\alpha$, this maximal function is called *a fractional maximal function of a measure* μ and is denoted by $M_\alpha\mu$, see [2], for example. We show that the L_p-norm of $M^f\mu$, $1 < p < \infty$, is equivalent to the L_p-norm of $I^f\mu$. Such an equivalence gives us the description of an equivalent norm in the generalized Bessel potential space $H_p^{f(|\cdot|^2),-2}(\mathbb{R}^n)$, which is the closure of the Schwartz space $S(\mathbb{R}^n)$ under the norm

$$\|u\|_{H_p^{f(|\cdot|^2),-2}(\mathbb{R}^n)} := \left\| F^{-1}(1+(f(|\cdot|^2))^{-1})Fu(\cdot)\right\|_{L_p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

see [3, 4] for more information about the construction of such spaces. Here F, F^{-1} are respectively the Fourier and the inverse Fourier transforms. Besides others the generalized Bessel potential spaces are interesting from the analytical point of view as they are the particular cases of the spaces of generalized smoothness, and appear as domains of generators of L_p-sub-Markovian semigroups: if f is a Bernstein function, then $-f(-\Delta)$ is the generator of an L_p-sub-Markovian semigroup, corresponding to a Lévy process $(X_t)_{t\geq 0}$ with Lévy exponent $f(|\xi|^2)$ (i.e., $Ee^{i\langle \xi, X_t\rangle} = e^{-tf(|\xi|^2)}$). The domain of $-f(-\Delta)$ is $H_p^{f(|\cdot|^2),2}(\mathbb{R}^n)$, which we can identify with the dual of $H_p^{f(|\cdot|^2),-2}(\mathbb{R}^n)$. In the case where f is a Bernstein function satisfying some growth restrictions and such that the convolution semigroup associated with it has monotone 0-potencial density, it was proved in [5], Theorem 1.1.2, that the kernel of the resolvent associated with $-f(-\Delta)$ is equivalent to the kernel of I^f.

For $u \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, we can also define the potential as

$$I^f u(x) := \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^n f\big(|x-y|^{-2}\big)} dy. \tag{3}$$

Therefore in the case $\mu(dy) = u(y)dy$, $u \in L_p(\mathbb{R}^n)$ positive, the generalization of the Muckenhoupt – Wheeden theorem gives us the equivalence of norms:

$$\|u\|_{H_p^{f(|\cdot|^2),-2}(\mathbb{R}^n)} \sim \left\|I^f u\right\|_p + \|u\|_p \sim \left\|M^f u\right\|_p + \|u\|_p,$$

where

$$M^f u(x) := \sup_{r>0} \frac{1}{f(r^{-2/n})\omega_n r^n} \int_{B(x,r)} u(y)dy. \tag{4}$$

Here and below the relation $\|\cdot\|_1 \sim \|\cdot\|_2$ means that there exist positive constants c_1 and c_2 such that $c_1\|\cdot\|_1 \le \|\cdot\|_2 \le c_2\|\cdot\|_2$.

The "classical" Muckenhoupt – Wheeden theorem, i.e., the equivalence of $L_p(\mathbb{R}^n)$-norms of Riesz potentials $I_\alpha\mu$ of a positive measure μ, $0 < \alpha < n$, and of the fractional maximal function M_α, is a useful tool in the theory of function spaces. This theorem plays an important role in the proof of such a remarkable fact that the positive cone of Triebel – Lizorkin spaces $F^\alpha_{pq}(\mathbb{R}^n)$, $1 < p < \infty$, $1 < q \le \infty$, $\alpha <$ < 0, is independent of q, see Corollary 4.3.9 from [2], also [6] for the original result. Further, the Muckenhoupt – Wheeden theorem is useful for getting estimates for non-linear potentials, in particular, it is employed to show the equivalence of different definitions of capacities, see § 4.4 – 4.5 [2] and the reference therein. Also, the weighted Muckenhoupt – Wheeden inequality applied to I_1 allows to obtain some norm inequalities for the Schrödinger operator $L = -\Delta - v$ for v of some type, which can be used for getting the eigenvalue estimates for L, see [7, 8]. Therefore the generalized version of the Muckenhoupt – Wheeden theorem may give rise to new results in the theory of function spaces and applications.

The main result of the paper is formulated in the following theorem.

Theorem 1. *Let* $1 < p < \infty$, $n \ge 2$, *and assume that the Bernstein function* f *satisfies* (6) *and* (7). *Then there exists a constant* c *such that for any positive measure* μ

$$c^{-1}\left\|M^f\mu\right\|_p \le \left\|I^f\mu\right\|_p \le c\left\|M^f\mu\right\|_p. \tag{5}$$

Since the left-hand side inequality is trivial, it remains to prove the right-hand side part. The proof is based on Lemma 1 and Lemma 2 below, see also [2, p. 73 – 74].

Assumptions and auxiliary results. In what follows we will assume that our Bernstein function satisfies the following assumptions:

1. There exists $\beta > 0$ such that for all $\lambda \ge 1$

$$c_1\lambda^\beta \le \frac{f(\lambda x)}{f(x)}, \quad x > 0; \tag{6}$$

2. There exists $0 < \sigma < n/2p$ such that for all $\lambda \le 1$

$$c_2\lambda^\sigma \le \frac{f(\lambda x)}{f(x)}, \quad x > 0. \tag{7}$$

Here c_1, c_2 are some positive constants, independent on x and λ.

Consider some examples.

Examples. 1. $f(x) = x^{\alpha}$, $0 \le \alpha \le 1$.

2. $f(x) = x^{\alpha}\ln(1+x^{\alpha})$, $0 \le \alpha < 1/2$.

Indeed, (6) is satisfied with $\beta \le \alpha$ due to monotonicity of $f(x)$; to prove (7) consider the function $g(x) = \ln(1+(\lambda x)^{\alpha}) - c\lambda^{\varepsilon}\ln(1+x^{\alpha})$, $0 < \alpha \le 1$.

$$g'(x) = \alpha x^{\alpha-1}\left[\frac{\lambda^{\alpha}}{1+(x\lambda)^{\alpha}} - \frac{c\lambda^{\varepsilon}}{1+x^{\alpha}}\right] > \frac{\alpha x^{\alpha-1}}{1+x^{\alpha}}\left[\lambda^{\alpha} - c\lambda^{\varepsilon}\right] > 0 \quad \text{if} \quad \alpha < \varepsilon.$$

Since $g(0) = 0$, then $g(x) > 0$ for all $x > 0$, and hence we have (7) with $\alpha < \sigma < n/2p$.

3. $f(x) = \sqrt{x}\left(1 - e^{-4\sqrt{x}}\right)$. Again, we have (6) with $\beta \le 1/2$ due to monotonicity. To show (7) consider the function $g(x) = 1 - e^{-4\sqrt{x\lambda}} - c\lambda^{\varepsilon}\left(1 - e^{-4\sqrt{x}}\right)$. Then for suitable $c > 0$

$$g'(x) = \frac{2}{\sqrt{x}}\left(\sqrt{\lambda} - c\lambda^{\varepsilon}\right) > 0 \quad \text{if} \quad \varepsilon < \frac{1}{2}.$$

Since $g(0) = 0$, we get (7) with $1/2 < \sigma < n/2p$.

4. $f(x) = \sqrt{x}\,\dfrac{I_{\nu+1}(\sqrt{x})}{I_{\nu}(\sqrt{x})}$, see [9]. Here I_{ν} is the modified Bessel function of the first kind, see [10]. Sinse

$$I_{\nu}(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu} \quad \text{as} \quad x \to 0,$$

$$I_{\nu}(x) \sim \frac{1}{\sqrt{2\pi x}}e^{x} \quad \text{as} \quad x \to \infty,$$

we have $f(x) \sim \nu x/2$ as $x \to 0$, and $f(x) \sim \sqrt{x}$ as $x \to \infty$, hence we can choose constants c_1 and c_2 such that (6) and (7) are satisfied.

5. $f(x) = \sqrt{x}\,\dfrac{K_{\nu-1}(\sqrt{x})}{K_{\nu}(\sqrt{x})}$, see [9]. Here K_{ν} is the modified Bessel function of the third kind, see [10]. Since

$$K_{\nu}(x) \sim \frac{\Gamma\nu}{2}\left(\frac{2}{x}\right)^{\nu} \quad \text{as} \quad x \to 0,$$

$$K_{\nu}(x) \sim \frac{\sqrt{\pi}}{\sqrt{2x}}e^{-x} \quad \text{as} \quad x \to \infty,$$

we have $f(x) \sim \dfrac{x\Gamma(\nu-1)}{2\Gamma(\nu)} = \dfrac{x}{2(\nu-1)}$, $\nu > 1$, as $x \to 0$, and $f(x) \sim \sqrt{x}$ as $x \to \infty$, hence as above we can choose constants c_1 and c_2 such that (6) and (7) are satisfied with $\beta \ge 1/2$ and $\sigma > 1$.

6. By the same arguments, (6) and (7) are satisfied for Bernstein functions $f(x) = \dfrac{xI_{\nu}(\beta\sqrt{x})}{I_{\nu}(\alpha\sqrt{x})}$ and $f(x) = \dfrac{xK_{\nu}(\alpha\sqrt{x})}{K_{\nu}(\beta\sqrt{x})}$, $\nu > 0$, $\alpha > \beta > 0$ (see [9]).

Below we will use the estimates for derivatives of a Bernstein function, see [4]:

$$\left|f^{(k)}(x)\right| \le \frac{k!\,f(x)}{x^{k}}, \quad k \ge 1, \quad x > 0. \tag{8}$$

For a positive measure μ define the *Hardy – Littlewood maximal finction*

$$M\mu(x) := \sup_{r>0} \frac{1}{w_n r^n} \int_{B(x,r)} \mu(dy). \tag{9}$$

In case $\mu(dy) = |u(y)|dy$ for some L_p-function u, $1 \le p \le \infty$, we will use the notation M_u.

We will need the Hardy – Littlewood – Wiener theorem, see, for example, [2], Theorem 1.1.1. Denote by "Vol" the volume of a set.

Theorem 2 (Hardy – Littlewood – Wiener [2])**.** *Let* $u \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$. *There exists a constant* A *depending only on* p *and* n *such that*

a) *if* $p = 1$, *then*

$$\mathrm{Vol}\{x : Mu(x) > \lambda\} \le \frac{A}{\lambda}\|u\|_1 \quad \textit{for all} \quad \lambda > 0;$$

b) *if* $1 < p \le \infty$, *then*

$$\|Mu\|_p \le A\|u\|_p. \tag{10}$$

Lemma 1. *Let* f *be a Bernstein function satisfying* (6) *and* (7), I^f *be as in* (3), *and* $1 \le p < \infty$. *Then*

$$\left|I^f u(x)\right| \le \frac{cMu(x)}{f\left(\left(\dfrac{Mu(x)}{\|u\|_p}\right)^{2p/n}\right)}. \tag{11}$$

Proof. Take $0 < \delta < 1$, split the integral:

$$I^f u(x) = \int_{|x-y|<\delta} \frac{u(y)}{|x-y|^n f\left(|x-y|^{-2}\right)}dy + \int_{|x-y|\ge\delta} \frac{u(y)}{|x-y|^n f\left(|x-y|^{-2}\right)}dy = I_1 + I_2,$$

and consider the terms I_1 and I_2 separately.

Changing the variables $y = r\zeta$, $r \in \mathbb{R}_+$, $\zeta \in S^{n-1}$, we obtain

$$I_1 = \int_0^\delta \int_{S^{n-1}} \frac{u(x-r\zeta)}{r^n f(r^{-2})} \sigma_n(d\zeta)\,dr,$$

where $\sigma_n(d\zeta)$ is the surface measure on S^{n-1}. Let

$$\rho(x, dr) = \int_{S^{n-1}} u(x-r\zeta)\sigma_n(d\zeta)\,dr$$

and

$$\rho(x,r) = \int_0^r \int_{S^{n-1}} u(x-\tau\zeta)\sigma_n(d\zeta)\,d\tau = \int_{B(x,r)} u(y)\,dy.$$

Using (8), we get

$$|I_1| = \left|\int_0^\delta \frac{\rho(x,dr)}{r^n f(r^{-2})}\right| =$$

$$= \left| \frac{\rho(x,r)}{r^n f(r^{-2})} \Big|_0^\delta - \int_0^\delta \rho(x,r) \left[\frac{-n}{r^{n+1} f(r^{-2})} + (n+2) \frac{f'(r^{-2})}{r^{n+3} f^2(r^{-2})} \right] dr \right| \le$$

$$\le \frac{|\rho(x,\delta)|}{\delta^n f(\delta^{-2})} + n \int_0^\delta \frac{|\rho(x,r)|}{r^{n+1} f(r^{-2})} dr \le$$

$$\le \omega_n Mu(x) \left[\frac{1}{f(r^{-2})} + n \int_0^\delta \frac{dr}{r f(r^{-2})} \right].$$

Since f satisfies (6), we have that the last integral is less than $\frac{1}{f(\delta^{-2})} \int_0^1 \frac{dx}{x^{1-2\beta}}$ and thus

$$I_1 \le c \frac{M u(x)}{f(\delta^{-2})}.$$

Further, by Hölder's inequality, we get

$$I_2 \le \|u\|_p \left(\int_{|x-y| \ge \delta} \left(\frac{1}{|x-y|^n f(|x-y|^{-2})} \right)^{p'} dy \right)^{1/p'} =$$

$$= \|u\|_p \left(\int_\delta^\infty \left(\frac{c_1}{r^n f(r^{-2})} \right)^{p'} r^{n-1} dr \right)^{1/p'} =$$

$$= \|u\|_p \left(\int_1^\infty \left(\frac{c_1}{(\delta\tau)^n f(\delta^{-2}\tau^{-2})} \right)^{p'} \delta^n \tau^{n-1} d\tau \right)^{1/p'} \le$$

$$\le c_1 \frac{\|u\|_p \delta^{n(1-p')/p'}}{f(\delta^{-2})} \left(\int_1^\infty \frac{d\tau}{\tau^{np'-n+1-2\sigma p'}} \right)^{1/p'} \le \frac{c_2 \|u\|_p \delta^{-(n/p)}}{f(\delta^{-2})},$$

where (7) is used in the third line.

For $\delta > 1$ the estimates are the same due to the restricttions on σ and β (we need not to pose the restriction $\beta < n/p$, since β can be arbitrary small in (6)).

Combining the estimates for I_1 and I_2 and choosing $\delta = \left(\frac{\|u\|_p}{Mu} \right)^{p/n}$, we arrive at (11).

Remark 1. Let $g_p(x) = \frac{x}{f(x^{2p/n})}$, $x > 0$. The function g_p^{-1} is convex, monotone increasing for $2p/n \le 1$ and monotone decreasing for $2p/n > 1$. Then we can get the inequalities analogous to the Sobolev inequality, but under some restrictions on the norm $\|u\|_p$.

1. If $2p/n \le 1$, then $cg_p^{-1}(x) \le g_p^{-1}(cx)$ if $c \ge 1$ and then for u, $\|u\|_p \le 1$, we have, due to the Hardy – Littlewood – Wiener theorem,

$$\frac{1}{\|u\|_p^p} \left\| g_p^{-1}(I^f u) \right\|_p^p \le \left\| g_p^{-1} \left(\frac{I^f u(x)}{\|u\|_p} \right) \right\|_p^p \le \frac{\|Mu\|_p^p}{\|u\|_p^p} \le C,$$

or

$$\left\| g_p^{-1}(I^f u) \right\|_p \leq \|u\|_p. \tag{12}$$

2. If $2p/n > 1$, then we have $g_p^{-1}(cx) \leq g_p^{-1}(x)$ for $c \geq 1$, and in this case if $\|u\|_p > 1$,

$$g_p^{-1}(I^f u) \leq g_p^{-1}\left(g_p\left(\frac{Mu}{\|u\|_p} \right) \|u\|_p \right) \leq \frac{Mu}{\|u\|_p},$$

whence (12) is satisfied if $\|u\|_p > 1$.

Remark 2. Lemma 1 and Theorem 2a) imply weak type estimates:

$$\mathrm{Vol}\left\{ x : \left| I^f u(x) \right| > \lambda \right\} \leq \frac{c}{g^{-1}\left(\lambda / \|u\|_1 \right)}, \tag{13}$$

where $g(x) = g_1(x) = \dfrac{x}{f(x^{2/n})}$. Indeed,

$$\left\{ x : \left| I^f u(x) \right| > \lambda \right\} \subset \left\{ x : Mu(x) > \|u\|_1 \, g^{-1}\left(\frac{\lambda}{\|u\|_1} \right) \right\}.$$

Applying Theorem 2a), we get (13). Moreover, (13) is valid for finite measures on $\mathbb{R}^n$, $\|u\|_1 = \mu(\mathbb{R}^n)$.

Remark 3. Note that the statements of Lemma 1 and Remark 2 can be naturally generalized to the case of finite measures on $\mathbb{R}^n$.

Remark 4. For the case of Riezs potentials, i.e. when $f(x) = x^{\alpha}$, $0 < \alpha < 1$, see [2], Proposition 3.1.2 and Theorem 3.1.4.

Lemma 2. *There exists* $a > 1$, $b > 0$, *such that for all* $\lambda > 0$, *for all* ε, $0 < \varepsilon < 1$,

$$\mathrm{Vol}\left\{ x : I^f \mu(x) > a\lambda \right\} \leq$$

$$\leq \frac{b}{g^{-1}(\varepsilon^{-1})} \mathrm{Vol}\left\{ x : I^f \mu(x) > \lambda \right\} + \mathrm{Vol}\left\{ x : M^f \mu(x) > \varepsilon\lambda \right\}. \tag{14}$$

Proof. Since μ is a positive measure, by Fatou's Lemma the potential (1) is lower semicontinuous. Then the set $\left\{ x : I^f \mu > \lambda \right\}$ is open. By Whitney decomposition theorem there exists a set of dyadic cubes $\{Q_i\}$ with disjoint interior such that for all Q_i there exists

$$x:\ \mathrm{dist}\,(x, Q_i) \leq 4 \,\mathrm{diam}\, Q_i. \tag{15}$$

For such x we have $I^f \mu(x) \leq \lambda$. Assume that $Q \in \{Q_i\}$, $a > 1$ and consider the set $\left\{ x \in Q : I^f \mu(x) > a\lambda \right\}$.

1. Suppose $Q \cap \left\{ x : M^f \mu(x) \leq \varepsilon\lambda \right\} \neq \varnothing$. Let P be a ball concentric with Q, with radius $6 \,\mathrm{diam}\, Q$. Let $\mu_1 := \mu|_P$, $\mu_2 := \mu - \mu_1$. By Remark 2,

$$\mathrm{Vol}\left\{ x : I^f \mu_1(x) > \frac{a\lambda}{2} \right\} \leq \frac{c}{g^{-1}\left(\lambda a \left(\int_{\mathbb{R}^n} d\mu_1 \right)^{-1} \right)}.$$

Let $x_0 \in Q$ be such that $M^f \mu(x_0) \leq \lambda\varepsilon$ and let $B(x_0) = B(x_0, 8 \,\mathrm{diam}\, Q)$ (then $P \subset B(x_0)$). Then

$$\int_{\mathbb{R}^n} d\mu_1 \le \int_P d\mu \le \int_{B(x_0)} d\mu \le M^f \mu(x_0) g(\mathrm{Vol}(B(x_0))) \le \lambda\varepsilon g(\mathrm{Vol}(B(x_0))).$$

By definition, f is increasing. Then for $\delta > 0$ there exists $c_\delta > 0$ such that for all $\delta \le \lambda \le 1$, $x \ge 1$,

$$f(x\lambda) \le c_\delta f(x) f(\lambda).$$

Since for $n \ge 2$, $g = g_2$ is also increasing, for such x and λ we get $g(\lambda)g(x) \le \le c_\delta g(\lambda x)$, and hence due to monotonocity (take $\lambda = g^{-1}(v)$, $x = g^{-1}(w)$), we have

$$g^{-1}(vw) \le g^{-1}\big(c_\delta g(g^{-1}(v) g^{-1}(w))\big) \le$$

$$\le g^{-1}\big(g(c_\delta g^{-1}(v) g^{-1}(w))\big) \le g^{-1}(c_\delta v) g^{-1}(w).$$

Further, for small diameters of Q we have $\int_{\mathbb{R}^n} d\mu_1 \le 1$, and $g(\mathrm{Vol}(B(x_0))) < 1$. Then from

$$\frac{1}{\varepsilon} < \frac{g(\mathrm{Vol}(B(x_0)))}{\int_{\mathbb{R}^n} d\mu_1},$$

for $a > 1$ such that $\lambda a > 0$, we get

$$g^{-1}\left(\frac{a}{\varepsilon}\right) \le g^{-1}\left(\frac{\lambda a g(\mathrm{Vol}(B(x_0)))}{\int_{\mathbb{R}^n} d\mu_1}\right) \le g^{-1}\left(\frac{\lambda a}{\int_{\mathbb{R}^n} d\mu_1}\right) c_\delta \mathrm{Vol}(B(x_0)),$$

where the constant c_δ depends on the size of the cubes $\{Q_i\}$ but is independent of the choice of Q_i (i.e., we may assume that the size of $\{Q_i\}$ is bounded from below by 2^{-M} with M fixed and large enough).

Then by covering

$$\mathrm{Vol}\left\{x \in Q : I^f \mu_1(x) > \frac{a\lambda}{2}\right\} \le \frac{b}{g^{-1}(\varepsilon^{-1})} \mathrm{Vol}(Q), \tag{16}$$

or, covering the whole set $\left\{x : I^f \mu_1(x) > \frac{a\lambda}{2}\right\} \subset \left\{x : I^f \mu(x) > \lambda\right\}$, $a > 1$,

$$\mathrm{Vol}\left\{x : I^f \mu_1(x) > \frac{a\lambda}{2}\right\} \le \frac{b}{g^{-1}(\varepsilon^{-1})} \mathrm{Vol}\left\{x : I^f \mu(x) > \lambda\right\}. \tag{17}$$

2. Take $x_1 \notin Q$; then we have (15). Because of the choice of P, there exists a constant L depending only on n and such that for all $y \in P^c$, $\forall x \in Q : |x_1 - y| \le \le L|x - y|$ (we may assume here $L \ge 1$). By the property of Bernstein functions

$$f\big(f|x-y|^2\big) \le f\big(L|x_1 - y|^2\big) \le Lf\big(|x_1 - y|^2\big),$$

and since $I^f \mu(x_1) \le \lambda$ (due to the conditions of Whitney decomposition, we have (15)), we get

$$I^f \mu_2(x) \le L^{n+1} I^f \mu_2(x_1) \le \lambda L^{n+1}.$$

Choose $a > 2L^{n+1}$; then if $I^f \mu_2(x) \le a\lambda/2$ and $I^f \mu(x) > a\lambda$, we obtain $I^f \mu_1(x) > a\lambda/2$. Thus, for all $x \in Q$ such that $Q \cap \left\{x : M^f \mu \le \varepsilon\lambda\right\} \ne \varnothing$, we can write the inclusion

$$\left\{x \in Q: I^f \mu(x) > a\lambda\right\} \subset \left\{x \in Q: I^f \mu_1(x) > \frac{a\lambda}{2}\right\}.$$

Let us summarize the statements proved above. The set $\left\{x: I^f \mu(x) > a\lambda\right\}$ can be covered by cubes of two types:

1) $Q: Q \cap \left\{x: M^f \mu \le \varepsilon\lambda\right\} \ne \varnothing$. In the case of such Q, for all $x \in Q$ we have

$$\left\{x: I^f \mu(x) > a\lambda\right\} \subset \left\{I^f \mu_1(x) > \frac{a\lambda}{2}\right\};$$

2) $\tilde{Q}: \tilde{Q} \subset \left\{x: M^f \mu > \varepsilon\lambda\right\}$.

Covering $\left\{x: I^f \mu(x) > a\lambda\right\}$ by such cubes, in view of (17), we get the estimate (14).

Lemma 2 is proved.

***Proof of Theorem* 1.** For any $r > 0$

$$I^f \mu(x) \ge \int\limits_{|x-y|\le r} \frac{\mu(dy)}{|x-y|^n f\left(|x-y|^{-2}\right)} \ge \frac{1}{r^n f(r^{-2})} \int\limits_{|x-y|\le r} \mu(dy),$$

and by the definition of M^f we get the lower bound.

Let us show the upper bound. Integrate (14):

$$\int\limits_0^R \mathrm{Vol}\left\{x: I^f \mu(x) > a\lambda\right\} \lambda^{p-1} d\lambda \le$$

$$\le \frac{b}{g^{-1}(\varepsilon^{-1})} \int\limits_0^R \mathrm{Vol}\left\{x: I^f \mu(x) > \lambda\right\} \lambda^{p-1} d\lambda +$$

$$+ \int\limits_0^R \mathrm{Vol}\left\{x: M^f \mu(x) > \varepsilon\lambda\right\} \lambda^{p-1} d\lambda.$$

Changing the variables, we get

$$a^{-p} \int\limits_0^{aR} \mathrm{Vol}\left\{x: I^f \mu(x) > \lambda\right\} \lambda^{p-1} d\lambda \le$$

$$\le \frac{b}{g^{-1}(\varepsilon^{-1})} \int\limits_0^R \mathrm{Vol}\left\{x: I^f \mu(x) > \lambda\right\} \lambda^{p-1} d\lambda +$$

$$+ \varepsilon^{-p} \int\limits_0^{\varepsilon R} \mathrm{Vol}\left\{x: M^f \mu(x) > \lambda\right\} \lambda^{p-1} d\lambda.$$

If μ is compactly supported, then all the integrals are finite. Choose ε so small that

$$\frac{b}{g^{-1}(\varepsilon^{-1})} \le \frac{a^{-p}}{2}.$$

Then

$$a^{-p}\int_0^{aR}\operatorname{Vol}\left\{x:\, I^f\mu(x)>\lambda\right\}\lambda^{p-1}d\lambda \;\le$$

$$\le\; 2\varepsilon^{-p}\int_0^{\varepsilon R}\operatorname{Vol}\left\{x:\, M^f\mu(x)>\varepsilon\lambda\right\}\lambda^{p-1}d\lambda\,,$$

and letting $R\to\infty$ we get

$$a^{-p}\int_{\mathbb{R}^n}\left|I^f\mu(x)\right|^p dx \;\le\; 2\varepsilon^{-p}\int_{\mathbb{R}^n}\left|M^f\mu(x)\right|^p d\lambda\,.$$

If μ has no compact support, approximate with $\mu_n=\mu|_{B(0,n)}$, $n=1,\ldots$. Then

$$\left\|I^f\mu_n\right\|_p \;\le\; C\left\|M^f\mu\right\|_p,$$

and we get the statement of Theorem 1 by letting $n\to\infty$.

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