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FACTORIZATION OF ONE CONVOLUTION-TYPE INTEGRO -DIFFERENTIAL EQUATION ON POSITIVE HALF LINE

ФАКТОРИЗАЦІЯ ОДНОГО ІНТЕГРО-ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ТИПУ ЗГОРТКИ НА ДОДАТНІЙ ПІВОСІ

Sufficient conditions for the existence of a solution of one class of convolution-type integro-differential equations on half line are obtained. The investigation is based on three factor decomposition of initial integro-differential operator.

Отримано достатні умови для існування розв'язку одного класу інтегро-диференціальних рівнянь типу згортки на півосі. Дослідження базуються на розкладі початкового інтегро-диференціального оператора на три множники.

1. Introduction. A number of problems of physical kinetics (see [1–3]) are described by the integro-differential equation

$$\frac{dS}{dx} + AS(x) = g(x) + \lambda(x)\left\{B\int_0^\infty K_1(x-t)\frac{dS}{dt}dt + C\int_0^\infty K_2(x-t)S(t)dt\right\}, \quad x \in R^+. \tag{1.1}$$

Here, S is an unknown solution from a class of functions absolutely continuous on R^+ and of slow growth in $+\infty$, i.e.,

$$S \in \mathfrak{M} \overset{\text{df}}{=} \left\{f \in AC(R^+) \text{ s.t.} \forall \varepsilon > 0,\ e^{-\varepsilon x} f(x) \to 0 \quad \text{as} \quad x \to +\infty\right\},$$

where $AC(R^+)$ is the space of functions absolutely continuous on R^+, A, B, C are nonpositive parameters, $0 \le \lambda(\cdot) \le 1$, and $\lambda \in W_\infty^1(R^+)$ (where $W_p^n(R^+)$ is the Sobolev space of functions f such that $f^{(k)} \in L_p(R^+)$, $k = 0, 1, 2, \ldots, n$). The functions g and K_j, $j = 1, 2$, satisfy the following conditions:

$$0 \le g \in L_1(R^+) \tag{1.2}$$

and $0 \le K_j \in L_1(R)$ such that

$$\int_{-\infty}^{\infty} K_j(x)dx = 1, \quad j = 1, 2. \tag{1.3}$$

The initial condition to equation (1.1) – (1.3) is joined

$$S(0) = s_0 \in R^+. \tag{1.4}$$

In the case where

$$K_1(x) \equiv 0, \quad A = 0, \quad \lambda(x) = 1, \quad K_2(x) = \int_1^\infty e^{-|x|s}\frac{ds}{s^2}, \tag{1.5}$$

the first results of studying equation (1.1) – (1.3) appeared in the works [3 – 5]. Later, in [6], the equation (1.1) was considered in the more general case where

$$K_1(x) \equiv 0, \quad \lambda(x) \equiv 1, \quad 0 \le K_2 \in L_1(R), \quad \| K_2 \|_{L_1} = 1, \tag{1.6}$$

and, under some additional conditions on functions K_2, g and parameters A, C, the structural theorems on existence were obtained. Note that in [5, 7], the solvability of equation (1.1), (1.5) in the space $W_1^1(R^+)$ is proved and, by means of the Ambartsumian – Chandrasekhar function, analytical formulae describing the structure of obtained solution are founded.

In the present work, structural theorems on existence are obtained by putting some additional conditions on functions λ, K_1 and K_2 for equation (1.1) – (1.4).

Below, we briefly describe our approach to the investigation. First, we construct three factor decomposition of the initial integro-differential operator $D + AI - BK_\lambda^1 D - CK_\lambda^2$ [where D is a differential operator, I is the unit operator, $(K_\lambda^j f)(x) = \lambda(x)\int_0^\infty K_j(x-t)f(t)\,dt$, $j = 1, 2$] in the form of product of one differential and two integral operators. Using this factorization, the problem is reduced to the successive solution of two integtal equations and one first-order simple differential equation. The former is the Volterra-type integral equation (it can be solved elementary) and the latter is the integral equation with the kernel $\lambda(x)w(x-t)$, where $w(\cdot) \in L_1(R)$ if $A > 0$ and $w(x) = \rho_0(x) + \rho_1(x)$ if $A = 0$ (here, $\rho_0 \in L_1(R)$, $\rho_1 \in M(R)$).

It should be also noted that above mentioned factorization allow us to construct a nontrivial solution (from class $\mathfrak{M}$) of the corresponding homogeneous equation for $A = C$, i.e.,

$$\frac{dS}{dx} + AS(x) = \lambda(x)\left\{ B\int_0^\infty K_1(x-t)\frac{dS}{dt}\,dt + A\int_0^\infty K_2(x-t)S(t)\,dt \right\}. \tag{1.7}$$

2. Notations and auxiliary information. Let E^+ be one of the following Banach spaces: $L_p(0,\infty)$, $1 \le p \le +\infty$, and $L_1 \equiv L_1(-\infty, +\infty)$. We denote by Ω a class of the Wiener – Hopf integral operators (see [8]): $W \in \Omega$ if $(Wf) = \int_0^\infty w(x-t)f(t)\,dt$, $w \in L_1$.

It is easy to check that the operator W acts in the space E^+ and the following estimation holds:

$$\| W \|_{E^+} \le \int_{-\infty}^{\infty} |w(x)|\,dx. \tag{2.1}$$

The kernel w of the operator W is called conservative if

$$0 \le w \in L_1, \quad \gamma \overset{\text{df}}{=} \int_{-\infty}^{\infty} w(x)\,dx = 1. \tag{2.2}$$

We also introduce the algebra $\Omega^\pm \in \Omega$ of lower and upper Volterra-type operators: $V_\pm \in \Omega^\pm$ if

$$(V_+ f)(x) = \int_0^x \nu_+(x-t)f(t)\,dt, \quad (V_- f)(x) = \int_x^\infty \nu_-(t-x)f(t)\,dt, \tag{2.3}$$

where $x \in (0, \infty)$, $\nu_\pm \in L_1(R^+)$.

It is easy to see that $\Omega = \Omega^+ \oplus \Omega^-$. We denote by Ω^λ a class of the following integral operators: $Q^\lambda \in \Omega^\lambda$ if

$$(Q^\lambda f)(x) = \lambda(x)\int_0^\infty q(x-t)\,f(t)\,dt, \tag{2.4}$$

where $0 \le \lambda(\cdot) \le 1$, $\lambda \in W^1_\infty(R^+)$, $q \in L_1(R)$.

It is known that if $W \in \Omega$, $V_\pm \in \Omega^\pm$, then $V_- W \in \Omega$ (see [9]). Below, we prove one generalization of this fact and make essential use of it in the further reasoning.

Lemma 2.1. *If* $Q^\lambda \in \Omega^\lambda$, *then the following possibilities take place*:

a) $Q^\lambda V_+ \in \Omega^\lambda$, *where* $V_+ \in \Omega^+$,

b) $V_- Q^\lambda \in \Omega^\lambda$, *where* $V_- \in \Omega^-$, *if and only if there exists a real function* $r(t)$ *on* R^+, *for which*

$$\lambda(x+t) = \lambda(x)r(t), \qquad r(t)\nu_-(t) \in L_1(R^+).$$

Proof. Let $f \in E^+$ be an arbitrary function. We have

$$(Q^\lambda V_+ f)(x) = \lambda(x)\int_0^\infty q(x-t)\int_0^t \upsilon_+(t-\tau)\,f(\tau)\,d\tau\,dt. \tag{2.5}$$

Changing the order of integration in (2.5), we obtain

$$(Q^\lambda V_+ f)(x) = \lambda(x)\int_0^\infty f(\tau)\int_\tau^\infty q(x-t)\,\upsilon_+(t-\tau)\,dt\,d\tau =$$

$$= \lambda(x)\int_0^\infty f(\tau)\int_0^\infty q(x-\tau-z)\,\upsilon_+(z)\,dz\,d\tau = \lambda(x)\int_0^\infty P(x-t)\,f(\tau)\,d\tau,$$

where

$$P(x) = \int_0^\infty q(x-z)\,\upsilon_+(z)\,dz. \tag{2.6}$$

It follows from Fubin's theorem (see [10]) that $P \in L_1(R)$. Now let $V_- \in \Omega^-$, $Q^\lambda \in \Omega^\lambda$. In this case, analogous discussions reduce to the following formula:

$$(V_- Q^\lambda f)(x) = \int_0^\infty f(\tau)\int_0^\infty \upsilon_-(z)\lambda(x+z)q(x-\tau+z)\,dz\,d\tau = \int_0^\infty \rho(x,\tau)\,f(\tau)\,d\tau,$$

where

$$\rho(x,\tau) = \int_0^\infty \upsilon_-(z)\lambda(x+z)q(x-\tau+z)\,dz.$$

Let $\lambda(x+z) = \lambda(x)r(z)$, where $r(z)\upsilon_-(z) \in L_1(R^+)$. Then

$$\rho(x,\tau) = \lambda(x)\int_0^\infty \upsilon_-(z)r(z)q(x-\tau+z)\,dz \overset{\text{df}}{=} \lambda(x)\rho_0(x-\tau),$$

where $\rho_0 \in L_1(R)$.

The inverse statement is proved by analogy.

The lemma is proved.

Let us consider the following homogeneous equation on half line:

$$B(x) = \lambda(x)\int_0^\infty K(x-t)B(t)\,dt \tag{2.7}$$

with respect to an unknown function $B \in L_1^{\text{loc}}(R)$, where $0 \le K \in L_1(R)$, $\|K\|_{L_1}$, $0 \le$ $\le \lambda(\cdot) \le 1$ is a measurable function.

Below, we need the following theorem proved in [11]:

Theorem [11]. 1. *If* $0 \le \lambda(x) \le 1$, $1-\lambda(x) \in L_1(R^+)$, $\nu(K) \overset{\text{df}}{=} \int_{-\infty}^{\infty} xK(x)\,dx <$ < 0, *then equation* (2.7) *possesses a nontrivial bounded solution* $B(x) \ne 0$ *and* $B(x) = O(1)$ *as* $x \to +\infty$.

2. *If* $0 \le \lambda(x) \le 1$, $x(1-\lambda(x)) \in L_1(R^+)$, $\nu(K) = 0$, *then equation* (2.7) *possesses a solution* $B(x) \ge 0$, $B(x) \ne 0$, *and besides,*

$$\int_0^x B(t)\,dt = O(x^2) \quad \text{as } x \to +\infty.$$

3. Factorization problem. We rewrite equation (1.1) in the operator form

$$\left(D + AI - BK_1^\lambda D - CK_2^\lambda\right)S = g. \tag{3.1}$$

We consider two possibilities: 1) $A > 0$, and 2) $A = 0$.

1. Let $A > 0$. We consider the following factorization problem: For operators D and $K_j^\lambda \in \Omega^\lambda$, $j = 1, 2$, and for arbitrary $\alpha > 0$, it is necessary to find operators $W^\lambda \in \Omega^\lambda$ and $U_\alpha \in \Omega^+$ such that the factorization

$$D + AI - BK_1^\lambda D - CK_2^\lambda = \left(I - W^\lambda\right)(I - U_\alpha)(D + \alpha I) \tag{3.2}$$

holds as an equality of integral operators acting in $W_1^1(R^+)$.

2. Let $A = 0$. For operators D and $K_j^\lambda \in \Omega^\lambda$, $j = 1, 2$, and for arbitrary $\alpha > 0$, it is necessary to find operators $V_\alpha \in \Omega^+$, $H^\lambda \in \Omega^\lambda$ such that the factorization

$$D - BK_1^\lambda D - CK_2^\lambda = \left(I - H^\lambda - V_\alpha\right)(D + \alpha I) \tag{3.3}$$

takes place as an equality of integral operators acting in $W_1^1(R^+)$.

The following lemma holds:

Lemma 3.1. *Suppose that* $A > 0$, $K_j^\lambda \in \Omega^\lambda$, $j = 1, 2$. *Then for each* $\alpha > 0$, *the factorization* (3.2) *takes place. Kernel functions of the operators* $W^\lambda \in \Omega^\lambda$ *and* $U_\alpha \in \Omega^+$ *have the forms*

$$w^\lambda(x,t) = \lambda(x)w(x-t),$$

where

$$w(x) = BK_1(x) + \int_{-\infty}^{x} \{cK_2(t) - ABK_1(t)\}e^{-A(x-t)}\,dt, \tag{3.4}$$

and

$$u_\alpha(x) = (\alpha - A)e^{-\alpha x}\theta(x), \quad \textit{where} \quad \theta(x) = \begin{cases} 1, & \textit{if } x \geq 0, \\ 1, & \textit{if } x < 0. \end{cases} \tag{3.5}$$

Moreover, if $K_1 \in W_1^1(R)$, *then* $w \in W_1^1(R)$.

Proof. We denote by Γ_α an inverse operator of the differential operator $D + \alpha I$ in the space $W_1^1(R^+) \cap \{f : f(0) = 0\}$. It is easy to verify that Γ_α belongs to Ω^+ and has the following form:

$$(\Gamma_\alpha f)(x) = \int_0^x e^{-\alpha(x-t)} f(t)\,dt, \quad \alpha > 0. \tag{3.6}$$

It follows from Lemma 2.1 that $Q^\lambda \overset{\text{df}}{=} K_j^\lambda \Gamma_\alpha \in \Omega^+$, $j = 1, 2$, and kernels of the operators Q_j^λ are given by formulae

$$q_j^\lambda(x,t) = \lambda(x) q_j(x-t), \quad q_j(x) = \int_{-\infty}^x K_j(t) e^{-\alpha(x-t)}\,dt \in W_1^1(R), \quad j = 1, 2. \tag{3.7}$$

We have

$$D + AI - BK_1^\lambda D - CK_2^\lambda = D + \alpha I - \alpha I + AI - BK_1^\lambda D - CK_2^\lambda = (I - P_\alpha)(D + \alpha I), \tag{3.8}$$

where

$$P_\alpha = BK_1^\lambda D\Gamma_\alpha + CK_2^\lambda \Gamma_\alpha + (\alpha - A)\Gamma_\alpha. \tag{3.9}$$

It is easy to see that $D\Gamma_\alpha = I - \alpha\Gamma_\alpha$, hence, $P_\alpha = R_\alpha^\lambda + U_\alpha$, where $R_\alpha^\lambda \in \Omega^\lambda$, $R_\alpha^\lambda = BK_1^\lambda + (CK_2^\lambda - \alpha BK_1^\lambda)\Gamma_\alpha$ and $U_\alpha \in \Omega^+$, the kernel of which is given by (3.5). We denote by $I + \Phi_\alpha$ the inverse of the operator $I - U_\alpha$ in $W_1^1(R^+)$. By means of simple calculations, it is easy to verify that $\Phi_\alpha \in \Omega^+$ and, moreover,

$$(\Phi_\alpha f)(x) = (\alpha - A)\int_0^x e^{-A(x-t)} f(t)\,dt, \quad A > 0. \tag{3.10}$$

Using (3.10), from (3.8) and (3.9) we have

$$D + AI - BK_1^\lambda D - CK_2^\lambda = \left(I - R_\alpha^\lambda(I + \Phi_\alpha)\right)(I - U_\alpha)(D + \alpha I) =$$

$$= (I - W^\lambda)(I - U_\alpha)(D + \alpha I),$$

where $W^\lambda \overset{\text{df}}{=} R_\alpha^\lambda + R_\alpha^\lambda \Phi_\alpha$.

Using Lemma 2.1, we conclude that $W^\lambda \in \Omega^\lambda$. It is easy to check that operator W^λ does not depend on α. Actually, let $f \in E^+$ be an arbitrary function. Then

$$(R_\alpha^\lambda \Phi_\alpha f)(x) = (\alpha - A)\lambda(x)\int_0^\infty r_\alpha(x-t)\int_0^t e^{-A(x-\tau)} f(\tau)\,d\tau\,dt,$$

where $\lambda(x) r_\alpha(x-t)$ is the kernel of the operator R_α^λ. Changing the order of integration in the last integral and taking into account (3.7), we have

$$(R_\alpha^\lambda \Phi_\alpha f)(x) = \lambda(x)\int_0^\infty Y_\alpha(x-\tau) f(\tau)d\tau,$$

where

$$Y_\alpha(x) = \int_{-\infty}^{x} \{CK_2(\tau) - A\alpha K_1(\tau)\} e^{-A(x-\tau)} d\tau - \int_{-\infty}^{x} \{CK_2(\tau) - A\alpha K_1(\tau)\} e^{-\alpha(x-\tau)} d\tau.$$

Hence, from (3.7) it follows that W^λ does not depend on α, its kernel is given by (3.4). Now we show that operator W^λ acts in the space $W_1^1(R^+)$. Really, let f be an arbitrary function from $W_1^1(R^+)$. We have

$$(W^\lambda f)(x) = \lambda(x)\int_0^\infty w(x-t) f(t)dt = \lambda(x)\int_{-\infty}^{x} w(\tau) f(x-\tau)d\tau.$$

We denote by $\rho(x)$ the function

$$\rho(x) = \lambda(x)\int_{-\infty}^{x} w(\tau) f(x-\tau)d\tau.$$

Applying Fubin's theorem and taking into account that $0 \le \lambda(x) \le 1$, $w \in L_1(R)$, and $f \in W_1^1(R)$, we obtain $\rho \in L_1(R)$. If $\lambda \in W_\infty^1(R)$, $f \in W_1^1(R)$, then from equality

$$\rho'(x) = \lambda'(x)\int_{-\infty}^{x} w(\tau) f(x-\tau)d\tau + \lambda(x)\left\{ w(x) f(0) + \int_{-\infty}^{x} w(\tau) f_x'(x-\tau)d\tau \right\}$$

it follows that $\rho' \in L_1(R)$. Therefore, $\rho \in W_1^1(R)$. From (3.4) it follows that if $K_1 \in$ $\in W_1^1(R)$, then $w \in W_1^1(R)$.

The lemma is proved.

It is simple to prove the following lemma:

Lemma 3.2. *If* $A = 0$, *then operator* $D - BK_1^\lambda D - CK_2^\lambda$ *permits factorization of type* (3.3), *where kernels of operators* $V_\alpha \in \Omega^+$, $H^\lambda \in \Omega^\lambda$ *are given, respectively, by formulae*

$$\nu_\alpha(x) = \alpha e^{-\alpha x}\theta(x), \qquad h^\lambda(x,t) = \lambda(x) h(x-t), \tag{3.11}$$

where

$$h(x) = BK_1(x) + \int_{-\infty}^{x} \{CK_2(t) - \alpha BK_1(t)\} e^{-\alpha(x-t)} dt. \tag{3.12}$$

Further, we essentially use the following lemma that establishes connection between first moments of functions w and K_j, $j = 1, 2$:

Lemma 3.3. *Suppose that*

$$\nu(K_j) \overset{\text{df}}{=} \int_{-\infty}^{+\infty} x K_j(x)dx < +\infty, \quad j = 1, 2,$$

exists. Then $\nu(w) < +\infty$ *exists and the following formula holds*:

$$\nu(w) = \frac{C - AB}{A^2} + \frac{C}{A}\nu(K_2) \quad \textit{for} \quad A > 0.$$

Proof. As $\nu(K_j) < +\infty$, $j = 1, 2$, then by Fubin's theorem we have

$$\nu(w) = \int_{-\infty}^{+\infty} x w(x)\, dx =$$

$$= B\nu(K_1) - AB\int_{-\infty}^{\infty} x \int_{-\infty}^{x} K_1(t)e^{-A(x-t)}dt\,dx + c\int_{-\infty}^{\infty} x\int_{-\infty}^{x} K_2(t)e^{-A(x-t)}dt\,dx =$$

$$= B\nu(K_1) - AB\int_{-\infty}^{\infty} K_1(t)e^{At}\int_{t}^{\infty} xe^{-Ax}dx\,dt + C\int_{-\infty}^{\infty} K_2(t)e^{At}\int_{t}^{\infty} xe^{-Ax}dx\,dt =$$

$$= \frac{C-AB}{A^2} + \frac{C}{A}\nu(K_2).$$

The lemma is proved.

Remark. If $C\nu(K_2) \le B - \dfrac{C}{A}$, $A > 0$, then $\nu(w) \le 0$.

4. Solution of problem (1.1) – (1.4) for $A = 0$. Let us consider equation (1.1) when $A = 0$. Using factorization (3.3), the equation (1.1) (for $A = 0$) we can write in the following form:

$$(I - H^{\lambda} - V_{\alpha})(D + \alpha I)S = g. \tag{4.1}$$

The solution of (4.1) is reduced to successive solution of the following equations:

$$(I - H^{\lambda} - V_{\alpha})\varphi = g, \tag{4.2}$$

$$(D + \alpha I)S = g. \tag{4.3}$$

We denote by $I + \Phi$ the resolvent of operator $I - V_{\alpha}$ in the space $L_1^{\text{loc}}(R^+)$. It is easy to check that $(\Phi f)(x) = \alpha\int_0^x f(t)dt$. From representation of operator Φ it follows that the operator Φ transfers the space $L_1(R^+)$ to the space $M(R^+)$, where $M(R)$ is the space of bounded functions on R^+. We represent the operator $I - H^{\lambda} - V_{\alpha}$ in the following form:

$$I - H^{\lambda} - V_{\alpha} = (I - G)(I - V_{\alpha}), \tag{4.4}$$

where $G = H^{\lambda} + H^{\lambda}\Phi$. It is easy to check that the operator $(Gf)(x)$ is determined as

$$(Gf)(x) = \lambda(x)\int_0^{\infty} G_0(x-t)f(t)dt,$$

where

$$G_0(x) = BK_1(x) + C\int_{-\infty}^{x} K_2(t)dt, \quad K_1 \in L_1(R), \quad \int_{-\infty}^{x} K_2(t)dt \in M(R).$$

Using factorization (4.4), the solution of equation (4.2) is reduced to successive solution of the following equations:

$$(I - G)\Psi = g, \tag{4.5}$$

$$(I - V_{\alpha})\varphi = \psi. \tag{4.6}$$

We rewrite the equation (4.5) in the open form

$$\psi(x) = g(x) + \lambda\int_0^\infty G_0(x-t)\psi(t)\,dt$$

and consider the following iteration process:

$$\psi^{(n+1)}(x) = g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi^{(n)}(t)\,dt, \qquad \psi^{(0)} = 0, \quad n = 0, 1, 2\ldots. \tag{4.7}$$

It is easy to see that $g(x) \le \psi^{(n)} \uparrow$ by n. We note that if $\lambda \in L_1(R^+)$, then $\psi^{(n)} \in L_1(R^+)$, $n = 0, 1, 2\ldots$. Really, for $n = 0$, we have $\psi^{(1)} = g(x) \in L_1(R)$. Assume that $\psi^{(n)} \in\, \in L_1(R^+)$ and prove that $\psi^{(n+1)} \in L_1(R)$. Then for arbitrary $r > 0$ we have

$$\int_0^r \psi^{(n+1)}(x)\,dx \le \int_0^\infty g(x)\,dx + \int_0^\infty \lambda(x)\int_0^\infty G_0(x-t)\psi^{(n)}(t)\,dt\,dx =$$

$$= \int_0^\infty g(x)\,dx + B\int_0^\infty \psi^{(n)}(t)\int_0^\infty K_1(x-t)\lambda(x)\,dx\,dt + C\int_0^\infty \psi^{(n)}(t)\int_0^\infty\int_{-\infty}^{x-t} K_2(\tau)\,d\tau\,\lambda(x)\,dx\,dt \le$$

$$\le \int_0^\infty g(x)\,dx + B\int_0^\infty \psi^{(n)}(t)\,dt + C\int_0^\infty \psi^{(n)}(t)\,dt\int_0^\infty \lambda(x)\,dx \Rightarrow \psi^{(n+1)} \in L_1(R^+).$$

It is also easy to check that

$$\int_0^\infty \psi^{(n+1)}(x)\,dx \le \int_0^\infty g(x)\,dx + \operatorname*{vrai\,max}_{t\in R^+}\int_{-t}^\infty \lambda(t+\tau)G_0(\tau)\,d\tau\int_0^\infty \psi^{(n+1)}(t)\,dt. \tag{4.8}$$

Now we suppose that

$$q_0 \overset{\text{df}}{=} \operatorname*{vrai\,max}_{t\in R^+}\int_{-t}^\infty \lambda(t+\tau)G_0(\tau)\,d\tau < 1. \tag{4.9}$$

Then from (4.8), taking into account (4.9), we receive

$$\int_0^\infty \psi^{(n+1)}(x)\,dx \le (1-q_0)^{-1}\int_0^\infty g(x)\,dx.$$

From B. Levi's theorem (see [10]) it follows that the sequence $\psi^{(n)}$ almost everywhere in R^+ has a limit $\psi(x) = \lim_{n\to\infty}\psi^{(n)}(x)$, and besides $\psi \in L_1(R^+)$.

We prove that $\psi(x)$ is the solution of equation (4.5). Actually, from (4.7) we have

$$\psi^{(n+1)}(x) \le g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi(t)\,dt, \quad n = 0, 1, 2\ldots. \tag{4.10}$$

Passing to the limit in the last inequality, we obtain

$$\psi(x) \le g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi(t)\,dt. \tag{4.11}$$

On the other hand,

$$g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi^{(n)}(t)\,dt \;\le\; \psi(x). \tag{4.12}$$

From Lebeg's theorem it follows that

$$g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi(t)\,dt \;\le\; \psi(x). \tag{4.13}$$

Combining inequalities (4.11) and (4.12), we get

$$\psi(x) \;=\; g(x) + \lambda(x)\int_0^\infty G_0(x-t)\psi(t)\,dt. \tag{4.14}$$

Now we pass to the solution of the equation (4.6):

$$\varphi(x) \;=\; \psi(x) + \alpha\int_0^x e^{-\alpha(x-t)}\varphi(t)\,dt. \tag{4.15}$$

It is obvious that

$$\varphi(x) \;=\; \psi(x) + \alpha\int_0^x \psi(t)\,dt. \tag{4.16}$$

Finally solving equation (4.3) and taking into account (1.4), we obtain

$$S(x) \;=\; s_0e^{-\alpha x} + \int_0^x e^{-\alpha(x-t)}\varphi(t)\,dt. \tag{4.17}$$

Using formula (4.16), we have

$$S(x) \;=\; s_0e^{-\alpha x} + \int_0^x \psi(t)\,dt. \tag{4.18}$$

In its turn, it follows that

$$\int_0^\infty g(x)\,dx \;\le\; S(+\infty) \;=\; \int_0^\infty \psi(t)\,dt \;\le\; \frac{\int_0^\infty g(x)\,dx}{1-q_0}. \tag{4.19}$$

The following theorem holds:

Theorem 4.1. *Let* $0 \le \lambda(x) \le 1$, $\lambda \in L_1(R^+)\cap W_\infty^1(R^+)$, *and let the following estimation be true*:

$$\operatorname*{vrai\,max}_{t\in R^+}\int_{-t}^\infty \lambda(t+\tau)G_0(\tau)\,d\tau \;<\; 1,$$

where $G_0(x) \;=\; BK_1(x) + C\int_{-\infty}^x K_2(t)\,dt.$

Then problem (1.1) – (1.4) *for* $A = 0$ *in the class* $\mathfrak{M}(R^+)$ *possesses a positive solution of the type* (4.18) *and inequality* (4.19) *is true.*

5. Solution of equation (1.1) – (1.4) for $A > 0$. In this section, we study equation (1.1) – (1.4) for $A > 0$. In this case, we consider the following three possibilities: 1) $A > C \ge 0$, 2) $A = C > 0$, 3) $0 < A < C$.

***5.1. Equation* (1.1) – (1.4) *in case* $A > C \ge 0$.** The following theorem is true:

Theorem 5.1. *Suppose that* a) $w(x) \geq 0,\ x \in R,$ b) $0 \leq \lambda(x) \leq 1,\ \lambda \in W_\infty^1(R^+)$. *Then the problem* (1.1) – (1.4) *for* $A > C \geq 0$ *in the space* $W_1^1(R^+)$ *has a positive solution of the type*

$$S(x) = s_0 e^{-\alpha x} + \int_0^x e^{-\alpha(x-t)} F(t)\,dt, \tag{5.1}$$

where $\alpha > 0$ *is the constant,* $0 \leq F \in L_1(R^+)$.

Proof. Using factorization (3.2), the equation (1.1) may be written in the form

$$(I - W^\lambda)(I - U_\alpha)(D + \alpha I)S = g. \tag{5.2}$$

Solution of (5.2) is reduced to successive solution of the following equations:

$$(I - W^\lambda)F = g, \tag{5.3}$$

$$(I - U_\alpha)\chi = F, \tag{5.4}$$

$$(D + \alpha I)S = \chi. \tag{5.5}$$

We rewrite the equation (5.3) in the open form and consider the iteration

$$F^{(n+1)}(x) = g(x) + \lambda(x)\int_0^\infty w(x-t)F^{(n)}(t)\,dt, \qquad F^{(0)} = 0, \quad n = 0, 1, 2 \ldots. \tag{5.6}$$

By induction, it is easy to check that

$$g(x) \leq F^{(n)} \in L_1(R^+), \quad n = 1, 2, \ldots, \quad F^{(n)} \uparrow \text{ by } n. \tag{5.7}$$

Therefore, we have

$$\int_0^\infty F^{(n+1)}(x)\,dx \leq \int_0^\infty g(x)\,dx + \int_0^\infty \lambda(x)\int_0^\infty w(x-t)F^{(n+1)}(t)\,dt\,dx =$$

$$= \int_0^\infty g(x)\,dx + \int_0^\infty F^{(n+1)}(t)\int_{-\infty}^\infty w(t)\,\lambda(t+z)\,dz\,dt \leq \int_0^\infty g(x)\,dx + \gamma\int_0^\infty F^{(n+1)}(t)\,dt,$$

where

$$\gamma = \int_{-\infty}^\infty w(x)\,dx = \frac{C}{A} < 1. \tag{5.8}$$

As (5.7) and (5.8) are satisfied, then form B. Levi's theorem it follows that the sequence $\{F^{(n+1)}(x)\}_0^\infty$ converges almost everywhere in R^+ to an integrable function $F(x)$. It is obvious that the function $F(x)$ is the solution of equation (5.6). Successively solving equations (5.4) and (5.5), we arrive to result (5.1).

The theorem is proved.

***5.2. Equation* (1.1) – (1.4) *in case* $A = C > 0$.** The following theorem holds:

Theorem 5.2. *Suppose that the following conditions are satisfied*: i) $w(x) \geq 0$, $x \in R$, ii) $0 \leq \lambda(x) \leq 1,\ \lambda \in W_\infty^1(R)$, iii) $\nu(K_j) < +\infty,\ j = 1, 2,$ *exists and moreover,* $\nu(K_2) \leq (B-1)/A$. *Then problem* (1.1) – (1.4) *for* $A = C > 0$ *in the class* $\mathfrak{M}$ *possesses the solution of the following structure* :

$$S(x) = s_0 e^{-\alpha x} + \int_0^x e^{-A(x-t)} \varphi(t)\, dt. \tag{5.9}$$

Here, $0 < \alpha = \text{const}$, $0 \leq \varphi \in L_1^{\text{loc}}(R^+)$,

$$\int_0^x \varphi(t)\, dt = o\left(\int_0^x f(t)\, dt\right) \quad \textit{for} \quad x \to +\infty,$$

where $f(x)$ *is the positive increasing function,* $f(0) = 1$ *and if* $\nu(K_2) < (B - 1)/A$, *then* $f(x) = O(1)$ *for* $x \to +\infty$, *and if* $\nu(K_2) = (B-1)/A$, *then* $f(x) = O(x)$ *for* $x \to +\infty$.

Proof. From the condition $A = C > 0$ it follows that $\gamma = 1$. Together with (5.3), we consider the following auxiliary equation:

$$\tilde{F}(x) = g(x) + \int_0^\infty w(x-t) \tilde{F}(t)\, dt, \tag{5.10}$$

$$f(x) = \int_0^\infty w(x-t) f(t)\, dt. \tag{5.11}$$

It was proved in [12, 13] that if $\nu(w) \leq 0$, $0 \leq g \in L_1(R^+)$, then equation (5.10) in $L_1^{\text{loc}}(R^+)$ has positive solution which, almost everywhere in $(0, +\infty)$, is the limit of the following simple iterations:

$$\tilde{F}^{(n+1)}(x) = g(x) + \int_0^\infty w(x-t) \tilde{F}^{(n)}(t)\, dt, \quad \tilde{F}^{(0)} = 0, \quad n = 0, 1, 2, \ldots, \tag{5.12}$$

and the asymptotic

$$\int_0^x \tilde{F}(t)\, dt = o\left(\int_0^x f(t)\, dt\right), \quad x \to +\infty, \tag{5.13}$$

is true, where f is a positive increasing solution of equation (5.11), $f(0) = 1$. Mentioned solution f satisfies also the following conditions: $f(x) = O(x)$, $(x \to \infty)$ for $\nu(w) = 0$ and $f(x) = O(1)$, $x \to \infty$, for $\nu(w) < 0$. We consider the following iteration for equation (5.3) (in the case $A = C > 0$):

$$F^{(n+1)}(x) = g(x) + \int_0^\infty w(x-t) F^{(n)}(t)\, dt, \quad \tilde{F}^{(0)} = 0, \quad n = 0, 1, 2, \ldots. \tag{5.14}$$

It is easy to show that

i) $g(x) \leq F^{(n)} \uparrow$ by n, ii) $F^{(n)} \leq \tilde{F}^{(n)}$ almost everywhere in $(0, +\infty)$.

Hence, almost everywhere in R^+, there exists $F(x) = \lim_{n \to \infty} F^{(n)}(x)$ and

$$0 \leq g(x) \leq F(x) \leq \tilde{F}(x). \tag{5.15}$$

It is obvious that $F(x)$ is the solution of equation (5.3) for $A = C > 0$ (the proof of last fact has analogy with Theorem 4.1). Using (5.13), (5.15) and Lemma 3.3, we obtain

$$\int_0^x F(t)dt = o\left(\int_0^x f(t)dt\right), \quad x \to +\infty.$$

Solving equations (5.4) and (5.5), we obtain (5.9).

The theorem is proved.

***5.3. Equation* (1.1) – (1.4) *in case* $C > A > 0$.** Doing analogous discussions as in Theorems 4.1 and 5.1, we get the following theorem:

Theorem 5.3. *Let* i) $w(x) \geq 0$, $x \in R$, ii) $0 \leq \lambda(x) \leq 1$, $\lambda(x) \in W_\infty^1(R^+)$, iii) *the inequality*

$$\operatorname*{vrai\,max}_{t \in R^+} \int_{-t}^{\infty} \lambda(t+\tau) w(\tau) d\tau < 1$$

takes place. Then problem (1.1) – (1.4) *for* $C > A > 0$ *in* $W_1^1(R^+)$ *possesses a solution of the type* (4.18).

6. Construction of nontrivial solution of homogeneous equation (1.7). The factorization (3.2) allows us to construct nontrivial solution of corresponding homogeneous equation when $A = C > 0$. Unfortunately, for other values of parameters A and C, up to now we were not able to construct a nontrivial solution. It is known only that, in the case $A > C > 0$, the homogeneous equation $F(x) = \lambda(x)\int_0^\infty w(x-t) \times F(t)dt$ in the class $\mathfrak{M}$ has no nontrivial solutions. It is also known that the homogeneous equation, in the case $A = C > 0$ and $\nu(w) > 0$, in $\mathfrak{M}$ has no nontrivial solutions either. On evristic level we conclude that for other values of parameters A and C nontrivial solutions do not exist.

We consider corresponding homogeneous equation (1.1) – (1.4) for $A = C > 0$ (see (1.7)).

Using factorization (3.2), we rewrite the equation (1.7) in the form

$$(I - W^\lambda)(I - U_\alpha)(D + \alpha I)S = 0. \tag{6.1}$$

The equation is equivalent to the successive solution of the following equations:

$$(I - W^\lambda)\rho_1 = 0, \tag{6.2}$$

$$(I - U_\alpha)\rho_2 = \rho_1, \tag{6.3}$$

$$(D + \alpha I)S = \rho_2. \tag{6.4}$$

We write equation (6.2) in the open form: $\rho_1(x) = \lambda(x)\int_0^\infty w(x-t)\rho_1(t)dt$.

As $A = C > 0$, then $\gamma = 1$. Using Theorem from [11] (see Sec. 2 of this paper), Lemma 3.3 and solving equations (6.3) and (6.4), we obtain the following results:

Theorem 6.1. A. *Suppose that* i) $w(x) \geq 0$, ii) $0 \leq \lambda(x) \leq 1$, $\lambda(x) \in W_\infty^1(R^+)$, $I - \lambda(x) \in L_1(R^+)$, iii) $\nu(K_2) < \dfrac{B-1}{A}$.

Then the problem (1.7), (1.3), (1.4) *for* $A = C > 0$ *in the class* $\mathfrak{M}$ *possesses a nontrivial solution of the type*

$$S(x) = s_0 e^{-\alpha x} + \int_0^x e^{-A(x-t)} \rho_1(t) dt, \tag{6.5}$$

where $\rho_1 \neq 0$ *and* $\rho_1(x) = O(1)$, $x \to \infty$.

B. *Let* i) $w(x) \geq 0$, $x \in R^+$, ii) $0 \leq \lambda(x) \leq 1$, $\lambda(x) \in W^1_\infty(R^+)$, $x(1-\lambda(x)) \in \in L_1(R^+)$, iii) $\nu(K_2) \leq \dfrac{B-1}{A}$. *Then the problem* (1.7), (1.3), (1.4) *for* $A = C > 0$ *in the class* $\mathfrak{M}$ *possesses a nontrivial solution of the type* (6.5), *where* $\rho_1 \geq 0$, $\rho_1 \neq 0$, *and has the asymptotic behaviour* $\int_0^x \rho_1(t)dt = O(x^2)$, $x \to +\infty$.

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