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LITTLEWOOD – PALEY THEOREM ON $L^{p(t)}(\mathbb{R}^n)$ SPACES*

ТЕОРЕМА ЛИТТЛВУДА – ПЕЛІ ПРО ПРОСТОРИ $L^{p(t)}(\mathbb{R}^n)$

We point out that when the Hardy – Littlewood maximal operator is bounded on the space $L^{p(t)}(\mathbb{R})$, $1 < a \le p(t) \le b < \infty$, $t \in \mathbb{R}$, the well-known characterization of spaces $L^p(\mathbb{R})$, $1 < p < \infty$, by the Littlewood – Paley theory extends to the space $L^{p(t)}(\mathbb{R})$. We show that if $n > 1$, the Littlewood – Paley operator is bounded on $L^{p(t)}(\mathbb{R}^n)$, $1 < a \le p(t) \le b < \infty$, $t \in \mathbb{R}^n$, if and only if $p(t) = \text{const}$.

Встановлено, що коли максимальний оператор Харді – Літтлвуда обмежений на просторі $L^{p(t)}(\mathbb{R})$, $1 < a \le p(t) \le b < \infty$, $t \in \mathbb{R}$, добре відома характеризація просторів $L^p(\mathbb{R})$, $1 < p < \infty$, теорією Літтлвуда – Пелі поширюється на простір $L^{p(t)}(\mathbb{R})$. Показано, що у випадку $n > 1$ оператор Літтлвуда – Пелі обмежений на $L^{p(t)}(\mathbb{R}^n)$, $1 < a \le p(t) \le b < \infty$, $t \in \mathbb{R}^n$, тоді і тільки тоді, коли $p(t) = \text{const}$.

1. Introduction. Let m be a bounded function on $\mathbb{R}^n$. The operator T defined by the Fourier transform equation $(Tf)\hat{}(x) = m(x)\hat{f}(x)$, $x \in \mathbb{R}^n$, is called a multiplier operator with multiplier m. Let ρ be an (n-dimensional) rectangle and χ_ρ the characteristic function of ρ. The operator S_ρ having multiplier $m = \rho$ and defined by the equation

$$(S_\rho f)\hat{}(x) = \chi_\rho(x)\hat{f}(x), \qquad x \in \mathbb{R}^n,$$

is called a partial sum operator.

Let a collection of disjoint rectangles $\Delta = \{\rho\}$ be a decomposition of $\mathbb{R}^n$ (i.e., $\bigcup_{\rho\in\Delta} = \mathbb{R}^n$). Given a function f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, define

$$G(f)(x) = \left(\sum_{\rho\in\Delta} \left|S_\rho f(x)\right|^2\right)^{1/2}, \qquad x \in \mathbb{R}^n.$$

Let $\{n_k\}_{k=-\infty}^{+\infty}$, $n_k > 0$, $k \in \mathbb{Z}$, be a lacunary sequence (i.e., there is an $a > 1$ such that $n_{k+1}/n_k \ge a$ for all k). Let Δ be the collection of all intervals of the form $[n_k, n_{k+1}]$ and $[-n_{k+1}, n_k]$, $k \in \mathbb{Z}$. Then Δ is called a lacunary decomposition of $\mathbb{R}$. When $n_k = 2^k$, $k \in \mathbb{Z}$, the resulting Δ is called the dyadic decomposition of $\mathbb{R}$.

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Let Δ_i, $i = 1, 2, \ldots, n$, be n lacunary decomposition of $\mathbb{R}$. Let Δ be the collection of the intervals of the form $\rho = \rho_1 \times \rho_2 \times \ldots \times \rho_n$ where $\rho_i \in \Delta_i$. Then Δ is called a lacunary decomposition of $\mathbb{R}^n$.

The important feature of the classical Littlewood – Paley theory is that a characterization of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. It is well known (see [1, 2]) that if Δ is a lacunary decomposition of $\mathbb{R}^n$ then $\|G(f)\|_p$ is equivalent to $\|f\|_p$ for $1 < p < \infty$; i.e., there are constants A and B such that

$$A\|f\|_p \le \|G(f)\|_p \le \|f\|_p.$$

The purpose of this paper is to obtain analogously characterizations of variable exponent Lebesgue spaces $L^{p(t)}(\mathbb{R}^n)$.

Given a measurable functions $p(\cdot)\colon \mathbb{R}^n \to [1, \infty)$, $L^{p(t)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(t)} = \inf\left\{\lambda > 0\colon \int \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

Given a locally integrable function f, we define the Hardy – Littlewood maximal function Mf by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes containing x with sides parallel to the coordinate axes. For conciseness, define $\mathcal{P}(\mathbb{R}^n)$ to be the set of measurable function $p(\cdot)\colon \mathbb{R}^n \to [1, \infty)$ such that

$$1 < a \le p(t) \le b < \infty\colon \quad t \in \mathbb{R}^n.$$

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(t)}(\mathbb{R}^n)$. Conditions for the boundedness of the Hardy – Littlewood maximal operator on spaces $L^{p(t)}(\mathbb{R}^n)$ have been studied in [3 – 8]. Diening [8] studied the necessary and sufficient conditions in terms of the conjugate exponent $p'(\cdot)$, $(1/p(t) + 1/p'(t) = 1$, $t \in \mathbb{R}^n)$. He has proved that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, he also proved that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ for some $q > 1$.

In harmonic analysis a fundamental operator is the Hardy – Littlewood maximal operator. In many applications a crutial step has been to show that operator M is bounded on a variable L^p space. Cruz-Uribe, Fiorenza, Martell and Perez [4] have showed that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(t)}(\mathbb{R}^n)$ whenever the Hardy – Littlewood maximal operator is bounded on $L^{p(t)}(\mathbb{R}^n)$.

If we consider, instead, the strong maximal operator $M_{\mathcal{R}}$ defined by

$$M_{\mathcal{R}}(f)(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(x)|\,dx$$

where R is any rectangle in $\mathbb{R}^n$, $n > 1$, with sides parallel to the coordinate axes then the situation is different. For the strong Hardy – Littlewood maximal operator $M_{\mathcal{R}}$ we prove following theorem.

Theorem 1. *Let* $1 \le p(t) \le b < \infty$, $t \in \mathbb{R}^n$. *The strong Hardy – Littlewood maximal operator* $M_{\mathcal{R}}$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *space if and only if* $p(t) = \mathrm{const} = p$ *and* $p > 1$.

For function $f \in L(\mathbb{R}^n)$, the expression

$$Hf(x) = \int\limits_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{1}{x_k - y_k} f(y)\,dy$$

is said to be n-dimensional $(n > 1)$ Hilbert operator.

Analogously we may prove following theorem.

Theorem 2. *Let* $1 \le p(t) \le b < \infty$, $t \in \mathbb{R}^n$. *Then n-dimensional Hilbert operator* $(n > 1)$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *space if and only if* $p(t) = \mathrm{const} = p$ *and* $p > 1$.

We prove following Littlewood – Paley type characterization of $L^{p(t)}(\mathbb{R}^n)$ space.

Theorem 3. 1. *Let* Δ *be a lacunary decomposition of* $\mathbb{R}$ *and* $p(\cdot) \in \mathcal{B}(\mathbb{R})$. *Then there are constants* c, $C > 0$ *such that for all* $f \in L^{p(t)}(\mathbb{R})$

$$c\|f\|_{p(t)} \le \|G(f)\|_{p(t)} \le C\|f\|_{p(t)}. \tag{1}$$

2. *Let* Δ *be the dyadic decomposition of* $\mathbb{R}^n$, $n > 1$. *If* $p(\cdot) \ne \mathrm{const}$ *then operator* G *is not bounded on* $L^{p(t)}(\mathbb{R}^n)$.

2. Proof of theorems. ***Proof of Theorem*** **1.** According to Jessen, Marcinkiewicz and Zygmund [9] $M_{\mathcal{R}}$ is bounded on all the L^p, $p > 1$, spaces and first part of Theorem 1 is trivial.

Let $M_{\mathcal{R}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Virtue of interpolation theorem (see [10]), we have $M_{\mathcal{R}}$ is bounded on $L^{p(\cdot)/\theta} = [L^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n)]_\theta$, $0 < \theta < 1$, and without restriction of generality we may assume that $1 < \inf_{\mathbb{R}^n} p(t)$. Let $1/p(t) + 1/p'(t) = 1$, $t \in \mathbb{R}^n$. Note that

$$\sup_R \frac{1}{|R|} \|\chi_R\|_{p(t)} \|\chi_R\|_{p'(t)} < \infty \tag{2}$$

condition is necessary for boundedness of $M_{\mathcal{R}}$ on $L^{p(t)}(\mathbb{R}^n)$ (see proof below).

We will give the proof of second part of Theorem 1 for the case $n = 2$ for simplicity, since the same argument holds when $n > 2$.

Let $\inf_{\mathbb{R}^2} p(t) < \sup_{\mathbb{R}^2} p(t)$. By Luzin's theorem we can construct pairwise disjoint family of set F_i with the following condition: 1) $\left|\mathbb{R}^2 \setminus \bigcup F_i\right| = 0$, 2) functions $p : F_i \to \mathbb{R}$ are continuous, 3) for every fixed i all points of F_i are points of density with respect to basis $\mathcal{R}$.

Note that, we can find pair of points $((x_0, y_1), (x_0, y_2))$- or $((x_1, y_0), (x_2, y_0))$-type from $\bigcup F_i$ such that $p(x_0, y_1) \ne p(x_0, y_2)$ or $p(x_1, y_0) \ne p(x_2, y_0)$. Without loss of generality, we may suppose that this pair is $((x_0, y_1), (x_0, y_2))$; $(x_0, y_1) \in F_1$, $(x_0, y_2) \in F_2$ and $y_1 < y_2$.

Let $0 < \varepsilon < 1$ be fixed. We may find $\delta > 0$ such that for any rectangles $Q_1 \ni (x_0, y_1)$, $Q_2 \ni (x_0, y_1)$ with diameters loss than δ the following inequalities are valid:

$$|Q_1 \cap F_1| > (1-\varepsilon)|Q_1|, \qquad |Q_2 \cap F_2| > (1-\varepsilon)|Q_2|, \tag{3}$$

$$p_{Q_1} = \sup_{Q_1 \cap F_1} p(x,y) < c_1 < c_2 < \inf_{Q_2 \cap F_2} p(x,y) = p_{Q_2} \tag{4}$$

for some constant c_1, c_2.

Let $Q_{1,\tau}$, $Q_{2,\tau}$ are rectangles with properties (3), (4) of the form $(x_0 - \tau, x_0 + \tau) \times \times (a, b)$, $(x_0 - \tau, x_0 + \tau) \times (c, d)$, where $a < b < c < d$.

We have continuously embedding $L^{p(t)}(Q_{2,\tau}) \hookrightarrow L^{p_{Q_2}}(Q_{2,\tau})$ and $L^{p'(t)}(Q_{1,\tau}) \hookrightarrow \hookrightarrow L^{p'_{Q_1}}(Q_{1,\tau})$, where $1/p'_{Q_1} + 1/p_{Q_1} = 1$ (see for example [11]). For rectangle $Q_\tau = = (x_0 - \tau, x_0 + \tau) \times (a, d)$ we have

$$A_\tau = \frac{1}{|Q_\tau|}\|\chi_{Q_\tau}\|_{p(t)}\|\chi_{Q_\tau}\|_{p'(t)} \geq \frac{1}{2\tau(d-a)}\left\|\chi_{Q_{2,\tau}\cap F_2}\right\|_{p(t)}\left\|\chi_{Q_{1,\tau}\cap F_1}\right\|_{p'(t)} \geq$$

$$\geq \frac{C}{2\tau(d-a)}(2\tau(d-c))^{1/p_{Q_2}}(2\tau(b-a))^{1-1/p_{Q_1}}.$$

Note that if $\tau \to 0$ (a, b, c, d is fixed) $A_\tau \to \infty$ and consequently (2) is not valid. This completes the proof.

***Proof of Theorem* 3.** The inequalities (1) are consequence of the extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez [4] and the weighted norm inequalities for $G(f)$ function given by Kurtz [12]. We describe this results.

Let $p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}\}$. By a weight we mean a nonnegative, locally integrable function ω. When $1 < p < \infty$, we say $\omega \in A_p$ if for every interval Q

$$\frac{1}{|Q|}\int_Q \omega(x)\,dx\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \leq C < \infty.$$

The infimum over the constants on the right-hand side of the last inequality we denote by $A_{p,\omega}$. By $\mathcal{F}$ will denote a family of ordered pairs of nonnegative, measurable functions (f, g). We say that an inequality

$$\int_{\mathbb{R}} f(x)^{p_0}\omega(x)\,dx \leq C\int_{\mathbb{R}} g(x)^{p_0}\omega(x)\,dx, \qquad 0 < p_0 < \infty, \tag{5}$$

holds for any $(f, g) \in \mathcal{F}$ and $\omega \in A_q$ (for some q, $1 < q < \infty$) if it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant C depends only on p_0 and the $A_{q,\omega}$ constant of ω.

Theorem 4. *Given a family* $\mathcal{F}$, *assume that* (5) *holds for some* $1 < p_0 < \infty$, *for every weight* $\omega \in A_{p_0}$ *and for all* $(f, g) \in \mathcal{F}$. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R})$ *be such that there exists* $1 < p_1 < p_-$, *with* $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R})$. *Then*

$$\|f\|_{p(t)} \leq C\|g\|_{p(t)}$$

for all $(f, g) \in \mathcal{F}$ *such that* $f \in L^{p(t)}(\mathbb{R})$.

Theorem 5 [12]. *Let* Δ *be a a lacunary decomposition of* $\mathbb{R}$, $1 < p < \infty$, *and*

$\omega \in A_p$. *Then there exist constant* c, C *depending only on* p, $A_{p,\omega}$, *and* Δ, *such that*

$$c\int_{\mathbb{R}} |f(x)|^p \omega(x)\,dx \;\le\; \int_{\mathbb{R}} (G(f)(x))^p \omega(x)\,dx \;\le\; C\int_{\mathbb{R}} |f(x)|^p \omega(x)\,dx.$$

From assumption of Theorem 3 we get that there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R})$ (see [8]). Let $L^{\infty}_{\text{comp}}(\mathbb{R})$ be the set of all bounded functions with compact support. From Theorems 4, 5 with the pairs $(\mathcal{W}f, |f|)$ we get right side inequality of (1) if $f \in L^{\infty}_{\text{comp}}(\mathbb{R})$. Note that $L^{\infty}_{\text{comp}}(\mathbb{R})$ is dense in $L^{p(t)}(\mathbb{R})$ (see [11]) and consequently this inequality is also valid for all $f \in L^{p(t)}(\mathbb{R})$. Analogously we obtain left side inequality of (1).

Let $n > 1$. Fix a rectangle $R = I_1 \times I_2 \times \ldots \times I_n$ and let f be positive on R and 0 elsewhere function. Let k_j be the greatest integer such that $2^{k_j} \le (4n|I_j|)^{-1}$ and ρ be the dyadic rectangle $[2^{k_1}, 2^{k_1+1}] \times \ldots \times [2^{k_n}, 2^{k_n+1}]$. Note that (see [12, p. 246]) for all $x \in R$

$$\left|S_\rho f(x)\right| \;\ge\; \frac{C}{|R|}\int_R f(x)\,dx.$$

Let the operator G is bounded on $L^{p(t)}(\mathbb{R}^n)$. Then for some constant C we have

$$\frac{1}{|Q|}\int_R f(x)\,dx\,\|\chi_R\|_{p(t)} \;\le\; C\|f\|_{p(t)}. \tag{6}$$

Note that $(L^{p(t)}(\mathbb{R}^n))^*$ is isomorphic to the space $L^{p'(t)}(\mathbb{R}^n)$, where $1/p(t) + 1/p'(t) = 1$, $t \in \mathbb{R}^n$ (see [11]). Therefore, for all rectangle R, from (6) we get condition (2). We use Theorem 1 to obtain the desired result.

3. Applications. We now consider applications of Theorem 3. In [7] is proved following theorem.

Theorem 6. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R})$ *and exponent* $p(\cdot)$ *is constant outside some large ball. Then operator* M *is bounded on* $L^{p(t)}(\mathbb{R})$ *if and only if* (2) *fulfilled for intervals.*

The estimate (2) is necessary for boundedness of operator M in $L^{p(t)}(\mathbb{R})$. Combining the Littlewood – Paley type characterization of $L^{p(t)}(\mathbb{R})$ space (Theorem 3) with the previous theorem we can obtain the following corollary.

Corollary. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R})$ *and exponent* $p(\cdot)$ *is constant outside some large ball. Let* Δ *be the dyadic decomposition of* $\mathbb{R}$. *The following are equivalent*:

1) $p(\cdot) \in \mathcal{B}(\mathbb{R})$;

2) *there are constants* c, $C > 0$ *such that for all* $f \in L^{p(t)}(\mathbb{R})$

$$c\|f\|_{p(t)} \;\le\; \|G(f)\|_{p(t)} \;\le\; C\|f\|_{p(t)}.$$

Let $\{f_k\}$ be a sequence of functions defined on $\mathbb{R}$. By $\sum_k f_k \in L^{p(t)}(\mathbb{R})$ we mean the partial sums $\sum_1^N f_k$ converge in $L^{p(t)}(\mathbb{R})$. We now will generalize Theorem 6 of Stein [13].

Theorem 7. *Let* $p(\cdot)\in\mathcal{B}(\mathbb{R})$, *and* S_k *be any collection of lacunary partial sum operators. Then* $f\in L^{p(t)}(\mathbb{R})$ *if and only if* $\sum_k \varepsilon_k S_k f$ *converges in* $L^{p(t)}(\mathbb{R})$ *for any sequence* $\{\varepsilon_k\}\in l^\infty$. *Moreover,* $\|f\|_{p(t)}$ *is equivalent to* $\sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1}\left\|\sum_k \varepsilon_k S_k f\right\|_{p(t)}$.

Proof. Let $p(\cdot)\in\mathcal{B}(\mathbb{R})$ and S_k be any collection of lacunary partial sum. For $f\in L^{p(t)}(\mathbb{R})$ we have $\left(\sum_k |S_k f|^2\right)^{1/2}\in L^{p(t)}(\mathbb{R})$. Note that if $\{\varepsilon_k\}\in l^\infty$ then $\left(\sum_k |\varepsilon_k S_k f|^2\right)^{1/2}\in L^{p(t)}(\mathbb{R})$ and

$$\left(\sum_k |\varepsilon_k S_k f|^2\right)^{1/2} \le \|\{\varepsilon_k\}\|_{l^\infty}\left(\sum_k |S_k f|^2\right)^{1/2}.$$

If $N > M$ using Theorem 3,

$$\left\|\sum_{M+1}^{N}\varepsilon_k S_k f\right\|_{p(t)} \le C\|\{\varepsilon_k\}\|_{l^\infty}\left\|\left(\sum_{M+1}^{N}|S_k f|^2\right)^{1/2}\right\|_{p(t)}$$

which implies $\left\{\sum_1^N \varepsilon_k S_k f\right\}_1^\infty$ is Cauchy in $L^{p(t)}(\mathbb{R})$. From this follows

$$\left\|\sum_k \varepsilon_k S_k f\right\|_{p(t)} \le c\|\{\varepsilon_k\}\|_{l^\infty}\left\|\left(\sum_k |S_k f|^2\right)^{1/2}\right\|_{p(t)}.$$

Assume that S_k be any collection of lacunary partial sum operators and $\sum_k \varepsilon_k S_k f \in L^{p(t)}(\mathbb{R})$ for all $\{\varepsilon_k\}\in l^\infty$. We will prove that $\left(\sum_k |S_k f|^2\right)^{1/2}\in L^{p(t)}(\mathbb{R})$ and there exists a constant $c > 0$ independent of f such that

$$\left\|\left(\sum_k |S_k f|^2\right)^{1/2}\right\|_{p(t)} \le c\sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1}\left\|\sum_k \varepsilon_k S_k f\right\|_{p(t)}. \tag{7}$$

First we will prove that $M = \sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1}\left\|\sum_k \varepsilon_k S_k f\right\|_{p(t)}$ is finite. Indeed, consider the collection of maps $\left\{G_N : l^\infty \to L^{p(t)}(\mathbb{R})\right\}$ defined by $G_N(\{\varepsilon_k\}) = \sum_{k=1}^N \varepsilon_k S_k f$. Let $G = G_\infty$. Each G_N is continuous and by assumption $G_N(\{\varepsilon_k\})$ converges to $G(\{\varepsilon_k\})$ in $L^{p(t)}(\mathbb{R})$ for each $\{\varepsilon_k\}\in l^\infty$. Therefore $\left\{\|G_N(\{\varepsilon_k\})\|_{p(t)}\right\}_{N=1}^\infty$ is bounded for each $\{\varepsilon_k\}\in l^\infty$. By the principle of uniform boundedness, there exists a constant $c > 0$ such that $\|G_N\| \le c$ for all N. It follows that $\|G\| \le c$.

To proof of (7) will use Khinchine's inequality for Rademacher series. Let $r_k(t) = \operatorname{sgn}(\sin 2^m \pi t)$, $m = 0, 1, 2, \ldots$, be the Rademacher functions, and set $f = \sum_0^\infty a_m r_m$. Then there are constants B_p and C_p such that for $0 < p < \infty$

$$B_p\left(\int_0^1 |f(t)|^p\,dt\right)^{1/p} \le \left(\sum_{m=0}^\infty |a_m|^2\right)^{1/2} \le C_p\left(\int_0^1 |f(t)|^p\,dt\right)^{1/p} \tag{8}$$

(see [2]). Let $\varepsilon_k = r_k(t)$ for $0 \le t < 1$. Then $\|\{\varepsilon_k\}\|_{l^\infty} = 1$ and

$$M \geq \left\| \sum_k r_k(x) S_k f \right\|_{p(t)}.$$

Using (8) for $p = 1$ and Fubini's theorem we have

$$\left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_{p(t)} \leq C_1 \left\| \int_0^1 \left| \sum_k r_k(x) S_k f \right| dx \right\|_{p(t)} \leq$$

$$\leq cC_1 \sup_{\|g\|_{p'(t)} \leq 1} \int_{\mathbb{R}} \left(\int_0^1 \left| \sum_k r_k(x) S_k f(z) \right| dx \right) |g(z)| dz =$$

$$= cC_1 \sup_{\|g\|_{p'(t)} \leq 1} \int_0^1 \left(\int_{\mathbb{R}} \left| \left(\sum_k r_k(x) S_k f(z) g(z) \right) \right| dz \right) dx \leq$$

$$\leq cC_1 \int_0^1 \left(\sup_{\|g\|_{q(t)} \leq 1} \int_{\mathbb{R}} \left| \left(\sum_k r_k(x) S_k f(z) \right) g(z) \right| dz \right) dx \leq$$

$$\leq cC_1 \int_0^1 \left\| \sum_k r_k(x) S_k f \right\|_{p(t)} dx \leq cC_1 M,$$

which proves (7). This completes the proof of Theorem 7.

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