UDC 517.9

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OSCILLATION OF CERTAIN FOURTH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

КОЛИВАННЯ ДЕЯКИХ ФУНКЦІОНАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

Some new criteria for the oscillation of fourth order nonlinear functional differential equations of the form

$$\frac{d^2}{dt^2}\left(a(t)\left(\frac{d^2x(t)}{dt^2}\right)^{\alpha}\right)+q(t)f(x[g(t)])=0, \quad \alpha>0,$$

are established.

Встановлено деякі нові критерії коливання нелінійних функціональних диференціальних рівнянь вигляду

$$\frac{d^2}{dt^2}\left(a(t)\left(\frac{d^2x(t)}{dt^2}\right)^{\alpha}\right)+q(t)f(x[g(t)])=0, \quad \alpha>0.$$

1. Introduction. In this paper we are concerned with the oscillatory behavior of fourth order nonlinear differential equations of the type

$$\frac{d^2}{dt^2}\left(a(t)\left(\frac{d^2x(t)}{dt^2}\right)^{\alpha}\right)+q(t)f(x[g(t)])=0, \tag{1.1}$$

where

(i) $a(t), q(t)\in C([t_0,\infty),\mathbb{R}^+=(0,\infty))$,

(ii) $g(t)\in C([t_0,\infty),\mathbb{R}=(-\infty,\infty))$ and $\lim_{t\to\infty} g(t)=\infty$,

(iii) $f\in C(\mathbb{R},\mathbb{R})$ and $xf(x)>0$ for $x\neq 0$, and

(iv) α is the ratio of two positive odd integers.

In what follows we shall assume that

$$\int^{\infty} a^{-1/\alpha}(s)ds=\infty. \tag{1.2}$$

By a solution of equation (1.1), we mean a function $x\in C^2([t_x,\infty),\mathbb{R})$ such that $a(t)\big(x''(t)\big)^{\alpha}\in C^2\big([t_x,\infty),\mathbb{R}\big)$ and satisfies the equation at every point $t\geq t_x\geq t_0\geq 0$. Here, we are concerned with proper solutions of equation (1.1), that is, those solutions $x(t)$ which satisfy $\sup\big\{|x(t)|: t\geq T\big\}>0$ for every $T\geq t_x$. Such a solution is said to be oscillatory if it has an infinite sequence of zeros clustering at infinity and nonoscillatory if it has at most a finite number of zeros in its interval of existence.

We introduce the notation L_i, $i=0,1,2,3,4$, for the lower order derivatives associated with the operator $L_4x(t)=\dfrac{d^2}{dt^2}\left(a(t)\left(\dfrac{d^2x(t)}{dt^2}\right)^{\alpha}\right)$:

$$\begin{gathered}L_0x(t)=x(t), \qquad L_1x(t)=\frac{d}{dt}L_0x(t), \qquad L_2x(t)=a(t)\left(\frac{d}{dt}L_1x(t)\right)^{\alpha}, \\ L_3x(t)=\frac{d}{dt}L_2x(t), \qquad L_4x(t)=\frac{d}{dt}L_3x(t).\end{gathered} \tag{1.3}$$

The classical Atkinson – Belohorec oscillation results [7] for the Emden – Fowler differential equation

$$x''(t) + q(t)\big|x(t)\big|^{\gamma} \operatorname{sgn} x(t) = 0, \tag{1.4}$$

where $0 < \gamma \neq 1$ is a constant and $q(t) \in C\big([t_0, \infty), \mathbb{R}^+\big)$ has been studied and generalized in various directions in the literature. One of the remarkable extensions of the oscillation due to Atkinson – Belohorec is for nonlinear differential equations of the type

$$\Big(\big|x'(t)\big|^{\alpha} \operatorname{sgn} x'(t)\Big)' + q(t)\big|x(t)\big|^{\beta} \operatorname{sgn} x(t) = 0, \tag{1.5}$$

where $\alpha,\ \beta > 0$ are constants and $q(t) \in C\big([t_0, \infty), \mathbb{R}^+\big)$ and was carried out by Elbert etc. [9] and Kusano etc. [13]. For related results the reader is referred to our book [5], and [1 – 4, 6] and the references cited therin.

Our main objective is to present a systematic study on the oscillation of equation (1.1) and establish some new oscillation criteria. In Section 2, we shall give the proof of an important lemma which is useful throughout this paper. Also, we present oscillation results when f satisfies the condition $f^{1-1/\alpha}(x)f'(x) \geq k > 0$ for $x \neq 0$, or $f(x) \operatorname{sgn} x \geq |x|^{\beta}$ for $x \neq 0$, where β is the ratio of two positive odd integers, $\beta > \alpha$, $\beta = \alpha$ and $\beta < \alpha$. Results that involve comparison with linear and half–linear differential equations are studied. Section 3 is devoted to the study of equation (1.1) when f satisfies either $\int^{\pm\infty} du/f^{1/\alpha}(u) < \infty$, or $\int_{\pm 0} du/f^{1/\alpha}(u) < \infty$. In Section 4 we give necessary and sufficient conditions for the oscillation of all bounded and unbounded solutions of equation (1.1) when $f(x) \operatorname{sgn} x \geq |x|^{\beta}$ for $x \neq 0$. In Section 5 we give a comparison result which allows us to extend our results to certain neutral differential equations and also, when f need not be a monotonic function. The obtained results are new, and extend and improve those known in the literature for the equation (1.5).

2. Oscillation and comparison results. Before we state our results, we shall need the following preliminaries: If $x(t)$ is an eventually positive solution of equation (1.1), then $L_4 x(t) \leq 0$ eventually, and since condition (1.2) holds, it follows that $L_i x(t)$, $i = 1, 2, 3$, are eventually of constant sign. We distinguish the following two cases:

(I) $L_i x(t) > 0$, $i = 0, 1, 2, 3$ and $L_4 x(t) \leq 0$ eventually,

(II) $L_0 x(t) > 0$, $L_1 x(t) > 0$, $L_2 x(t) < 0$, $L_3 x(t) > 0$ and $L_4 x(t) \leq 0$ eventually.

Let (I) hold. Since $L_3 x(t) > 0$ is decreasing (say) for $t \geq t_0 \geq 0$, we have

$$L_2 x(t) - L_2 x(t_0) = \int_{t_0}^{t} L_3 x(s) ds,$$

or

$$a(t) \left(\frac{d}{dt} L_1 x(t)\right)^{\alpha} \geq (t - t_0) L_3 x(t) \quad \text{for} \quad t \geq t_0,$$

or

$$x''(t) \geq \left(\frac{t - t_0}{a(t)}\right)^{1/\alpha} L_3^{1/\alpha} x(t) \quad \text{for} \quad t \geq t_0. \tag{2.1}$$

Integrating (2.1) from t_0 to t and using (I) and the decreasing property of $L_3 x(t)$ on $[t_0, \infty)$, we have

$$x'(t) \geq \left(\int_{t_0}^{t} \left(\frac{u - t_0}{a(u)}\right)^{1/\alpha} du\right) L_3^{1/\alpha} x(t), \quad t \geq t_0, \tag{2.2}$$

and

$$x(t) \geq \left(\int_{t_0}^{t} (t-u) \left(\frac{u-t_0}{a(u)} \right)^{1/\alpha} du \right) L_3^{1/\alpha} x(t), \quad t \geq t_0. \tag{2.3}$$

Let (II) hold. Then for $t \geq u \geq t_0$ and the decreasing property of $L_3 x(t) > 0$, we obtain

$$L_2 x(t) - L_2 x(u) = \int_{u}^{t} L_3 x(\tau) d\tau,$$

or

$$-a(u)\big(x''(u)\big)^{\alpha} \geq (t-u) L_3 x(t),$$

or

$$-x''(u) \geq \left(\frac{t-u}{a(u)} \right)^{1/\alpha} L_3^{1/\alpha} x(t), \quad t \geq u \geq t_0. \tag{2.4}$$

Integrating (2.4) from λt to $t \geq t_0$ for some λ, $0 < \lambda < 1$, using (II) and the decreasing property of $L_3 x(t),\ t \geq t_0$, we get

$$x'(\lambda t) \geq \left(\int_{\lambda t}^{t} \left(\frac{t-u}{a(u)} \right)^{1/\alpha} du \right) L_3^{1/\alpha} x(t) \tag{2.5}$$

and for $t \geq T/\lambda \geq t_0$,

$$x(t) \geq x(\lambda t) \geq \left(\int_{T}^{t} (\lambda t - u) \left(\frac{t-u}{a(u)} \right)^{1/\alpha} du \right) L_3^{1/\alpha} x(t). \tag{2.6}$$

For $t \geq T/\lambda \geq t_0$ and for some constant λ, $0 < \lambda < 1$, we let

$$h(t, T; a; \lambda) = \min \left\{ \int_{T}^{\lambda t} \left(\frac{u-T}{a(u)} \right)^{1/\alpha} du, \ \int_{\lambda t}^{t} \left(\frac{t-u}{a(u)} \right)^{1/\alpha} du \right\},$$

$$H(t, T; a; \lambda) = \min \left\{ \int_{T}^{t} (t-u) \left(\frac{u-T}{a(u)} \right)^{1/\alpha} du, \ \int_{T}^{\lambda t} (\lambda t - u) \left(\frac{t-u}{a(u)} \right)^{1/\alpha} du \right\}.$$

Combining the above results, we are ready to state the following interesting lemma.

Lemma 2.1. *Let $x(t)$ be a positive solution of equation* (1.1) *for $t \geq t_0$. Then for some constant λ, $0 < \lambda < 1$, and all large $t \geq T/\lambda \geq t_0$,*

$$x'(\lambda t) \geq h(t, T; a; \lambda) L_3^{1/\alpha} x(t) \tag{2.7}$$

and

$$x(t) \geq x(\lambda t) \geq H(t, T; a; \lambda) L_3^{1/\alpha} x(t). \tag{2.8}$$

We shall also need the following lemma given in [12].

Lemma 2.2. *If X and Y are nonnegative, then*

$$X^{\lambda} + (\lambda - 1) Y^{\lambda} - \lambda X Y^{\lambda - 1} \geq 0, \quad \lambda > 1,$$

where equality holds if and only if $X = Y$.

For $t \geq t_0$, we define

$$g_*(t,t_0;a) = \int\limits_{t_0}^{t}\int\limits_{t_0}^{s}\left(\frac{u}{a(u)}\right)^{1/\alpha} duds.$$

We shall also assume that

$$f^{1/\alpha-1}(x)f'(x) \geq k > 0 \quad \text{for} \quad x \neq 0, \quad k \quad \text{is a real constant} \tag{2.9}$$

and there exists a function $\sigma(t) \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ such that

$$\sigma(t) = \inf\big\{t,\ g(t)\big\}, \qquad \sigma'(t) > 0 \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \lim_{t\to\infty}\sigma(t) = \infty. \tag{2.10}$$

Our first result is embodied in the following theorem.

Theorem 2.1. *Let conditions* (1.2), (2.9) *and* (2.10) *hold. If there exist a function* $\rho(t) \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ *and a constant* λ, $0 < \lambda < 1$, *such that for* $\sigma(t) > T/\lambda$, *for some* $T \geq t_0$,

$$\limsup_{t\to\infty}\int\limits_{T}^{t}\left[\rho(s)q(s) - \frac{1}{(\lambda k)^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(\rho'(s))^{\alpha+1}}{[\rho(s)\sigma'(s)h(\sigma(s),t_0;a;\lambda)]^\alpha}\right]ds = \infty, \tag{2.11}$$

where h is as in Lemma 2.1, *then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. From equation (1.1), we see that $L_4x(t) \leq 0$ for $t \geq t_0$ and so $L_ix(t)$, $i = 1,2,3$, are eventually of one sign, and either (I) or (II) holds. In view of Lemma 2.1, there exist a $t_1 \geq t_0$ and a λ, $0 < \lambda < 1$, such that

$$x'(\lambda t) \geq h(t,t_1;a;\lambda)L_3^{1/\alpha}x(t) \quad \text{for} \quad t \geq t_1/\lambda. \tag{2.12}$$

We define

$$w(t) = \rho(t)\frac{L_3x(t)}{f\big(x[\lambda\sigma(t)]\big)}, \qquad t \geq t_2 \geq t_1. \tag{2.13}$$

Then for $t \geq t_2$, we have

$$\begin{aligned} w'(t) &= \rho(t)\frac{(L_3x(t))'}{f(x[\lambda\sigma(t)])} + \rho'(t)\frac{L_3x(t)}{f(x[\lambda\sigma(t)])} - \\ &\quad -\rho(t)\frac{L_3x(t)f'(x[\lambda\sigma(t)])x'[\lambda\sigma(t)]\lambda\sigma'(t)}{f^2(x[\lambda\sigma(t)])} = \\ &= -\rho(t)q(t)\frac{f(x[g(t)])}{f(x[\lambda\sigma(t)])} + \frac{\rho'(t)}{\rho(t)}w(t) - \\ &\quad -\lambda\rho(t)\sigma'(t)\frac{f'(x[\lambda\sigma(t)])}{f^{1-1/\alpha}(x[\lambda\sigma(t)])}\frac{L_3x(t)x'[\lambda\sigma(t)]}{f^{1+1/\alpha}(x[\lambda\sigma(t)])}. \end{aligned} \tag{2.14}$$

There exists a $t_2 \geq t_1$ such that $\sigma(t) > t_1/\lambda$ and

$$x'\big[\lambda\sigma(t)\big] \geq h(\sigma(t),t_1;a;\lambda)L_3^{1/\alpha}x(t) \quad \text{for} \quad t \geq t_2. \tag{2.15}$$

Using (2.9) and (2.15) and the fact that $x(t)$ is increasing for $t \geq t_2$ in (2.14), we obtain

$$w'(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \lambda k\rho^{-1/\alpha}(t)\sigma'(t)h(\sigma(t),t_1;a;\lambda)w^{1+1/\alpha}(t), \quad t \geq t_2. \tag{2.16}$$

Set

$$X = \Big(\lambda k \rho^{-1/\alpha}(t)\sigma'(t)h(\sigma(t), t_1; a; \lambda)\Big)^{\alpha/(\alpha+1)} w(t), \quad \lambda = \frac{\alpha+1}{\alpha} > 1,$$

and

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha} \left[\Big(\lambda k \rho^{-1/\alpha}(t)\sigma'(t)h(\sigma(t), t_1; a; \lambda)\Big)^{-\alpha/(\alpha+1)}\right]^{\alpha}$$

in Lemma 2.2 to conclude that for $t \geq t_2$,

$$\frac{\rho'(t)}{\rho(t)} w(t) - \lambda k \rho^{-1/\alpha}(t)\sigma'(t)h(\sigma(t), t_1; a; \lambda) w^{1+1/\alpha}(t) \leq$$

$$\leq \frac{1}{(\lambda k)^{\alpha}} \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}} \frac{(\rho'(t))^{\alpha+1}}{\big[\rho(t)\sigma'(t)h(\sigma(t), t_1; a; \lambda)\big]^{\alpha}}.$$

Thus, we have

$$w'(t) \leq -\rho(t)q(t) + \frac{1}{(\lambda k)^{\alpha}} \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}} \frac{(\rho'(t))^{\alpha+1}}{\big[\rho(t)\sigma'(t)h(\sigma(t), t_1; a; \lambda)\big]^{\alpha}}, \quad t \geq t_2. \tag{2.17}$$

Integrating (2.17) from t_2 to t, we get

$$0 < w(t) \leq$$

$$\leq w(t_2) - \int_{t_2}^{t} \left[\rho(s)q(s) - \frac{1}{(\lambda k)^{\alpha}} \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}} \frac{(\rho'(s))^{\alpha+1}}{[\rho(s)\sigma'(s)h(\sigma(s), t_1; a; \lambda)]^{\alpha}}\right] ds.$$

Taking the $\limsup$ of both sides of the above inequality as $t \to \infty$ and applying condition (2.12), we obtain $w(t) \to -\infty$ as $t \to \infty$, which is a contradiction. This completes the proof.

Theorem 2.2. *Let* $\alpha \geq 1$, *conditions* (1.2) *and* (2.10) *hold, and*

$$f(x)\operatorname{sgn} x \geq |x|^{\beta} \quad \textit{for} \quad x \neq 0, \tag{2.18}$$

where β *is the ratio of two positive odd integers. If there exist a function* $\rho(t) \in C^1\big([t_0, \infty), \mathbb{R}^+\big)$ *and a constant* λ, $0 < \lambda < 1$, *such that*

$$\limsup_{t\to\infty} \int_{T}^{t} \left[\rho(s)q(s) - \frac{(\rho'(s))^2}{4\lambda\beta\sigma'(s)\rho(s)h(\sigma(s), t_0; a; \lambda)H^{\alpha-1}(\sigma(s), t_0; a; \lambda)C(s)}\right] ds =$$

$$= \infty, \tag{2.19}$$

where h *and* H *are as in Lemma* 2.1 *and* $\sigma(t) > T/\lambda > t_0$, *and*

$$C(t) = \begin{cases} c_1, \ c_1 \quad \textit{is any positive constant}, & \textit{when} \quad \beta > \alpha, \\ 1, & \textit{when} \quad \beta = \alpha, \\ c_2 g_*^{\beta-\alpha}(t, t_0; a), \quad c_2 \quad \textit{is any positive constant}, & \textit{when} \quad \beta < \alpha, \end{cases}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 > 0$. Proceeding as in the proof of Theorem 2.1, there exists a $T > \overline{T} \geq t_0$ such that for $\sigma(T) > \overline{T}/\lambda$ and some $\lambda \in (0, 1)$, we have

$$x'[\lambda\sigma(t)] \geq h(\sigma(t), \overline{T}; a; \lambda) L_3^{1/\alpha} x(t), \quad t \geq T, \tag{2.20}$$

and

$$x(t) \ge x\big[\sigma(t)\big] \ge x\big[\lambda\sigma(t)\big] \ge H\big(\sigma(t), \overline{T}; a; \lambda\big) L_3^{1/\alpha} x(t), \quad t \ge T. \tag{2.21}$$

Next, there exists a constant $b > 0$ such that

$$L_3 x(t) \le b \quad \text{for} \quad t \ge T.$$

Integrating this inequality from T to t, one can easily find that there exist a constant $b_1 > 0$ and a $T_1 \ge T$ such that

$$x\big[\lambda\sigma(t)\big] \le x(t) \le b_1 g_*(t, T; a) \quad \text{for} \quad t \ge T_1. \tag{2.22}$$

We define the function $w(t)$ as in (2.13) and proceed as in the proof of Theorem 2.1 to obtain (2.14) with $f(x)$ replaced by x^β. Using (2.20) and (2.21) in (2.14), for $t \ge T$ we get

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) -$$

$$-\lambda\beta \frac{\sigma'(t)}{\rho(t)} h(\sigma(t), T; a; \lambda) H^{\alpha-1}(\sigma(t), T; a; \lambda) x^{\beta-\alpha}[\lambda\sigma(t)] w^2(t). \tag{2.23}$$

Now, we need to consider the following three cases:

Case 1. If $\beta > \alpha$, then there exist a constant $b_1 > 0$ and a $T_2 \ge T$ such that

$$x[\lambda\sigma(t)] \ge b_1 \quad \text{for} \quad t \ge T_2. \tag{2.24}$$

Thus, the inequality (2.23) becomes

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) -$$

$$-\lambda\beta b_1^{\beta-\alpha} \frac{\sigma'(t)}{\rho(t)} h(\sigma(t), T; a; \lambda) H^{\alpha-1}(\sigma(t), T; a; \lambda) w^2(t), \quad t \ge T_2. \tag{2.25}$$

Case 2. If $\beta = \alpha$, then inequality (2.23) becomes

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) -$$

$$-\lambda\beta \frac{\sigma'(t)}{\rho(t)} h(\sigma(t), T; a; \lambda) H^{\alpha-1}(\sigma(t), T; a; \lambda) w^2(t), \quad t \ge T. \tag{2.26}$$

Case 3. If $\beta < \alpha$, then by (2.22) we get

$$x^{\beta-\alpha}[\lambda\sigma(t)] \ge \gamma g_*^{\beta-\alpha}(t, T; a), \quad \gamma = b_1^{\beta-\alpha} \quad \text{for} \quad t \ge T_1 \tag{2.27}$$

and inequality (2.23) takes the form

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) -$$

$$-\lambda\beta\gamma \frac{\sigma'(t)}{\rho(t)} g_*^{\beta-\alpha}(t, T; a) h(\sigma(t), T; a; \lambda) H^{\alpha-1}(\sigma(t), T; a; \lambda) w^2(t), \quad t \ge T. \tag{2.28}$$

Let $T^* = \max\{T, T_1, T_2\}$, so that we can combine inequalities (2.25), (2.26) and (2.28), to obtain for $t \ge T^*$,

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)} w(t) -$$

$$-\lambda\beta\frac{\sigma'(t)}{\rho(t)}C(t)h(\sigma(t),T;a;\lambda)H^{\alpha-1}(\sigma(t),T;a;\lambda)w^2(t) = \tag{2.29}$$

$$= -\rho(t)q(t) - \left[\sqrt{\lambda\beta\frac{\sigma'(t)}{\rho(t)}C(t)h(\sigma(t),T;a;\lambda)H^{\alpha-1}(\sigma(t),T;a;\lambda)}\,w(t) - \right.$$

$$\left. - \frac{\rho'(t)}{2\rho(t)\sqrt{\lambda\beta\frac{\sigma'(t)}{\rho(t)}C(t)h(\sigma(t),T;a;\lambda)H^{\alpha-1}(\sigma(t),T;a;\lambda)}}\right]^2 +$$

$$+ \frac{(\rho'(t))^2}{4\lambda\beta\sigma'(t)\rho(t)C(t)h(\sigma(t),T;a;\lambda)H^{\alpha-1}(\sigma(t),T;a;\lambda)} \leq$$

$$\leq -\left[\rho(t)q(t) - \frac{(\rho'(t))^2}{4\lambda\beta\sigma'(t)\rho(t)C(t)h(\sigma(t),T;a;\lambda)H^{\alpha-1}(\sigma(t),T;a;\lambda)}\right]. \tag{2.30}$$

Integrating (2.30) from T^* to t, we have

$$0 < w(t) \leq w(T^*) -$$

$$-\int\limits_{T^*}^{t}\left[\rho(s)q(s) - \frac{(\rho'(s))^2}{4\lambda\beta\sigma'(s)\rho(s)C(s)h(\sigma(s),T;a;\lambda)H^{\alpha-1}(\sigma(s),T;a;\lambda)}\right]ds.$$

Taking the $\limsup$ of both sides of the above inequality as $t \to \infty$ and applying condition (2.19), we see that $w(t) \to -\infty$ as $t \to \infty$, which is a contradiction. This completes the proof.

Next, we have the following result for equation (1.1) when $0 < \alpha \leq 1$.

Theorem 2.3. *Let* $0 < \alpha \leq 1$, *conditions* (1.2), (2.10) *and* (2.18) *hold, and assume that there exist a function* $\rho(t) \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ *and a constant* λ, $0 < \lambda < 1$, *such that for* $\sigma(t) > T/\lambda$ *for some* $T \geq t_0$,

$$\limsup_{t\to\infty}\int\limits_{T}^{t}\left[\rho(s)q(s) - \frac{(\rho'(s))^2Q^{1-1/\alpha}(s)}{4\lambda\beta\sigma'(s)h(\sigma(s),t_0;a;\lambda)\tilde{C}(s)}\right]ds = \infty, \tag{2.31}$$

where h *is as in Lemma* 2.1, $Q(t) = \displaystyle\int_t^\infty q(s)ds$, *and*

$$\tilde{C}(t) = \begin{cases} c_1,\ c_1 \quad \textit{is any positive constant}, & \textit{if} \quad \beta > \alpha, \\ 1, & \textit{if} \quad \beta = \alpha, \\ c_2 g_*^{\beta/\alpha-1}(t,t_0;a), \quad c_2 \ \textit{is any positive constant}, & \textit{if} \quad \beta < \alpha, \end{cases}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 > 0$. Define the function $w(t)$ by (2.13) with $f(x) = x^\beta$ and proceed as in the proof of Theorems 2.1 and 2.2 to obtain (2.14), (2.20) and (2.22) for $t \geq T$. Using (2.20) in (2.14) one obtains

$$w'(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) -$$

$$-\lambda\beta\sigma'(t)\rho^{-1/\alpha}(t)w^2(t)w^{1/\alpha-1}(t)h(\sigma(t),\overline{T};a;\lambda)x^{\beta/\alpha-1}[\lambda\sigma(t)], \quad t \geq T. \tag{2.32}$$

It is easy to see that

$$w(t) \geq \rho(t)Q(t) \quad \text{for} \quad t \geq T.$$

Using this inequality in (2.32), we obtain

$$w'(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) -$$

$$-\lambda\beta\frac{\sigma'(t)}{\rho(t)}Q^{1/\alpha-1}(t)h(\sigma(t),\overline{T};a;\lambda)w^2(t)x^{\beta/\alpha-1}[\lambda\sigma(t)], \quad t \geq T. \tag{2.33}$$

The rest of the proof is similar to that of Theorem 2.2 and hence omitted.

Our next results involve comparison with related linear and half-linear second order differential equations, so that known oscillation theorems from the literature can be employed directly. To obtain these comparison criteria we need the following lemmas given in [5].

Lemma 2.3. *The half-linear differential equation*

$$\big(a(t)\big(x'(t)\big)^{\alpha}\big)' + q(t)x^{\alpha}(t) = 0, \tag{2.34}$$

where a, q *and* α *are as in equation* (1.1) *is nonoscillatory if and only if there exist a number* $T \geq t_0$ *and a function* $v(t) \in C^1\big([t_0,\infty),\mathbb{R}\big)$ *which satisfies the inequality*

$$v'(t) + \alpha a^{-1/\alpha}(t)|v(t)|^{1+1/\alpha} + q(t) \leq 0 \quad \textit{on} \quad [T,\infty).$$

Lemma 2.4. *Let* $h(t) \in C\big([T,\infty),\mathbb{R}^+\big)$, $T \geq t_0$. *If there exists a function* $v(t) \in$ $\in C^1\big([T,\infty),\mathbb{R}\big)$ *such that*

$$v'(t) + h(t)v^2(t) + q(t) \leq 0 \quad \textit{for every} \quad t \geq T,$$

then the second order linear differential equation

$$\left(\frac{1}{h(t)}x'(t)\right)' + q(t)x(t) = 0$$

is nonoscillatory.

First, we relate the oscillation of equation (1.1) to that of half-linear equations of type (2.34).

Theorem 2.4. *Let the hypotheses of Theorem* 2.1 *hold with* $\rho(t) = 1$ *and condition* (2.11) *is replaced by: the half-linear second order equation*

$$(A(t)(y'(t))^{\alpha})' + q(t)y^{\alpha}(t) = 0 \tag{2.35}$$

is oscillatory, where

$$A(t) = \left(\frac{\lambda k}{\alpha}\sigma'(t)h(\sigma(t),t_0;a;\lambda)\right)^{-\alpha}.$$

Then the conclusion of Theorem 2.1 *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceed as in the proof of Theorem 2.1 with $\rho(t) = 1$ to obtain (2.16) which takes the form

$$w'(t) \leq -q(t) - \lambda k\sigma'(t)h(\sigma(t),t_1;a;\lambda)w^{1+1/\alpha}(t) \quad \text{for} \quad t \geq t_2.$$

Applying Lemma 2.3 to the above inequality, we conclude that the equation (2.35) is nonoscillatory, which is a contradiction and completes the proof.

In the following results we shall compare the oscillation of equation (1.1) with that of linear second order ordinary differential equations.

Theorem 2.5. *Let the hypotheses of Theorem* 2.2 *hold with* $\rho(t) = 1$ *and condition* (2.19) *is replaced by*: *the linear second order equation*

$$\left(\frac{1}{r(t)}z'(t)\right)' + q(t)z(t) = 0 \tag{2.36}$$

is oscillatory, where

$$r(t) = \lambda\beta\sigma'(t)C(t)h(\sigma(t), t_0; a; \lambda)H^{\alpha-1}(\sigma(t), t_0; a; \lambda).$$

Then the conclusion of Theorem 2.2 *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceed as in the proof of Theorem 2.2 with $\rho(t) = 1$ to obtain (2.29) which takes the form

$$w'(t) \leq -q(t) - \lambda\beta\sigma'(t)C(t)h(\sigma(t), T; a; \lambda)H^{\alpha-1}(\sigma(t), T; a; \lambda)w^2(t), \quad t \geq T^*.$$

Applying Lemma 2.4 to the above inequality, we find that the equation (2.36) is nonoscillatory, a contradiction and the proof is complete.

Theorem 2.6. *Let the hypotheses of Theorem* 2.3 *hold with* $\rho(t) = 1$ *and condition* (2.31) *is replaced by*: *the linear second order equation*

$$\left(\frac{1}{b(t)}x'(t)\right)' + q(t)x(t) = 0 \tag{2.37}$$

is oscillatory, where

$$b(t) = \lambda\beta\sigma'(t)Q^{1/\alpha-1}(t)h(\sigma(t), t_0; a; \lambda)\tilde{C}(t).$$

Then the conclusion of Theorem 2.3 *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 > 0$. Proceed as in the proof of Theorem 2.3 with $\rho(t) = 1$ to obtain the inequality (2.33) which takes the form

$$w'(t) \leq -q(t) - \lambda\beta\sigma'(t)\tilde{C}(t)Q^{1/\alpha-1}(t)h(\sigma(t), \overline{T}; a; \lambda)w^2(t), \quad t \geq T.$$

The rest of the proof is similar to that of Theorem 2.5 and hence omitted.

Next, we have the following comparison results.

Theorem 2.7. *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (2.18) *hold. If the first order delay equation*

$$y'(t) + q(t)H^{\beta}(\sigma(t), T; a; \lambda)y^{\beta/\alpha}[\sigma(t)] = 0 \tag{2.38}$$

for some $T \geq t_0$ *and* $\lambda \in (0, 1)$ *is oscillatory, then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.2 we obtain the inequality (2.22) which takes the form

$$x[\sigma(t)] \geq H(\sigma(t), T; a; \lambda)L_3^{1/\alpha}x[\sigma(t)] \tag{2.39}$$

for some $T \geq t_0$ and a constant $\lambda \in (0, 1)$. Now, using (2.18) and (2.39) in equation (1.1), we find

$$L_4x(t) + q(t)H^{\beta}(\sigma(t), T; a; \lambda)L_3^{\beta/\alpha}x\big[\sigma(t)\big] \leq 0, \quad t \geq T.$$

Letting $y(t) = L_3x(t)$ in the above inequality, we get

$$y'(t) + q(t)H^{\beta}\big(\sigma(t), T; a; \lambda\big)y^{\beta/\alpha}\big[\sigma(t)\big] \leq 0 \quad \text{for} \quad t \geq T. \tag{2.40}$$

Integrating (2.40) from $t \geq T$ to u and letting $u \to \infty$, we find

$$y(t) \geq \int_t^\infty q(s)H^\beta(\sigma(s), T; a; \lambda)y^{\beta/\alpha}\big[\sigma(s)\big]ds, \quad t \geq T.$$

As in [16] it is easy to conclude that there exists a positive solution $y(t)$ of the equation (2.38) with $\lim_{t\to\infty} y(t) = 0$, a contradiction to the fact that equation (2.38) is oscillatory. This completes the proof.

The following corollary is immediate.

***Corollary* 2.1.** *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (2.18) *hold. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t q(s)H^\beta(\sigma(s), T; a; \lambda)ds > \frac{1}{e} \quad \textit{when} \quad \alpha = \beta,$$

or

$$\int^\infty q(s)H^\beta(\sigma(s), T; a; \lambda)ds = \infty \quad \textit{when} \quad \alpha < \beta$$

for some $t \geq T$ *and a constant* $\lambda \in (0, 1)$, *then equation* (1.1) *is oscillatory.*

Theorem 2.8. *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (2.18) *hold. If the second order equation*

$$y''(t) + Q^*(t)y^{\beta/\alpha}[\sigma(t)] = 0 \tag{2.41}$$

is oscillatory, where

$$Q^*(t) = \left(\frac{1}{a(t)}\int_t^\infty\int_s^\infty q(u)duds\right)^{1/\alpha},$$

then all bounded solutions of equation (1.1) *are oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. In this case $x(t)$ satisfies (II).

Integrating equation (1.1) from $t \geq t_0$ to u, using conditions (2.10) and (2.18) and letting $u \to \infty$, we obtain

$$L_3x(t) \geq x^\beta\big[\sigma(t)\big]\int_t^\infty q(s)ds.$$

Once again, integrating this inequality from $t \geq t_0$ to u and letting $u \to \infty$, we have

$$-x''(t) \geq Q^*(t)x^{\beta/\alpha}[\sigma(t)],$$

or

$$x''(t) + Q^*(t)x^{\beta/\alpha}[\sigma(t)] \leq 0 \quad \text{for} \quad t \geq t_1 \geq t_0.$$

By applying a known comparison criterion (see [16]), one can easily see that equation (2.41) has an eventually positive bounded solution, which contradicts the hypothesis and completes the proof.

The following corollary is now obvious.

***Corollary* 2.2.** *Let conditions* (1.2), (2.12) *and* (2.18) *hold. If*

(i) $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* $\int^{\infty} \sigma(s)Q^*(s)ds = \infty$ *when* $\beta > \alpha$,

(ii) $\sigma'(t) > 0$ *for* $t \geq t_0$ *and there is a function* $\rho(t) \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ *such that*

$$\int^{\infty} \left[\rho(s)Q^*(s) - \frac{(\rho'(s))^2}{4\rho(s)}\right] ds = \infty \quad \textit{when} \quad \beta = \alpha,$$

(iii) $\sigma'(t) \geq 0$ *for* $t \geq t_0$,

$$\int^{\infty} (\sigma(s))^{\beta/\alpha} Q^*(s)ds = \infty \quad \textit{when} \quad \beta < \alpha,$$

where Q^* *is as in Theorem* 2.8, *then all bounded solutions of equation* (1.1) *are oscillatory.*

3. Further oscillation results. In this section we shall present some sufficient conditions for the oscillation of equation (1.1) when $f(x)$ satisfies conditions of the type

$$\int^{\pm\infty} \frac{du}{f^{1/\alpha}(u)} < \infty, \tag{3.1}$$

or

$$\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty. \tag{3.2}$$

Theorem 3.1. *Let conditions* (1.2), (2.10) *and* (3.1) *hold. Moreover, assume that there exists a function* $\rho(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\rho'(t) \geq 0 \quad \textit{and} \quad \left(\frac{(\rho'(t))^{1/\alpha}}{\sigma'(t)h(\sigma(t),t_0;a;\lambda)}\right)' \leq 0 \tag{3.3}$$

for all large $t \geq t_0$ *and some constant* $\lambda \in (0,1)$. *If*

$$\int^{\infty} \rho(s)q(s)ds = \infty, \tag{3.4}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.1, we define $w(t)$ as in (2.13) and obtain (2.14) and (2.15) for $t \geq t_3$. Now

$$w'(t) \leq -\rho(t)q(t) + \rho'(t)\frac{L_3x(t)}{f(x[\lambda\sigma(t)])} \quad \text{for} \quad t \geq T \geq t_3. \tag{3.5}$$

Using (2.15) in (3.5), we find

$$w'(t) \leq -\rho(t)q(t) + \left(\frac{(\rho'(t))^{1/\alpha}}{\lambda\sigma'(t)h(\sigma(t),t_1;a;\lambda)}\frac{\lambda x'[\lambda\sigma(t)]\sigma'(t)}{f^{1/\alpha}(x[\lambda\sigma(t)])}\right)^{\alpha} \quad \text{for} \quad t \geq T.$$

Now by the second Bonnet mean value theorem for a fixed $t \geq T$ and for some $\xi \in [T,t]$, we have

$$\int_T^t \left(\frac{(\rho'(s))^{1/\alpha}}{\lambda\sigma'(s)h(\sigma(s),t_1;a;\lambda)}\right)\left(\frac{\lambda x'[\lambda\sigma(s)]\sigma'(s)}{f^{1/\alpha}(x[\lambda\sigma(s)])}\right)ds =$$

$$= \left(\frac{(\rho'(T))^{1/\alpha}}{\lambda\sigma'(T)h(\sigma(T), t_1; a; \lambda)} \right) \int_T^{\xi} \frac{\lambda x'[\lambda\sigma(s)]\sigma'(s)}{f^{1/\alpha}(x[\lambda\sigma(s)])} ds =$$

$$= \left(\frac{(\rho'(T))^{1/\alpha}}{\lambda\sigma'(T)h(\sigma(T), t_1; a; \lambda)} \right) \int_{x[\lambda\sigma(T)]}^{x[\lambda\sigma(\xi)]} \frac{du}{f^{1/\alpha}(u)} \leq$$

$$\leq \left(\frac{(\rho'(T))^{1/\alpha}}{\lambda\sigma'(T)h(\sigma(T), t_1; a; \lambda)} \right) \int_{x[\lambda\sigma(T)]}^{\infty} \frac{du}{f^{1/\alpha}(u)} = M, \tag{3.6}$$

where M is a positive constant. Using (3.6) in (3.5) and integrating from T to t, we obtain

$$\int_T^t \rho(s)q(s)ds \leq -w(t) + w(T) + M^\alpha.$$

Letting $t \to \infty$ in the above inequality, we arrive at a contradiction to condition (3.4). This completes the proof.

Theorem 3.2. *Let condition* (3.3) *in Theorem* 3.1 *be replaced by*: *for all large* $t \geq t_0$ *and some constant* $\lambda \in (0, 1)$,

$$\rho'(t) \geq 0 \quad \text{and} \quad \int^{\infty} \left| \left(\frac{(\rho'(s))^{1/\alpha}}{\sigma'(s)h(\sigma(s), t_0; a; \lambda)} \right)' \right| ds < \infty. \tag{3.7}$$

Then the conclusion of Theorem 3.1 *holds.*

Next, we present the following oscillation criteria for equation (1.1) when

$$Q(t) = \int_t^{\infty} q(s)ds < \infty. \tag{3.8}$$

Theorem 3.3. *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$, (3.1) *and* (3.8) *hold. If for all large* $t \geq t_0$ *and some constant* $\lambda \in (0, 1)$,

$$\int^{\infty} h(\sigma(s), t_0; a; \lambda)\sigma'(s)Q^{1/\alpha}(s)ds = \infty, \tag{3.9}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) and assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. Define $w(t)$ as in (2.13) with $\rho(t) = 1$. Then, we obtain

$$\int_{t_2}^t q(s)ds \leq \frac{L_3 x(t)}{f(x[\lambda\sigma(t)])}$$

and hence for any $t \geq t_2$,

$$Q(t) \leq \frac{L_3 x(t)}{f(x[\lambda\sigma(t)])},$$

or

$$Q^{1/\alpha}(t) \leq \frac{L_3^{1/\alpha} x(t)}{f^{1/\alpha}(x[\lambda\sigma(t)])}. \tag{3.10}$$

Proceeding as in the proof of Theorem 2.1, we obtain (2.15) for $t \geq t_3$. Using (2.15) in (3.10), we get

$$\lambda h(\sigma(t), t_1; a; \lambda)\sigma'(t)Q^{1/\alpha}(t) \leq \frac{\lambda x'[\lambda\sigma(t)]\sigma'(t)}{f^{1/\alpha}(x[\lambda\sigma(t)])} \quad \text{for} \quad t \geq t_3. \tag{3.11}$$

Integrating (3.11) from t_3 to t, we obtain

$$\lambda \int\limits_{t_3}^{t} h(\sigma(s), t_1; a; \lambda)\sigma'(s)Q^{1/\alpha}(s)ds \leq$$

$$\leq \int\limits_{t_3}^{t} \frac{\lambda x'[\lambda\sigma(s)]\sigma'(s)}{f^{1/\alpha}(x[\lambda\sigma(s)])}ds =$$

$$= \int\limits_{x[\lambda\sigma(t_3)]}^{x[\lambda\sigma(t)]} \frac{du}{f^{1/\alpha}(u)} \leq \int\limits_{x[\lambda\sigma(t_3)]}^{\infty} \frac{du}{f^{1/\alpha}(u)} < \infty,$$

which contradicts condition (3.9). This completes the proof.

Theorem 3.4. *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$, (3.1) *and* (3.8) *hold and suppose that*

$$f'(x)f^{(1-\alpha)/\alpha}(x) = \gamma(x), \tag{3.12}$$

where $\gamma(x)$ *is a positive nondecreasing function for* $x \neq 0$. *If for all large* $T \geq t_0 \geq 0$, *some constant* $\lambda \in (0, 1)$ *and every constant* $c > 0$,

$$\int\limits^{\infty} h(\sigma(s), t_0; a; \lambda)\sigma'(s) \times$$

$$\times \left[Q(s) + c\int\limits_{s}^{\infty} \sigma'(u)h(\sigma(u), t_0; a; \lambda)Q^{(1+\alpha)/\alpha}(u)du\right] ds = \infty, \tag{3.13}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) and assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. We define $w(t)$ as in (2.13) with $\rho(t) = 1$ and as in the proof of Theorem 2.1, we obtain (2.14) which takes the form

$$w'(t) \leq$$

$$\leq -q(t) - \lambda\sigma'(t)f'(x[\lambda\sigma(t)])f^{(1-\alpha)/\alpha}(x[\lambda\sigma(t)])\frac{L_3x(t)x'[\lambda\sigma(t)]}{f^{1+1/\alpha}(x[\lambda\sigma(t)])} \quad \text{for} \quad t \geq t_2. \tag{3.14}$$

Using (2.15) in (3.14), we get

$$w'(t) \leq -q(t) - \lambda\sigma'(t)h(\sigma(t), t_1; a; \lambda)\gamma(x[\lambda\sigma(t)])w^{1+1/\alpha}(t) \quad \text{for} \quad t \geq t_2. \tag{3.15}$$

Since $x(t)$ is increasing on $[t_2, \infty)$ and $\gamma(x)$ is a nondecreasing function, there exists a constant m and a $T \geq t_2$ such that

$$x[\lambda\sigma(t)] \geq m \quad \text{for} \quad t \geq T. \tag{3.16}$$

Using (3.16) in (3.15), we find

$$w'(t) \le -q(t) - \lambda\sigma'(t)\gamma(m)h(\sigma(t),t_1;a;\lambda)w^{1+1/\alpha}(t) \quad \text{for} \quad t \ge T. \tag{3.17}$$

Integrating (3.17) from t to $u \ge t$ and letting $u \to \infty$, we have

$$L_3x(t) \ge$$

$$\ge f\big(x\big[\lambda\sigma(t)\big]\big)\left[Q(t) + \lambda\gamma(m)\int\limits_t^\infty \sigma'(s)h(\sigma(s),t_1;a;\lambda)w^{1+1/\alpha}(s)ds\right] \quad \text{for} \quad t \ge T. \tag{3.18}$$

Clearly, we have

$$w(t) \ge Q(t) \quad \text{for} \quad t \ge T. \tag{3.19}$$

Using (3.19) in (3.18), we obtain

$$\frac{x'[\lambda\sigma(t)](\lambda\sigma'(t))}{f^{1/\alpha}(x[\lambda\sigma(t)])} \ge \lambda h(\sigma(t),t_1;a;\lambda)\times$$

$$\times\left[Q(t) + \lambda\gamma(m)\int\limits_t^\infty \sigma'(s)h(\sigma(s),t_1;a;\lambda)Q^{1+1/\alpha}(s)ds\right]^{1/\alpha} \quad \text{for} \quad t \ge T.$$

Integrating the above inequality from T to t and using condition (3.1), we obtain a contradiction to condition (3.13). This completes the proof.

***Corollary* 3.1.** *Let condition* (3.12) *in Theorem* 3.4 *be replaced by*

$$f'(x)f^{(1-\alpha)/\alpha}(x) \ge k > 0 \quad \textit{for} \quad x \ne 0, \tag{3.20}$$

where k is a constant and let $c = k$ in condition (3.13). *Then the conclusion of Theorem* 3.4 *holds.*

Theorem 3.5. *Let conditions* (1.2), (2.10) *with $\sigma'(t) \ge 0$ for $t \ge t_0$ and* (3.2) *hold, and assume that f satisfies*

$$-f(-xy) \ge f(xy) \ge f(x)f(y) \quad \textit{for} \quad xy > 0. \tag{3.21}$$

If for all large $t \ge t_0$ and some constant $\lambda \in (0,1)$,

$$\int^\infty q(s)f(H(\sigma(s),t_0;a;\lambda))ds = \infty, \tag{3.22}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 \ge 0$. Proceeding as in Theorem 2.2 we obtain the inequality (2.21), which takes the form

$$x[g(t)] \ge H(\sigma(t),t_0;a;\lambda)L_3^{1/\alpha}x(t) \quad \text{for} \quad t \ge T \ge t_0 \quad \text{and some constant} \quad \lambda \in (0,1). \tag{3.23}$$

Using condition (3.21) and inequality (3.23) in equation (1.1), we get

$$-\frac{d}{dt}L_3x(t) \ge q(t)f(x[g(t)]) \ge q(t)f\left(H(\sigma(t),t_0;a;\lambda)L_3^{1/\alpha}x(t)\right) \ge$$

$$\ge q(t)f(H(\sigma(t),t_0;a;\alpha))f\left(L_3^{1/\alpha}x(t)\right) \quad \text{for} \quad t \ge T.$$

Substituting $u(t)$ for $L_3x(t),\ t \ge T$ we have

$$-\frac{du(t)}{dt} \geq q(t)f(H(\sigma(t), t_0; a; \lambda))f\left(u^{1/\alpha}(t)\right) \quad \text{for} \quad t \geq t_0. \tag{3.24}$$

Dividing both sides of (3.24) by $f(u^{1/\alpha}(t))$ and integrating from T to t, we obtain

$$\int_T^t q(s)f(H(\sigma(s), t_0; a; \lambda))ds \leq \int_t^T \frac{u'(s)ds}{f(u^{1/\alpha}(s))} = \int_{u(t)}^{u(T)} \frac{du}{f(u^{1/\alpha})}.$$

Letting $t \to \infty$, we conclude

$$\int_T^\infty q(s)f(H(\sigma(s), t_0; a; \lambda))ds \leq \int_0^{u(T)} \frac{du}{f(u^{1/\alpha})} < \infty,$$

which contradicts condition (3.22) and completes the proof.

Theorem 3.6. *Let conditions* (1.2), (2.10) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (2.18) *with* $\beta < \alpha$ *and assume that*

$$0 < Q(t) = \int_t^\infty q(s)ds < \infty. \tag{3.25}$$

If for all large $t \geq t_0$, *some constant* $\lambda \in (0, 1)$ *and every constant* $c > 0$,

$$\limsup_{t\to\infty} H(\sigma(t), t_0; a; \lambda)\left[Q(t) + c\int_t^\infty h(\sigma(s), t_0; a; \lambda)\sigma'(s)Q^{1+1/\beta}(s)ds\right]^{1/\beta} = \infty, \tag{3.26}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define, $w(t) = L_3x(t)/x^\beta[\lambda\sigma(t)]$ for $t \geq t_1 \geq t_0$. Then for $t \geq t_1$, we get

$$w'(t) \leq -q(t) - \beta\lambda\sigma'(t)L_3^{-1/\alpha}x(t)x'\big[\lambda\sigma(t)\big]w^{1+1/\alpha}(t)x^{(\beta-\alpha)/\alpha}\big[\lambda\sigma(t)\big].$$

As in the proof of Theorem 2.1 we obtain the inequality (2.15) for $t \geq t_2$. Using (2.15) in the above inequality, we have

$$w'(t) \leq -q(t) - \beta\lambda\sigma'(t)h(\sigma(t), t_1; a; \lambda)w^{1+1/\alpha}(t)x^{(\beta-\alpha)/\alpha}\big[\lambda\sigma(t)\big] \quad \text{for} \quad t \geq t_2. \tag{3.27}$$

Integrating (3.27) from $t \geq t_2$ to $u \geq t$ and letting $u \to \infty$, we get

$$L_3x(t) \geq x^\beta[\lambda\sigma(t)] \times$$

$$\times\left[Q(t) + \lambda\beta\int_t^\infty h(\sigma(s), t_1; a; \lambda)\sigma'(s)w^{1+1/\alpha}(s)x^{(\beta-\alpha)/\alpha}[\lambda\sigma(s)]ds\right], \tag{3.28}$$

and hence $Q(t) \leq w(t)$ for $t \geq t_2$.

There exist a constant $b > 0$ and a $t_3 \geq t_2$ such that

$$L_3x(t) \leq b \quad \text{for} \quad t \geq t_3. \tag{3.29}$$

Using (3.29) in (3.28), we obtain

$$x^{(\beta-\alpha)/\alpha}[\lambda\sigma(t)] \geq b^{(\beta-\alpha)/\alpha\beta}Q^{(\alpha-\beta)/\alpha\beta}(t) \quad \text{for} \quad t \geq t_3. \tag{3.30}$$

Next, we proceed as in the proof of Theorem 2.2 and obtain (2.21) with $\overline{T} = t_1$ for $t \geq t_3 \geq t_2$. Now, for $t \geq t_3$,

$$L_3^{1/\beta}x(t) \geq H(\sigma(t), t_1; a; \lambda) L_3^{1/\alpha}x(t) \times$$
$$\times \left[Q(t) + \lambda\beta b^{(\beta-\alpha)/\alpha\beta} \int\limits_t^\infty h(\sigma(s), t_1; a; \lambda)\sigma'(s) Q^{1+1/\beta}(s) ds \right]^{1/\beta},$$

or

$$b^{(\alpha-\beta)/\alpha\beta} \geq L_3^{(\alpha-\beta)/\alpha\beta}x(t) \geq$$
$$\geq H(\sigma(t), t_1; a; \lambda) \left[Q(t) + \lambda\beta b^{(\beta-\alpha)/\alpha\beta} \int\limits_t^\infty h(\sigma(s), t_1; a; \lambda)\sigma'(s) Q^{1+1/\beta}(s) ds \right]^{1/\beta}.$$

Taking the $\limsup$ of this inequality as $t \to \infty$ we arrive at a contradiction to condition (3.26) and complete the proof.

The following result is concerned with the oscillation of advanced equation (1.1), i.e., when $g(t) \geq t$ for $t \geq t_0$. We shall need the following lemma due to Werbowski [17].

Lemma 3.1. *Consider the integrodifferential inequality with deviating argument*

$$y'(t) \geq \int\limits_t^\infty Q(t, s) y[g(s)] ds, \tag{3.31}$$

where $Q \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *and* $g(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $g(t) \geq t$ *for* $t \geq t_0 \geq 0$. *If*

$$\liminf_{t\to\infty} \int\limits_t^{g(t)} \int\limits_s^\infty Q(s, u) du ds > \frac{1}{e},$$

then inequality (3.31) *has no eventually positive solution.*

Theorem 3.7. *Let conditions* (1.2), (2.18) *with* $\beta = \alpha$ *and* (3.25) *hold, and assume that* $g(t) \geq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int\limits_t^{g(t)} P(s) ds > \frac{1}{e}, \tag{3.32}$$

where

$$P(t) = \min\left\{ \int\limits_t^\infty \left[\frac{1}{a(s)} \int\limits_s^\infty Q(\tau) d\tau \right]^{1/\alpha} ds,\ Q^{1/\alpha}(t) h(t, t_0; a; \lambda) \right\}$$

for all large $t \geq t_0$ *and some constant* $\lambda \in (0, 1)$, *then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. There exists a $t_1 \geq t_0$ such that $x(t)$ satisfies Case (I), or Case (II) for $t \geq t_1$. Now we consider:

Case (I). Integrating equation (1.1) from $t \geq t_1$ to $u \geq t$ and letting $u \to \infty$, we obtain

$$L_3 x(t) \geq \left(\int\limits_t^\infty q(s) ds \right) f(x[g(t)]) = Q(t) f(x[g(t)]). \tag{3.33}$$

Once again, we integrate the above inequality from $t \geq t_1$ to $u \geq t$ and let $u \to \infty$, we find

$$-x''(t) \geq \left(\frac{1}{a(t)}\int\limits_t^\infty Q(s)ds\right)^{1/\alpha} f^{1/\alpha}(x[g(t)]) \geq$$

$$\geq \left(\frac{1}{a(t)}\int\limits_t^\infty Q(s)ds\right)^{1/\alpha} x[g(t)] \quad \text{for} \quad t \geq t_1.$$

Integrating the above inequality from $t \geq t_1$ to u and letting $u \to \infty$, we have

$$x'(t) \geq \int\limits_t^\infty \left(\frac{1}{a(s)}\int\limits_s^\infty Q(\tau)d\tau\right)^{1/\alpha} x[g(s)]ds. \tag{3.34}$$

Inequality (3.34) in view of condition (3.32) and Lemma 3.1 has no eventually positive solution, which is a contradiction.

Case (II). Proceeding as in Theorem 2.1 and Case (I) we obtain (2.12) and (3.33) for $t \geq T \geq t_1$. Now, using (2.12) in (3.33) and the fact that $x'(t)$ is increasing on $[t_1, \infty)$, we get

$$x'(t) \geq x'(\lambda t) \geq h(t, t_1; a; \lambda) L_3^{1/\alpha} x(t) \geq$$

$$\geq h(t, t_1; a; \lambda) Q^{1/\alpha}(t) f^{1/\alpha}(x[g(t)]) \geq$$

$$\geq h(t, t_1; a; \lambda) Q^{1/\alpha}(t) x[g(t)] \quad \text{for} \quad t \geq T. \tag{3.35}$$

Inequality (3.35) in view of condition (3.32) and a result in [15] has no eventually positive solution, a contradiction. This completes the proof.

4. Necessary and sufficient conditions. In this section we shall establish some necessary and sufficient conditions for the oscillation of a special case of equation (1.1), namely, the equation

$$L_4 x(t) + q(t) x^\beta[g(t)] = 0, \tag{4.1}$$

where β is the ratio of two positive odd integers.

The following theorem is concerned with a necessary and sufficient condition for the oscillation of all unbounded solutions of the sublinear equation (4.1), i.e., when $\beta < \alpha$.

Theorem 4.1. *Let condition* (1.2) *hold and* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$ *and let* $\beta < \alpha$. *All unbounded solutions of equation* (4.1) *are oscillatory if and only if*

$$\int^\infty q(s) H_1^\beta(g(s), T; a) ds = \infty \tag{4.2}$$

for all large $T \geq t_0$, *where*

$$H_1(t, T; a) = \int\limits_T^t (t-u)\left(\frac{u-T}{a(u)}\right)^{1/\alpha} du.$$

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of equation (4.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Clearly, $x(t)$ satisfies Case (I). Now, the proof of the "if" part is similar to that of Theorem 3.5 and hence omitted. To prove the "only if" part it suffices to assume that

$$\int^\infty q(s) H_1^\beta(g(s), T; a) ds < \infty, \tag{4.3}$$

and show the existence of a nonoscillatory solution of equation (4.1).

Let $c > 0$ be an arbitrary constant and choose $T_1 > T \geq t_0$ sufficiently large so that

$$\int_{T_1}^{\infty} q(s)H_1^{\beta}(g(s), T; a)ds < c^{1-\beta/\alpha}. \tag{4.4}$$

Define the set X by

$$X = \Big\{x \in C[T_1, \infty)\colon c_1 H_1(t, T; a) \leq x(t) \leq c_2 H_1(t, T; a), \quad t \geq T_1\Big\},$$

where $c_1 = (c/2)^{1/\alpha}$ and $c_2 = (2c)^{1/\alpha}$.

Clearly, X is a closed convex subset of the locally convex space $C[T_1, \infty)$ of continuous functions on $[T_1, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_1, \infty)$. Next, let S be a mapping defined on X as follows: For $x \in X$,

$$(Sx)(t) =$$

$$= \int_{T_1}^{t} (t - s) \left\{ \frac{1}{a(s)} \left[c(s - T_1) + \int_{T_1}^{s} \int_{u}^{\infty} q(\tau) x^{\beta}[g(\tau)] d\tau du \right] \right\}^{1/\alpha} ds \quad \text{for} \quad t \geq T_1. \tag{4.5}$$

Clearly, S is well-defined and continuous on X. It can be shown without any difficulty that S maps X into itself and $S(X)$ is relatively compact in $C[T_1, \infty)$. Therefore, by the Schauder – Tychonoff fixed point theorem, S has a fixed element x in X, which satisfies

$$x(t) = \int_{T_1}^{t} (t - s) \left\{ \frac{1}{a(s)} \left[c(s - T_1) + \int_{T_1}^{s} \int_{u}^{\infty} q(\tau) x^{\beta}[g(\tau)] d\tau du \right] \right\}^{1/\alpha} ds, \quad t \geq T_1.$$

Differentiation shows that $x = x(t)$ is a positive solution of equation (4.1) on $[T, \infty)$ such that $\lim_{t\to\infty} x(t)/H_1(t, T; a) = \gamma > 0$, where γ is a constant. For more details, we refer the reader to [6].

Theorem 4.1 can be reformulated as follows:

Theorem 4.1′. *Let condition* (1.2) *hold, and* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$ *and let* $\beta < \alpha$. *Equation* (4.1) *has a nonoscillatory solution* $x(t)$ *such that* $\lim_{t\to\infty} x(t)/H_1(t, T; a) =$ *nonzero constant, and* $t \geq T$ *(large)* $\geq t_0$ *if and only if*

$$\int^{\infty} q(s)H_1^{\beta}(g(s), T; a)ds < \infty, \tag{4.6}$$

where H_1 *is as in Theorem* 5.1.

Next, we present the following necessary and sufficient condition for the oscillation of all bounded solutions of the superlinear equation (4.1), i.e., with $\beta > \alpha$.

Theorem 4.2. *Let condition* (1.2) *hold,* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$ *and* $\beta > \alpha$. *All bounded solutions of equation* (4.1) *are oscillatory if and only if*

$$\int^{\infty} s \left(\frac{1}{a(s)} \int_{s}^{\infty} \int_{u}^{\infty} q(\tau) d\tau du \right)^{1/\alpha} ds = \infty. \tag{4.7}$$

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (4.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Clearly, $x(t)$ satisfies Case (II). The proof of the "if" part is similar to that of Theorem 2.8 and Corollary 2.2(i) and hence omitted.

The “only if” part of the theorem is proved as follows: Let $c > 0$ be given arbitrarily and choose a large $T \geq t_0$ such that

$$\int\limits_T^\infty s \left(\frac{1}{a(s)} \int\limits_s^\infty \int\limits_u^\infty q(\tau) d\tau du \right)^{1/\alpha} ds \leq \frac{1}{2} c^{1-\beta/\alpha}.$$

We define the set Y and the mapping F by

$$Y = \left\{ x \in C[T, \infty) : \ \frac{c}{2} \leq x[g(t)] \leq c, \quad t \geq T \right\}$$

and

$$Fx(t) = c - \int\limits_t^\infty \int\limits_s^\infty \left[\frac{1}{a(u)} \int\limits_u^\infty \int\limits_v^\infty q(\tau) x^\beta[g(\tau)] d\tau dv \right]^{1/\alpha} duds$$

respectively. Now, it is easy to prove that F maps Y into itself and F is a continuous mapping. Also, $F(Y)$ is relatively compact in $C[T, \infty)$. Therefore, by Schauder – Tychnoff fixed point theorem there exists an element $x \in Y$ such that $x = Fx$. It is clear that this fixed point $x = x(t)$ is a positive solution of equation (4.1) on $[T, \infty)$ such that $x(\infty) = c$. This completes the proof.

Once again we can reformulate Theorem 4.2 as follows:

Theorem 4.2′. *Let condition* (1.2) *hold,* $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$, *and* $\beta > > \alpha$. *Equation* (4.1) *has a nonoscillatory solution* $x(t)$ *such that* $\lim_{t\to\infty} x(t) =$ *nonzero constant, if and only if*

$$\int^\infty s \left(\frac{1}{a(s)} \int\limits_s^\infty \int\limits_u^\infty q(\tau) d\tau du \right)^{1/\alpha} ds < \infty.$$

Remark **4.1.** 1. It is easy to see that the results obtained for equation (4.1) can be easily extended to equation (1.1).

2. We note that if equation (4.1) has a bounded eventually positive solution $x(t)$, then $x(t)$ satisfies (II) and so there exist a constant $c > 0$ and a $t_1 \geq t_0$ such that

$$\frac{c}{2} \leq x[g(t)] \leq c \quad \text{for} \quad t \geq t_1.$$

In this case, we see that all bounded solutions of equation (4.1) are oscillatory if

$$\int^\infty \int\limits_s^\infty Q^*(u) duds = \infty,$$

where $Q^*(t)$ is as in Theorem 2.8. The details are easy and left to the reader.

5. Comparison and extensions. In this section we shall obtain a comparison result which is useful in extending our previous results to the neutral equations of the form

$$L_4(x(t) + p(t)x[\tau(t)]) + q(t)f(x[g(t)]) = 0, \tag{5.1}$$

where the operator L_4 and the functions q, g and f are as in equation (1.1), and $p(t)$, $\tau(t) \in C([t_0, \infty), \mathbb{R})$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Now, we present the following comparison result.

Theorem 5.1. *Let condition* (1.2) *hold. If the inequality*

$$L_4x(t) + q(t)f(x[g(t)]) \le 0 \quad (\ge 0) \tag{5.2}$$

has an eventually positive (*negative*) *solution, then equation* (1.1) *also has eventually positive* (*negative*) *solution.*

Proof. Let $x(t)$ be an eventually positive solution of inequality (5.2). It is easy to see that there exists a $t_0 \ge 0$ such that $x(t)$ satisfies either (I), or (II) for $t \ge t_0$. Integrating inequality (5.2) from $t \ge t_0$ to $u \ge t$ and letting $u \to \infty$, we have

$$L_3x(t) \ge \int_t^\infty q(s)f(x[g(s)])ds.$$

Integrating again from $t \ge t_0$ to $u \ge t$, we obtain

$$L_2x(u) - L_2x(t) \ge \int_t^u \int_{s_1}^\infty q(s)f(x[g(s)])dsds_1. \tag{5.3}$$

Now, we distinguish the two cases:

Case (I). Since $L_ix(t) > 0$, $i = 0, 1, 2$, and $t \ge t_0$, we replace t by t_0 and u by t in (5.3), to obtain

$$L_2x(t) \ge \int_{t_0}^t \int_{s_1}^\infty q(s)f(x[g(s)])dsds_1,$$

or

$$x''(t) \ge \left(\frac{1}{a(t)} \int_{t_0}^t \int_{s_1}^\infty q(s)f(x[g(s)])dsds_1\right)^{1/\alpha}$$

and hence

$$x(t) \ge x(t_0) + \int_{t_0}^t \int_{t_0}^{s_3} \left(\frac{1}{a(s_2)} \int_{t_0}^{s_2} \int_{s_1}^\infty q(s)f(x[g(s)])dsds_1\right)^{1/\alpha} ds_2ds_3 =:$$

$$=: c + \Phi(t, x[g(t)]), \tag{5.4}$$

where $x(t_0) = c$.

Now, it is easy to show the existence of a positive solution to the integral equation

$$w(t) = c + \Phi(t, w[g(t)]) \quad \text{for} \quad t \ge t_0.$$

We define the sequence $\{w_n(t)\}$, $n = 0, 1, 2, \ldots$, such that $w_0(t) = x(t)$ for $t \ge t_0$,

$$w_{n+1}(t) = \begin{cases} c + \Phi(t, w_n[g(t)]) & \text{for} \quad t \ge t_0, \\ c & \text{for} \quad t \le t_0. \end{cases}$$

Then one can easily see that $w_n(t)$ is well defined and

$$0 \le w_n(t) \le x(t), \quad c \le w_{n+1}(t) \le w_n(t).$$

Thus, by the Lebesgue monotone convergence theorem there exists $w(t)$ such that $w(t) = = \lim_{n\to\infty} w_n(t)$ for $t \ge t_0$, and

$$w(t) = c + \Phi(t, w[g(t)]) \quad \text{for} \quad t \geq t_0.$$

If we differentiate (5.4) four times, we obtain equation (1.1).

Case (II). Since $L_2x(t) < 0$ for $t \geq t_0$, we let $u \to \infty$ in (5.3) to have

$$-x''(t) \geq \left(\frac{1}{a(t)} \int\limits_t^\infty \int\limits_{s_1}^\infty q(s) f(x[g(s)]) ds ds_1 \right)^{1/\alpha}$$

and hence

$$x(t) \geq x(t_0) + \int\limits_{t_0}^t \int\limits_{s_3}^\infty \left(\frac{1}{a(s_2)} \int\limits_{s_2}^\infty \int\limits_{s_1}^\infty q(s) f(x[g(s)]) ds ds_1 \right)^{1/\alpha} ds_2 ds_3 =:$$

$$=: c + \Psi(t, x[g(t)]),$$

where $x(t_0) = c$. The rest of the proof is similar to that of Case (I) and hence omitted. This completes the proof.

Next, we shall employ Theorem 5.1 to extend the obtained results to the neutral equation (5.1). In fact, we have the following comparison theorem.

Theorem 5.2. *Let conditions* (1.2) *and* (3.21) *hold. Moreover, assume that* $0 \leq p(t) \leq 1$, $\tau(t) < t$, $\tau'(t) > 0$, $g(t) \leq t$ *and* $g'(t) \geq 0$ *for* $t \geq t_0$ *and* $p(t) \not\equiv 1$ *eventually. If the equation*

$$L_4x(t) + q(t) f(1 - p[g(t)]) f(x[g(t)]) = 0 \tag{5.5}$$

is oscillatory, then equation (5.1) *is oscillatory.*

Theorem 5.3. *Let conditions* (1.2) *and* (3.21) *hold and assume that* $p(t) \geq 1$, $\tau(t) > t$, $\tau'(t) > 0$, $\tau^{-1} \circ g(t) \leq t$, $(\tau^{-1} \circ g(t))' \geq 0$ *for* $t \geq t_0$ *and* $p(t) \not\equiv 1$ *eventually. If the equation*

$$L_4x(t) + q(t) f(P[g(t)]) f(x[\tau^{-1} \circ g(t)]) = 0 \tag{5.6}$$

is oscillatory, where

$$P(t) = \frac{1}{p[\tau^{-1}(t)]} \left(1 - \frac{1}{p[\tau^{-1} \circ \tau^{-1}(t)]} \right),$$

then equation (5.1) *is oscillatory.*

***Proof of Theorems* 5.2 *and* 5.3.** Let $x(t)$ be a nonoscillatory solution of equation (5.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define

$$y(t) = x(t) + p(t) x[\tau(t)], \quad t \geq t_0.$$

Then for $t \geq t_0$,

$$L_4y(t) + q(t) f(x[g(t)]) = 0. \tag{5.7}$$

It is easy to check that there exists a $t_1 \geq t_0$ such that $y'(t) > 0$ for $t \geq t_1$. Now, by using the hypotheses of Theorem 5.2, we find

$$x(t) = y(t) - p(t) x\big[\tau(t)\big] = y(t) - p(t)\Big[y[\tau(t)] - p[\tau(t)] x\big[\tau \circ \tau(t)\big]\Big] \geq$$

$$\geq y(t) - p(t) y\big[\tau(t)\big] \geq \big(1 - p(t)\big) y(t) \quad \text{for} \quad t \geq t_1. \tag{5.8}$$

Using (5.8) and condition (3.21) in equation (5.7), we have

$$L_4y(t) + q(t)f(1 - p\big[g(t)\big]\big)f\big(y[g(t)]\big) \leq 0 \quad \text{for} \quad t \geq t_1. \tag{5.9}$$

Next, by using the hypotheses of Theorem 5.3, we find

$$x(t) = \frac{1}{p[\tau^{-1}(t)]}\big(y[\tau^{-1}(t)] - x[\tau^{-1}(t)]\big) =$$

$$= \frac{y[\tau^{-1}(t)]}{p[\tau^{-1}(t)]} - \frac{1}{p[\tau^{-1}(t)]}\left(\frac{y[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}\circ\tau^{-1}(t)]} - \frac{x[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}\circ\tau^{-1}(t)]}\right) \geq$$

$$\geq \frac{y[\tau^{-1}(t)]}{p[\tau^{-1}(t)]} - \frac{y[\tau^{-1}\circ\tau^{-1}(t)]}{p[\tau^{-1}(t)]p[\tau^{-1}\circ\tau^{-1}(t)]} \geq$$

$$\geq \frac{1}{p[\tau^{-1}(t)]}\left(1 - \frac{1}{p[\tau^{-1}\circ\tau^{-1}(t)]}\right)y[\tau^{-1}(t)] = P(t)y[\tau^{-1}(t)] \quad \text{for} \quad t \geq t_1. \tag{5.10}$$

Using (5.10) and condition (3.21) in equation (5.7), we get

$$L_4y(t) + q(t)f\big(P[g(t)]\big)f\big(y\big[\tau^{-1}\circ g(t)\big]\big) \leq 0 \quad \text{for} \quad t \geq t_1.$$

Inequalities (5.9) and (5.10) have positive solutions and hence equations (5.5) and (5.6) have also positive solutions which contradicts the hypotheses and completes the proof.

Finally, we shall extend the results of this paper to equations (1.1) and (5.1) when the function f need not be a monotone function. For this, we let

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if} \quad t_0 > 0, \\ (-\infty, 0) \cup (0, \infty) & \text{if} \quad t_0 = 0, \end{cases}$$

$$C(\mathbb{R}) = \big\{f\colon \mathbb{R} \to \mathbb{R} \quad \text{is continuous and} \quad xf(x) > 0 \quad \text{for} \quad x \neq 0\big\}$$

and

$$C_B(\mathbb{R}_{t_0}) = \Big\{f \in C(\mathbb{R})\colon f \quad \text{is of bounded variation on every interval} \quad [a,b] \subseteq \mathbb{R}_{t_0}\Big\}.$$

We shall need the following Lemma [14].

Lemma 5.1. *Suppose $t_0 > 0$ and $f \in C(\mathbb{R})$. Then $f \in C_B(\mathbb{R}_{t_0})$ if and only if $f(x) = H(x)G(x)$ for all $x \in R_{t_0}$, where $G\colon \mathbb{R}_{t_0} \to (0,\infty)$ is nondecreasing on $(-\infty, t_0)$ and nonincreasing on (t_0, ∞) and $H\colon \mathbb{R}_{t_0} \to \mathbb{R}$ is nondecreasing on $\mathbb{R}_{t_0}$.*

Theorem 5.4. *Let condition (1.2) hold and assume that $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, and let G and H be a pair of continuous components with H being the nondecreasing one. In addition, let $g(t) \leq t$ and $g'(t) \geq 0$ for $t \geq t_0$. If for every constant $k > 0$ and all sufficiently large $T \geq t_0$, the equation*

$$L_4y(t) + q(t)G(kg_*(g(t), T; a))H(y[g(t)]) = 0$$

is oscillatory, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.2 we obtain (2.22) for $t \geq T_1 \geq T$. Now,

$$L_4x(t) + q(t)G\big(b_1g_*(g(t), T; a)\big)H\big(x[g(t)]\big) \leq L_4x(t) + q(t)G\big(x[g(t)]\big)H\big(x[g(t)]\big) =$$

$$= L_4x(t) + q(t)f\big(x[g(t)]\big) = 0, \quad t \geq T_1.$$

The rest of the proof is similar to that of Theorems 5.2 and 5.3 and hence omitted.

6. General remarks.

1. The results of this paper are presented in a form which is essentially new and are of higher degree of generality.

2. The results of this paper can be extended to higher order functional differential equations of the form

$$(a(t)(x^{(n)}(t))^{\alpha})^{(n)} + \delta q(t)f(x[g(t)]) = 0,$$

where $\delta = \pm 1,\ n > 0$ is an integer.

3. The results of this paper can be extended to forced functional differential equations of the type

$$\Big(a(t)\big(x^{(n)}(t)\big)^{\alpha}\Big)^{(n)} + q(t)f(x[g(t)]) = e(t),$$

where $e(t) \in C([t_0, \infty), \mathbb{R})$.

4. When $a(t) \equiv 1$ for $t \geq t_0$, one can easily see that

$$h(t, t_0; 1; 1/2) = c_1 t^{1+1/\alpha} \quad \text{for all large} \quad t$$

and

$$H(t, t_0; 1; 1/2) = c_2 t^{2+1/\alpha} \quad \text{for all large} \quad t,$$

where c_1 and c_2 are positive constants and can be calculated easily. Here, we omit the details.

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Received 03.11.2006