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WHITNEY'S JETS FOR SOBOLEV FUNCTIONS

СТРУМЕНІ УЇТНІ ДЛЯ ФУНКЦІЙ СОБОЛЕВА

We present two fundamental facts of the jet theory for Sobolev spaces $W^{m,p}$. One of them is that the formal differentiation of k-jets theory is compatible with the pointwise definition of Sobolev $(m-1)$-jet spaces on regular subsets of Euclidean spaces $\mathbb{R}^n$. The second result describes the Sobolev embedding operator of Sobolev jet spaces increasing the order of integrability of Sobolev functions up to the critical Sobolev exponent.

Встановлено два фундаментальних факти теорії струменів для просторів Соболева $W^{m,p}$. Перший із них полягає в тому, що формальне диференціювання k-струменів є сумісним з поточковим визначенням соболевських просторів $(m-1)$-струменя на регулярних підмножинах евклідових просторів $\mathbb{R}^n$. Другий результат описує соболевські оператори вкладення соболевських просторів струменів, що покращують порядок сумовності соболевських функцій аж до критичного показника.

1. Introduction. We present here two main theorems — Theorem 3.1 and 3.2 below — as the fundamental facts of the jet theory for Sobolev spaces $W^{m,p}$. The first one expresses the basic fact that the formal differentiation $\mathbf{D}^j: J^k(U) \to J^{k-|j|}(U)$ of k-jets theory is compatible with the pointwise definition of Sobolev $(m-1)$-jet spaces $\mathrm{VLC}(m, p, U)$ on regular subsets U of Euclidean spaces $\mathbb{R}^n$.

Classically and in any theory describing the Sobolev spaces $W^{m,p}(U)$ [1 – 3] by any process of consecutive differentiations $D^\alpha f \in L^p$, e. g. for $|\alpha| < m$ Theorem 3.1 is essentially part of the definition. However with the spaces $\mathrm{VLC}(m, p, U)$ defined by the pointwise inequality (1.1) the deduction of the inequalities (3.2) from (3.1) or (1.1) is far from obvious and quite nontrivial.

Theorem 3.2 describes the Sobolev embedding operator of Sobolev jet spaces increasing the order of integrability of Sobolev functions up to the critical Sobolev exponent.

The case $\mathrm{VLC}(1, p, U)$ is identical with the Hajłasz space $M^{1,p}$ [4, 5] which has been extended by P. Hajłasz for the general case of $M^{1,p}(X, d, \mu)$ spaces on general measure metric spaces (X, d, μ) satisfying the doubling condition. The Hajłasz theorem is one of the cornerstones of the analysis on general metric spaces (fractals etc.).

Theorems 3.1 and 3.2 play a crucial role in incorporating the Sobolev space theory into the general framework of Whitney type theory for classical function spaces, with the full exploitation, in a rather deep way, of the classical Taylor approximating formulas.

The classes $\mathrm{VLC}(m, p, U)$ are described in the form of a pointwise inequality

$$\left|R^{m-1}F(y, x)\right| \le b(x, y)|x-y|^m, \quad (x, y) \in U \times U, \tag{1.1}$$

describing the behavior of the $(m-1)$-order Taylor remainder $R^{m-1}F(x, y)$ of the Sobolev jet F near the diagonal $\Delta = \{x, x\}$, $\Delta \subset U \times U$, in terms of the distance $|x-y|$ of the point (x, y) in the cartesian product $U \times U$ from the diagonal Δ. We call (1.1) the $\mathrm{S}^m\mathrm{RC}$ condition — the Sobolev remainder condition — describing the behavior of the $(m-1)$-order Taylor remainder $R^{m-1}F(y, x)$ of the Sobolev $(m-1)$-jet F.

In the literature only the “splitted” case

$$b(x,\ y) \equiv a(x) + a(y) \quad \text{for some} \quad a \in L^p(U), \quad p > 1,$$

was considered [4 – 6]. Our approach allows also to consider somewhat more general case, not necessarily “splitted”.

As methodological novelty let us mention the general idea of averaging pointwise inequalities and systematic use of local fractional maximal and sharp maximal functions. Markov inequality is crucial also, we refer to [7, 8].

In the seminal papers [9, 10] of H. Whitney the space of continuously differentiable functions $C^m(U)$ on a subset U of $\mathbb{R}^n$ was described by their m-jets. Recall that the space $J^m(U)$ of m-jets F — also called Whitney fields on U — is defined as a collection of functions, also called *components* (*or coefficients*) *of the jet*,

$$F = \left\{ f_j(x) \right\}, \qquad x \in U, \quad |j| \le m,$$

indexed by multiindices $j = (j_1, \ldots, j_n)$, $j_l \ge 0$, $\Sigma_l j_l = |j|$, from a linear class $\mathcal{A}$ of some usual function space of analysis. This can be the class of measurable, bounded, continuous, Lebesgue integrable $L^p_{\text{loc}}(U, \mu)$, μ — some regular Borel measure on U, $p \ge 1$, etc. The class $\mathcal{A}$ is required to admit the multiplication by polynomials in $x \in \mathbb{R}^n$ (restricted to U).

The jets $F \subset J^m(U)$ define the formal Taylor polynomials in $x \in \mathbb{R}^n$ (centered at $y \in U$)

$$T_j^{k-|j|} F(y, x) = T_{y,j}^{k-|j|} F(x) = \sum_{|l+j| \le k} f_{j+l}(y) \frac{(x-y)^l}{l!},$$

$$k \le m, \qquad l \ge 0, \qquad |l| \le k - |j|, \tag{1.2}$$

$$T^k F(y, x) = T_y^k F(x) = \sum_{|l| \le k} f_l(y) \frac{(x-y)^l}{l!}$$

(also called Taylor fields) and the formal Taylor remainders $R_j^{k-|j|} F(y, x)$ defined by the formulas

$$f_j(x) \equiv T_{y,j}^{k-|j|} F(x) + T_j^{k-|j|} F(y, x), \qquad R^k F(y, x) \equiv R_0^k F(y, x),$$

$$(y,\ x) \in U \times U, \qquad |j| \le m.$$

On the jet spaces $J^m(U)$ there are three basic natural operations:

(I) The formal jet differential

$$\mathbf{D}^i : J^m(U) \to J^{m-|i|}(U)$$

defined by the formula

$$\{\mathbf{D}^i F\}_k = f_{i+k} \quad \text{for} \quad |k| \le m.$$

Obviously $\mathbf{D}^i F \equiv 0$ for $|i| > m$.

(II) The reduction $\mathbf{C}^l$, $l \in \mathbb{N}$,

$$\mathbf{C}^l : J^m(U) \to J^{m-l}(U), \qquad l \le m,$$

$$(\mathbf{C}^l F)_k = f_k, \qquad |k| \le m - l.$$

(III) The restriction or trace operation. For a subset Σ of U the trace $\mathrm{Tr}_\Sigma :$ $J^m(U) \to J^m(\Sigma)$ is defined componentwise

$$\{F_\Sigma\}_k = \{f_k|_\Sigma\}.$$

The restriction operation is unconditionally meaningful and well understood for jet spaces with continuous coefficients or when Σ is an open subset of an open domain U in $\mathbb{R}^n$. For general Sobolev type jet spaces, e. g. VLC-spaces, defined below, the restriction operation is defined and understood only under very special conditions on the set Σ. Classically Σ should be a smooth submanifold, typically a hyperplane, of $U \subset \mathbb{R}^n$. In this context for Sobolev spaces the d-sets Σ of U are an important class of admissible subsets (see [11]).

It is convenient to distinguish between the polynomial (or space) variables $x \in \mathbb{R}^n$ and the field variables $y \in U$ of the Taylor fields $T_{y,j}^{k-|j|}F(x)$. Thus in the definition of m-jets in $J^m(U)$ the components f_j of the jet F are to be considered as (local) functions of the field variables.

The Taylor fields (1.2) $T^kF(y, x)$ considered as functions on the product $U \times \mathbb{R}^n$ are polynomials in $x \in \mathbb{R}^n$ with coefficients in $\mathcal{A}$. Moreover,

$$T_{y,j}^{k-|j|}F(x) = D_x^j T_y^k F(x), \qquad T_y^k F(x) \equiv T_{y,0}^k F(x),$$

where D_x^i are the standard differential operators acting on polynomials in $x \in \mathbb{R}^n$.

Though the Taylor remainders $R_j^{k-|j|}F(y,x)$ are defined on the product $U \times U$ as functions of the field variables x, y and are in the class $\mathcal{A}$ only, their difference

$$R_j^{k-|j|}F(y,x) - R_j^{k-|j|}F(z,x) \equiv T_{z,j}^{k-|j|}F(x) - T_{y,j}^{k-|j|}F(x)$$

is a polynomial in x for arbitrary y, $z \in U \times U$.

H. Whitney considered the case when $U = K$ is a compact subset K of $\mathbb{R}^n$ and $\mathcal{A}$ the ring of continuous functions on K, $\mathcal{A} = C(K)$. By definition an m-jet $F \subset J^m(K)$ is in $C^m(K)$ if

$$F = G|_K = D^m g\Big|_K \tag{1.3}$$

is the restriction to K of an m-jet $G = \{D^j g : |j| \le m\}$ of a function $g \in C^m(\mathbb{R}^n)$. Numerous natural and delicate questions related with the above definition and the inverse extension operators $\mathrm{Ext} : C^m(K) \to C^m(\mathbb{R}^n)$ have been discussed in detail in the vast literature of the subject [9 – 13].

A necessary and sufficient condition for (1.3) was formulated and proved by H. Whitney in the form of the famous Whitney's remainder condition $\mathrm{WRC}^m(K)$ on the behavior of the Taylor remainders $R_j^{m-|j|}F(y,x)$ near the diagonal $\Delta \subset K \times K$:

$$R_j^{m-|j|}F(y,x) = o(|x-y|^{m-|j|}), \qquad |j| \le m,$$

when $(y, x) \to (x_0, x_0)$ uniformly on K, $x_0 \in K$.

Actually it is enough to require

$$R^m F(y,x) = o(|x-y|^m), \qquad j = 0$$

(see [7, 14]).

In the present paper U will be mainly an open subset of $\mathbb{R}^n$, possibly $U = \mathbb{R}^n$ or a model (closed) cube $Q \subset \mathbb{R}^n$. Functions in the Sobolev space $W^{m,p}(Q)$, $p > 1$, will be described by their $(m-1)$-jets. The corresponding precise definition is formulated below as jet spaces VLC.

Remark. Most of our concepts and results are meaningful and hold when U is considered to be a regular subset of $\mathbb{R}^n$, e. g. bounded John domain or extension domain for Sobolev spaces.

The notation throughout this paper is either standard in the literature on Sobolev spaces or self-explanatory. As a general reference for the Sobolev spaces $W^{m,p}(G)$, $G \subset \mathbb{R}^n$, we propose W. P. Ziemer's monograph [15]. Since this paper is a continuation of [4, 6, 7, 16], our notation and the basic definitions are consistent with those in the quoted papers. For convenience we recall that for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, D^α is the classical or distributional differential operator, $|\alpha| = \Sigma_i \alpha_i$. For a locally integrable function g and a measurable subset $E \subset \mathbb{R}^n$, $|E|$ will be its Lebesgue measure and $⨍_E g = g_E = |E|^{-1} \int_E g$ the average value of the function g over E. $B(a, r)$ will denote the open ball in $\mathbb{R}^n$ of radius r centered at a. The letter Q will be used either for our fixed model open cube, where all our function spaces "live" or for a generic subcube of "small" diameter $\delta(Q)$ or sidelength $s(Q)$, containing all the points x, y, z of $\mathbb{R}^n$ which appear in the considered pointwise inequality. Usually $s(Q) \approx |x-y|$, $x, y, z \in Q$, $x \neq y \neq z$, $2Q \subset B(x, r)$, $r \approx 4|x-y|$, where $2Q$ is the notation for the cube concentric with Q and of the doubled diameter. Given two quantities A and B, we write $A \approx B$ if $C_1 A \leq B \leq C_2 A$ for some positive constants C_1 and C_2. In the case of the Lipschitz moduli $a_Q(x)$ or the local Hardy – Littlewood maximal functions $M_Q g(x)$, the subscript Q indicates the local character: $x \in Q$, $s(Q) \approx 2|x-y|$, of the pointwise estimates and the concepts used: local maximal functions, local Riesz potentials, local Lipschitz moduli, etc. All our pointwise estimates are essentially local, i.e., the important information, conveyed by the pointwise estimates of the Taylor remainders $R_j^k F(x, y)$, is relevant only when $|x-y| \to 0$ or $|x-y| < \delta$, i.e., in an open neighborhood of the diagonal $\{(x, x)\}$ of $Q \times Q$ for sufficiently small δ.

Finally we explicitly state that the constants in our formulas can change values in the same string of estimates. Writing $C(n, m, p)$ we emphasize that the constant depends on the indicated parameters only.

2. Fractional maximal function. The local fractional maximal function of a locally integrable function $f\colon \mathbb{R}^n \to [-\infty, \infty]$ is defined by

$$M_\alpha f(x) = M_{\alpha,R} f(x) = \sup_{0<r<R} r^\alpha ⨍_{B(x,r)} |f(y)| dy, \qquad 0 \leq \alpha \leq n.$$

For $\alpha = 0$ we obtain the local Hardy – Littlewood maximal function. The function $M_{\alpha,R} f$ is closely related to the Riesz potential $(R = \infty)$ and its localized form

$$I_{\alpha,R} f(x) = \int_{|x-y| \leq R} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

Generally the Riesz potential $I_\alpha f$ majorizes the maximal function $M_\alpha f$

$$M_\alpha f(x) \leq C_1 I_\alpha f(x),$$

but for $1 < p < \infty$

$$\| I_\alpha f \|_{L^p} \leq C_2 \| M_\alpha f \|_{L^p}$$

by a known result of Muckenhoupt – Wheeden [17] (with the corresponding inequalities for local versions). The constants C_1, C_2 are universal and depend on n, p only and possibly on R in some controlled way. For $0 < \alpha p < n$ the Hardy – Littlewood – Pólya theorem on fractional integration gives

$$\| I_\alpha f \|_{L^q} \leq C(n, p, \alpha) \| f \|_{L^p}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{2.1}$$

It follows that the local fractional operators $M_{\alpha,R}$ also increase the integrability exponent

$$M_{\alpha,R} : L^p_{\text{loc}}(Q) \to L^{p_\alpha}_{\text{loc}}(Q), \qquad p_\alpha = \frac{np}{n - \alpha p} > p. \tag{2.2}$$

The fractional maximal operator $M_{\alpha,R}$ is meaningful on arbitrary measure metric space (X, d, μ). If the Borel measure μ on X is an s-measure for some real $s > 0$ in the sense that

$$C_1 r^s \leq \mu(B(x, r)) \leq C_2 r^s, \tag{2.3}$$

then (2.2) holds if the L^p spaces are understood as $L^p(X, \mu)$. The real number s in (2.3) is then interpreted as the Hausdorff dimension, $\dim_H(X, d, \mu) = s$, of the regular measure metric space (X, d, μ). The mapping property (2.2) then holds with the Euclidean dimension n replaced by the Hausdorff dimension s satisfying the condition

$$\alpha p < s. \tag{2.4}$$

In this paper our main concern is naturally the case of Sobolev spaces in the Euclidean spaces $\mathbb{R}^n$: the explicit expressions for the remainders $R^m F(x, y)$ are meaningful only in this case. The above remarks on Sobolev spaces on measure metric spaces are included only to see the broader perspective of the special cases of the theory ($m = 1$, d-sets in $\mathbb{R}^n$, $d < n$ [11], etc.). In this context it is important to notice that the proof [17] of the inequality (2.2) is presented independently of the Hardy – Littlewood – Pólya – Sobolev inequalities (2.1) and thus the inequalities (2.2) antecede the inequalities (2.1).

The proof of the Hardy – Littlewood type result for the fractional operators $M_\alpha f$ for measure metric spaces satisfying the doubling condition

$$\mu(B(x, 2r)) \leq C \mu(B(x, r)) \tag{2.5}$$

should be presented in connection with this aspect of [17], rather than the traditional reference to the classical Hardy – Littlewood – Pólya theorem [2, 3].

The result of P. Hajłasz [5] is somehow more subtle since it does not use the full strength of the doubling condition (2.3) – (2.5).

The local fractional sharp maximal function of a locally integrable function f is defined by

$$f^{\#}_{\beta,R}(x) = \sup_{0<r<R} r^{-\beta} \fint_{B(x,r)} \left| f - f_{B(x,r)} \right| dx, \quad \beta \geq 0.$$

For $R = \infty$ we simply write $f^{\#}_{\beta}(x)$. The case $\beta = 0$ is the Feffermann – Stein sharp maximal function and instead of $f^{\#}_{0,R}(x)$ we write $f^{\#}_{R}(x)$ for the local version of the Feffermann – Stein operator.

It has been understood during the last few decades that the fractional maximal functions are a convenient way to describe various subtle properties of Sobolev function spaces. They are also intimately connected with the pointwise inequalities characterization of these spaces [6, 16, 18 – 21], etc.

Let us briefly recall the case of first order Sobolev spaces $W^{1,p}(U)$. A. P. Calderón characterized the $W^{1,p}(\mathbb{R}^n)$ spaces by the pointwise condition $f \in L^p$, and $f_1^{\#}(x) \in L^p(\mathbb{R}^n)$ [19]. They are also characterized by the pointwise inequality

$$|f(x) - f(y)| \le (a(x) + a(y))|x - y|, \quad x, y \in U,$$

with $a \in L^p(U)$, $p > 1$, and the closely related averaged Poincaré inequality [4, 22]

$$\fint_{B(x,r)} \left| f(x) - f_{B(x,r)} \right| dy \le Cr \fint_{B(x,r)} g(y)\,dy \tag{2.6}$$

with g estimated by

$$g(y) \le a(y) + M_Q a(y)$$

for some local maximal function $M_Q a$ of $a \in L^p$ ($g \in L^p$), Q — cube of diameter $Q \le 4|x-y|$, containing x and y.

Maximizing the both sides of (2.6) we obtain [18]

$$f_R^{\#}(x) \le CM_{1,R}\,g(x)$$

which implies, by (2.2), that $f_R^{\#}(x) \in L^{p_1}$ and consequently $M_R f$ and $f \in L^{p_1}$. This is the Sobolev imbedding of $W^{1,p}(U)$ into $L^{p_1^*}(U)$. This procedure is meaningful for the case when U is replaced by a measure metric space (X, d, μ) with the doubling measure μ, and the Sobolev imbedding holds with the geometric dimension n replaced by the Hausdorff dimension s (for $p < s$) [5] (see also the proof of Proposition 4.1 below).

The described properties of the fractional maximal operator $M_{1,R}$ for measure metric spaces lead then to an alternative proof of a weaker version of the Hajłasz theorem (Hajłasz in [5] does not use the full strength of the doubling condition). For us it will be important to understand the integrability properties of the function $b(x, y) = (a(x) + a(y))|x - y|$ on the product space $U \times U$. They are controlled by the iterated local maximal functions or iterated Steklov means

$$\fint_{B(x,r)} \left(\fint_{B(y,r)} b(x,y)\,dx \right) dy \le \fint_{B(x,r)} M_R^1 b(y)\,dy \le M_R^2(M_R^1 b)(x)$$

or (2.7)

$$\fint_{B(y,r)} \left(\fint_{B(x,r)} b(x,y)\,dy \right) dx \le \fint_{B(y,r)} M_R^1 b(x)\,dx \le M_R^2(M_R^1 b)(y),$$

where $M_R^1 b(y) = \sup\limits_{r<R} \fint_{B(y,r)} b(x,y)\,dx$ and $M_R^2 b(x) = \sup\limits_{r<R} \fint_{B(x,r)} b(x,y)\,dy$.

The representation $b(x, y) = (a(x) + a(y))|x - y|$ implies that each of the iterated $M_R^2 M_R^1$ maximal functions is controlled by either $M_R(M_{1,R}a)$ or $M_{1,R}(M_R a)$ evaluated at x and y, respectively (see also the proof of Proposition 4.2 below). Thus by (2.2) we have the following proposition.

Proposition 2.1. *The iterated Steklov means* (2.7) *of the function* $b(x, y) = (a(x) + a(y))|x - y|$ *with* $a \in L^p$ *are in* L^{p_1}.

3. The classes $\mathbf{VLC}(m,p,Q)$. The $(m-1)$-jet $F \in J^{m-1}(Q)$, $F = \{f_i : |i| \leq m-1\}$, $f_i \in L^p(Q)$ is said to be a $(m-1)$-*jet in* Q *with variable Lipschitz coefficient,* $F \in \mathrm{VLC}(m,p,Q)$ for short, if for some function $a_Q = a_Q(F) \in L^p(Q)$ the pointwise inequality holds

$$\left| R^{m-1} F(x,y) \right| \leq |x-y|^m [a_Q(x) + a_Q(y)], \quad x, y \in Q. \tag{3.1}$$

Intimately related with (3.1) is the a priori stronger condition requiring the inequalities

$$\left| R_i^{m-1-|i|} F(x,y) \right| \leq |x-y|^{m-|i|} [a_Q^i(x) + a_Q^i(y)], \quad a_Q^i \in L^p(Q), \tag{3.2}$$

to be satisfied for all $|i| \leq m-1$ $(R_0^k F \equiv R^k F)$.

Functional coefficients $a_Q(x)$, $a_Q^i(x)$ in (3.1) and (3.2) will be called the *Lipschitz moduli* of the jet F. The Lipschitz moduli in (3.1) and (3.2) are obviously not defined uniquely. The equivalence relation $a_Q \approx b_Q$ in the sense of the pointwise inequality $C_1 a_Q(x) \leq b_Q(x) \leq C_2 a_Q(x)$, C_1, C_2 — positive constants, preserves all essential information contained in (3.1) and (3.2).

The concepts of the Taylor algebra recalled above give a convenient tool to formulate the pointwise inequalities satisfied by the $(m-1)$-jet $F = \{f_\alpha \equiv D^\alpha f : |\alpha| \leq m-1\}$ of a function f in the Sobolev class $W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Indeed, the pointwise inequalities from [4] (Theorem 2) and [6] (formulas (3.5), (3.6)) may be stated as the following proposition.

Proposition 3.1. *If* $F = \{D^\alpha f : |\alpha| \leq m-1\}$, $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$, *then*

$$\left| R^{m-1} F(x,y) \right| \leq C|x-y|^m (a_Q(x) + a_Q(y)), \quad a_Q \equiv a_Q^0 \tag{3.3}$$

and

$$R_i^{m-1-|i|} F(x,y) \leq C|x-y|^{m-|i|} (a_Q^i(x) + a_Q^i(y)), \quad |i| \leq m-1. \tag{3.4}$$

The local Lipschitz moduli a_Q, a_Q^i in the right-hand side of (3.3) and (3.4) in view of [4, 6, 7, 16] can be chosen as dominated by the local Hardy – Littlewood maximal function $M_Q(|\nabla^m f|)(x)$ of the highest gradient $|\nabla^m f|$ of f. Also the constants C in (3.3) and (3.4) are universal and depend on n and m only. Q is a model cube, containing x and y of the $\operatorname{diam} Q \approx \delta$, $\delta \geq |x-y|$ (usually $\delta \leq 5|x-y|$).

The jet spaces $\mathrm{VLC}(m,p,Q)$ can be supplied with Banach space norms in a natural way. For instance, the $(m-1)$-jet space $\mathrm{VLC}(m,p,Q)$ can be given the norm

$$\|F\| = \max(\|f_i\|_{L^p(Q)}, |i| \leq m-1) + \inf \|a_Q\|_{L^p(Q)}$$

where the inf is taken over all admissible moduli in (3.3) $\Big($or $\sum_{|i| \leq k} \|a_Q^i\|_{L^p(Q)}$ for the space $\widetilde{\mathrm{VLC}}(m,p,Q)$ defined by the inequalities (3.2)$\Big)$.

For simplicity we use the same notation $\mathrm{VLC}(m,p,Q)$ for spaces of jets or spaces of functions generating the jets. The actual object is defined by the context. The arising normed spaces are complete Banach spaces.

The space $\mathrm{VLC}(0,p,Q)$ is identified with jet space $J^0(Q) \equiv L^p(Q)$.

The space $\mathrm{VLC}(1,p,Q)$ is the space $M^{1,p}(Q)$ of P. Hajłasz whose immediate natural generalizations to the case of measure metric spaces (X,d,μ) were studied by P. Hajłasz [5] in an original and profound way.

For $m = 1$ and the jet space $J^0(Q) = L^p(Q)$, $R^0 f(x, y) \equiv f(x) - f(y)$. Inequality (3.3) has the form

$$|f(x) - f(y)| \leq |x - y|\big(a(x) + a(y)\big), \quad a \in L^p(Q).$$

This makes sense for the general measure metric space (X, d, μ) as

$$|f(x) - f(y)| \leq d(x, y)[a(x) + a(y)], \quad a \in L^p(X, d\mu),$$

$$\|f\|_{(X,d,\mu)} = \|f\|_{L^p(X)} + \inf\|a\|_{L^p}$$

and produces the Banach space $M^{1,p}(X, d, \mu)$.

The jet spaces $\mathrm{VLC}(m, p, Q)$ as Banach spaces have been identified with the classical Sobolev spaces $W^{m,p}(Q)$, Q — a cube in $\mathbb{R}^n$. This is Theorem 9.1 in [16], which we recall here referring for the detailed proof to [16].

Theorem. *Let F be a $(m-1)$-jet, $F = \{f_\alpha : |\alpha| \leq m - 1\}$,*

$$F \in \mathrm{VLC}(m, p, Q), \quad p > 1, \quad m \geq 1.$$

Then in any cube $Q_0 \Subset Q$ there exists a function f in the Sobolev class $W^{m,p}(Q_0)$ such that the $(m-1)$-jet $J^{m-1} f \in J^{m-1}(Q_0)$ coincides with $F|_{Q_0}$ or

$$D^\alpha f\big|_{Q_0} = f_\alpha|_{Q_0} \quad \text{for} \quad |\alpha| \leq m - 1$$

and the extended m-jet, $\tilde{F} = \{D^\alpha f : |\alpha| \leq m\} \in J^m(Q_0)$, satisfies the inequality

$$\big\|\tilde{F}\big\|_{L^p(Q_0)} \leq C\|F\|_{\mathrm{VLC}(m,p,Q)}$$

or

$$\big\|D^\alpha \tilde{F}\big\|_{L^p(Q_0)} \leq C\|F\|_{\mathrm{VLC}(m,p,Q)} \quad \text{for} \quad |\alpha| = m.$$

Moreover, for any $\alpha + \beta$, $|\alpha + \beta| \leq m$, $|\alpha| \leq m$, $|\beta| \leq m$,

$$D^{\alpha+\beta} f = D^\alpha(D^\beta f), \qquad f_{\alpha+\beta} = D^\alpha f_\beta, \qquad |\beta| \leq m - 1, \tag{3.5}$$

with the differential operators D^α, D^β understood in the generalized Sobolev, or distributional, sense.

Remark. The component f_α, $|\alpha| = m$, of the extended m-jet $\tilde{F}$ is obtained also from f_α, $|\alpha| = m - 1$, by approximate differentiation [7, 8].

***Corollary* 3.1.** *The reduced jets $\mathbf{C}^1(F)$, $F \subset \mathrm{VLC}(m, p, Q)$ are in $\mathrm{VLC}(m - 1, p_1, Q)$ where p_1 is the Sobolev conjugate of p, $1/p_1 = 1/p - 1/n$.*

This is the simplest of the Sobolev imbedding theorems for the spaces $W^{m,p}(Q)$, for the case $mp < n$.

However, our general idea in this paper is to study the properties of the jet spaces VLC independently of the identification with the classical Sobolev spaces $W^{m,p}(Q)$.

We state now and prove the two main theorems of this paper — Theorems 3.1 and 3.2.

Theorem 3.1. *The formal differential operator*

$$\mathbf{D}^i : J^m(U) \to J^{m-|i|}(U)$$

acts as a bounded operator

$$\mathbf{D}^i : \mathrm{VLC}(m, p, Q) \to \mathrm{VLC}(m - |i|, p, Q).$$

Proof. We have

$$R^{m-|i|-1}(\mathbf{D}^i F) \equiv R_i^{m-|i|-1} F$$

with $F \in \mathrm{VLC}(m, p, Q)$ satisfying the pointwise inequality (3.1). We need to assess that the inequality (3.2) holds. This we formulate as Lemma 3.1 below, which is the general case of the main Lemma 8.1 in [16]. We enclose the proof from [16], because the formulation here is simpler and explicitly drops the unnecessary assumption of quasisuperharmonicity.

Lemma 3.1. *If*

$$\begin{gathered} \left| R^{m-1} F(x, y) \right| \le |x-y|^m [a_Q(x) + a_Q(y)], \quad x, \ y \in Q, \\ \operatorname{diam} Q = \delta, \qquad \delta \approx 2|x-y|, \end{gathered} \tag{3.6}$$

then

$$\begin{gathered} \left| R_i^{m-1-|i|} F(y, z) \right| \le |z-y|^{m-|i|} [a_{Q'}^i(y) + a_{Q'}^i(z)], \quad y, \ z \in Q', \\ \delta' = \operatorname{diam} Q', \qquad Q' \subset Q, \qquad \delta' \approx 2|y-z| \approx \frac{1}{2}\delta, \end{gathered}$$

with $a_{Q'}^i \approx a_Q$ *or*

$$a_{Q'}^i(y) \le C_i(n, m) [a_Q(y) + M_{Q'}(a_Q)(y)], \quad y \in Q',$$

for some constant C_i *depending on* n *and* m *only.*

Proof. From the Taylor algebra

$$R_i^{m-1-|i|} F(y, z) = D_x^i [R^{m-1} F(x, y) - R^{m-1} F(x, z)]_{x=y} \equiv D_x^i P(x; y, z)_{x=y} \tag{3.7}$$

with

$$P(x; y, z) \equiv R^{m-1} F(x, y) - R^{m-1} F(x, z) \tag{3.8}$$

which is a polynomial in x of order $\le m-1$, with coefficients obtained by restriction to Q, of the coefficients of the local Whitney jet F. These are functions from $L^1(Q_{\delta'})$. For y, z fixed, $y \ne z$, consider the balls $B(y, r)$ and $B(z, r)$, $r = |y-z|$ and the spherical segment

$$S \equiv S_r = B(y, r) \cap B(z, r), \qquad S_r = S_r(y, z).$$

By elementary geometry

$$|B(z, r)| = |B(y, r)| \le \sigma |S_r|$$

with $1 < \sigma < \sigma(n)$ independent of r.

By Markov's inequality, [7, 8, 16, 23], applied to the subset S of the ball $B(y, r)$, $r = |y-z|$, we obtain

$$\left| D_x^i P(x; y, z) \right|_{x=y} \le \frac{C(n)}{r^{|i|}} \fint_S |P(x'; y, z)| dx'. \tag{3.9}$$

The integrand $|P(x'; y, z)|$ for $x' \in S$ is estimated in view of (3.8) and (3.6) as

$$\begin{gathered} |P(x'; y, z)| \le \left| R^{m-1} F(x', y) \right| + \left| R^{m-1} F(x', z) \right| \le \\ \le [a_{Q'}(x') + a_{Q'}(y)] |x'-y|^m + [a_{Q'}(x') + a_{Q'}(z)] |x'-z|^m. \end{gathered}$$

For $x' \in S_r$ we have

$$|x'-y| \le |y-z| \quad \text{and} \quad |x'-z| \le |y-z|$$

and the integral in the right-hand side of (3.9) splits into the sum $I_1(r) + I_2(r)$ with

$$I_1(r) \equiv \frac{C}{r^{|i|}} r^m \fint_S [a_{Q'}(x') + a_{Q'}(y)]dx',$$

$$I_2(r) \equiv \frac{C}{r^{|i|}} r^m \fint_S [a_{Q'}(x') + a_{Q'}(z)]dx'.$$

In $I_1(r)$ we consider S as a subset of the ball $B(z, r)$ and in $I_2(r)$ we consider it as a subset of $B(y, r)$ and we get

$$I_1(r) \le Cr^{m-|i|}(C_1 M_Q(a_{Q'})(z) + a_{Q'}(y)) \le CC_1 r^{m-|i|}[M_Q(a_{Q'})(z) + a_{Q'}(y)] \tag{3.10}$$

and

$$I_2(r) \le CC_2 r^{m-|i|}[C_1 a_{Q'}(z) + M_Q a_{Q'}(y)]$$

this ends the proof of the lemma with the explicit estimates for the constants $C_i(n, m) = \max(CC_1, CC_2)$

Theorem 3.2. *The reduction operators* $\mathbf{C}^l$ *act as a bounded operators* $(pm < n)$

$$\mathbf{C}^l : \mathrm{VLC}(m, p, Q) \to \mathrm{VLC}(m - l, p_l, Q)$$

with p_l *defined by the Sobolev relation*

$$\frac{1}{p_l} = \frac{1}{p} - \frac{l}{n} \quad \textit{for} \quad l = 1, \ldots, m.$$

For $m = 1$ Theorem 3.2 reduces to Theorem 3.3 or the case of the spaces $W^{1,p}(Q) = M^{1,p}(Q)$ treated by P. Hajłasz [5].

It is convenient for us to formulate the case $m = 1$ in the form of a separate theorem, which we state here in the general Hajłasz form for general measure metric spaces $M^{1,p}(X, d, \mu)$.

Theorem 3.3 (P. Hajłasz [5]). *If* μ *is a doubling measure with the doubling constant* C, *and* $s = \dim_H(X, d, \mu)$ *the Hausdorff dimension* $(C = 2^s)$, *then*

$$M^{1,p}(X) \hookrightarrow L^{p^*}(X), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{s}, \quad p < s,$$

and

$$M^{1,p}(X) \subset \mathrm{Lip}(\alpha, X), \qquad \alpha = 1 - \frac{s}{p}, \quad p > s.$$

Proof. [5] or Proposition 4.2 below.

***Proof of Theorem* 3.2.** Let $F = \{f_0, \ldots, f_{m-1}\}$ be in $\mathrm{VLC}(m, p, Q)$. Then $D^{m-1}F = f_{m-1}$ and by Theorem 3.1 f_{m-1} is a (vector valued) function in $\mathrm{VLC}(1, p, Q)$. By the quoted Theorem 3.3,

$$\| f_{m-1} \|_{L^{p_1}} \le C \| F \|_{\mathrm{VLC}(m,p,Q)}.$$

We have

$$R^{m-1}F \equiv R^{m-2}(\mathbf{C}^1 F) + f_{m-1}(y)\frac{(x-y)^{m-1}}{(m-1)!}$$

and we see that the $(m-2)$-jet $\mathbf{C}^1 F$ satisfies the pointwise inequality

$$R^{m-2}(\mathbf{C}^1 F)(x, y) \le C|x-y|^{m-1}\Big(|f_{m-1}(y)| + \big(a_Q(F)(x) + a_Q(F)(y)\big)|x-y|\Big) \tag{3.11}$$

with $|f_{m-1}(y)| \in L^{p_1}(Q)$, $a_Q(F) \in L^p(Q)$.

Symmetrizing the right-hand side of (3.11) by adding the term $|f_{m-1}(x)|$ and applying the iterated maximal function operation as in Proposition 2.1 and Lemma 3.1 we deduce from (3.11) the pointwise inequality

$$R^{m-2}(\mathbf{C}^1 F)(x, y) \le C|x-y|^{m-1}\big[b_Q(x) + b_Q(y)\big]$$

with $b_Q(x)$ controlled by

$$b_Q(x) \le C\Big(|f_{m-1}(x)| + M\big(M_1(a_Q(F))(x)\big)\Big) \in L^{p_1}(Q).$$

Since $\mathbf{C}^l$ may be represented as a composition of $\mathbf{C}^1$, $\mathbf{C}^l = (\mathbf{C}^1)^l$ and the formulas $1/p_k = 1/p_{k-1} - 1/n$, $k = 1, \dots, l$, also respect composition, we see that the described procedure ends the proof.

4. The averaging of the pointwise inequalities. Averaging of the pointwise inequalities as a method of generating new inequalities out of known ones has been systematically applied probably for the first time in [16], see also [8]. The proof of Lemma 8.1 in [16], in some slightly modified form reproduced also above in the proof of our Lemma 3.1, gives a general scheme of the averaging process and is in principle applicable to the pointwise inequalities in function spaces (X, d, μ) satisfying the doubling condition for the measure μ.

In this context it seems proper now to formulate a following general proposition for the simplest case of the pointwise inequalities characterizing the Sobolev space $W^{1,p}(Q)$, $p > 1$, or Hajłasz – Sobolev spaces $M^{1,p}(X, d, \mu)$.

Proposition 4.1. *Let a real valued function f satisfy the pointwise estimate*

$$|f(x) - f(y)| \le$$
$$\le |x-y|\Big[a(x) + a(y) + |x-y|\big(a_1(x) + a_1(y)\big) + |x-y|^2\big(a_2(x) + a_2(y)\big) + \dots\Big] \tag{4.1}$$

for some finite sequence of nonnegative valued functions

$$a(x) \equiv a_0(x), \qquad a_1, a_2, \dots, a_i \in L^{p_{-i}}, \quad \frac{1}{p_{-i}} = \frac{1}{p} + \frac{i}{n}, \quad i = 1, 2, \dots,$$

then

$$|f(x) - f(y)| \le C|x-y|[\tilde{a}(x) + \tilde{a}(y)] \tag{4.2}$$

with

$$\tilde{a}(x) \le C\big(M_0 a(x) + M_0 M_1(a_1)(x) + \dots + M_0 M_i(a_i)(x) + \dots\big), \tag{4.3}$$

where M_i is the local fractional maximal function of order i, $M_0 f \equiv M_Q f$.

Proof. Integrating inequality (4.1) over a ball $B(x_0, r)$ with respect to x and y we obtain

$$⨍_B |f(y) - f_B|\,dy \le Cr\left(⨍_B a\,dx + r⨍_B a_1\,dx + \dots + r^{k-1}⨍_B a_{k-1}\,dx\right). \tag{4.4}$$

Hence, for $r \le R_Q$ we conclude for some $R > 0$ the inequality

$$⨍_B |f(y) - f_B|\,dy \le f^{\#}_{R_Q}(x) \le$$
$$\le C\big(M_{1,R_Q} a + M_{2,R_Q} a_1 + \dots + M_{k,R_Q} a_{k-1}\big) \tag{4.5}$$

for the local fractional maximal functions $M_{i,R}a_{i-1}(x)$ and the local sharp maximal function $f_R^{\#}$ and the inequality for the local fractional sharp maximal function $f_{1,R}^{\#}$

$$\frac{1}{r}\fint_{B(x,r)}|f(y)-f_B|dy \;\le\; f_{1,R}^{\#}(x) \;\le\; C\big(M_R a(x)+M_{1,R}a_1+\ldots+M_{k-1,R}a_{k-1}\big). \quad (4.6)$$

In view of (2.2) all terms in the right-hand side of (4.5) and (4.6) are in $L_{\text{loc}}^{p_1}$ and L_{loc}^{p} respectively.

Thus (4.5) implies $f_R^{\#} \in L_{\text{loc}}^{p_1}$, hence $f \in L^{p_1}$ — which is one of the conclusions of the Hajłasz theorem for generalized $M^{1,p}$ pointwise inequalities (4.1). In an analogous way (4.6) implies $f_{1,R}^{\#}(x) \in L_{\text{loc}}^{p}$ and we recognize the A. Calderón [19] condition characterizing the Sobolev classes $W^{1,p}(Q)$.

Application of the telescoping argument at the Lebesgue points of the function f to the generalized Poincaré inequality (4.4), probably first used by L. Hedberg in [24], allows to recover the pointwise inequality (4.2) in the same way as in the proof of Theorem 3.2 [22] (see also [25]). This finishes the proof of the proposition.

By the imbedding theorem (2.2) for fractional maximal functions all terms in the right-hand side of (4.3) are in the space $L^p(Q)$.

Closely related with Proposition 4.1 is following proposition.

Proposition 4.2. *Assume that a function $f \in L^p(Q)$ satisfies instead of* (4.2) *the inequality*

$$|f(x)-f(y)| \;\le\; b(x,y)|x-y|, \quad x,y \in Q.$$

Then for a small concentric cube $Q_0 \subset Q$ (e.g. $Q \subset 4Q_0$ where $4Q_0$ is the concentric 4-*times extended sube Q_0*),

$$|f(x)-f(y)| \;\le\; Cr\left(\fint_{B(x,r)} b(x,z)dz \;+\; \fint_{B(y,r)} b(z,y)dz\right)$$

with some constant $C = C(n)$.

Proof. Let $x, y \in Q_0$ and $r = |x-y|$. Then the balls $B(x,\ r)$ and $B(y,\ r)$ are both contained in Q and so is the spherical segment $S = B(x,\ r)\bigcap B(y,\ r)$. The obvious inequality

$$|f(x)-f(y)| \;\le\; |f(x)-f(z)|+|f(z)-f(y)| \quad (4.7)$$

averaged over $z \in S$ gives

$$|f(x)-f(y)| \;\le\; \fint_S (b(x,z)|x-z| + b(z,y)|z-y|)dz \;\le$$

$$\le\; Cr\left[\fint_{B(x,r)} b(x,z)dz \;+\; \fint_{B(y,r)} b(z,y)dz\right] \quad (4.8)$$

for some constant C depending on n only.

For $b(x,\ y)$ symmetric: $b(x,\ y) = b(y,\ x)$, what we can always assume, (4.8) has the “splitted form”, for $b(x,\ y)$ “splitted”, $b(x,\ y) = a(x)+a(y)$, the Steklov means $\fint_{B(x,r)} b(x,z)dz$ have the form $a(x)+M_Q a$ and we recognize the terms in the right-hand side of (3.10) in our proof of Lemma 3.1. If $b(x,\ y)$ does not appear in a symmetric form, say $b(x,\ y) = a(x)+a(y)+c(x)|x-y|$, the second integral in (4.8) will produce the term of the form $M_1c(z)$ of a fractional maximal function of the coefficient $c(x)$. The described averaging process can be "iterated" and applied to the pointwise inequalities (4.8). It leads then to the iterated fractional maximal functions appearing in (2.7) in the sketch of the proof of Proposition 2.1.

Proposition 4.2 and its proof generalize to the higher order Taylor remainder estimates

$$\left| R^{m-1} F(y, x) \right| \leq b(x, y) |x - y|^m .$$

Instead of the obvious inequality (4.7) the proof requires somewhat more sophisticated formula (3.7) and the Markov inequality (3.9) for $|i| = 0$.

The pointwise inequality (4.2) characterizes the behavior of the zero-order remainder term $R^0 f(x, y) \equiv f(x) - f(y)$ in the neighborhood of the diagonal Δ of the cartesian product $Q \times Q$. Inequality (4.1) can be viewed as a more subtle description of the asymptotic of $R^0 f(x, y)$ in the neighborhood of the diagonal in terms of powers of the distance $|x - y|$ of the point $(x, y) \in Q \times Q$ from the diagonal Δ, $\operatorname{dist}((x, y), \Delta) \sim$ $\sim |x - y|$.

Proposition 4.1 states that the a priori more general asymptotics (4.1) can be always reduced to the "simplest" case (4.2).

Actually the averaging process described in Proposition 4.1 can be continued one step further reducing (4.2) to the pointwise inequality

$$|f(x) - f(y)| \leq C|x - y|^0 (\tilde{\tilde{a}}(x) + \tilde{\tilde{a}}(y)) \equiv C(\tilde{\tilde{a}}(x) + \tilde{\tilde{a}}(y))$$

with the estimate

$$\tilde{\tilde{a}}(x) \leq C\big(M(M_1(a))(x) + M(M_2(a_1))(x) + \ldots + M(M_{i+1}(a_i))(x)\big)$$

with all the terms locally in $L^{p_1}(Q)$.

Thus the extended averaging process gives essentially the Sobolev and Sobolev – Hajłasz embedding theorem for $p < n$ ($p < s = \dim_H(X, d, \mu)$ — for measure metric spaces).

Analogous asymptotic $S^m \mathrm{RC}$ conditions can be formulated for $m > 1$ and Proposition 4.1 generalizes to this context also.

When complemented by the recent extension results of Hajłasz – Koskela – Tuominen [26] or P. Shvartsman [27], the poinwise theory approach to Sobolev spaces $W^{m,p}(Q)$, $p > 1$, as presented in [5, 7, 16] and outlined above for Sobolev jet spaces $\mathrm{VLC}(m, Q, p)$ extends to regular (n-Ahlfors regular) open subsets of $\mathbb{R}^n$. In this way the Sobolev jet space theory seems to be brought to a satisfactory shape, according to the expectations expressed in the final comments of §10 of [16]. It can be expected that the averaging process described above when applied with respect to a d-measure μ supported by a d-set $\Sigma \subset Q$ ($\Sigma \subset \mathbb{R}^n$) [11, 28] will be useful for fully understanding from the pointwise point of view the case of fractional Sobolev – Besov spaces on n-Ahlfors regular subsets of $\mathbb{R}^n$ (see also [21], wich almost fulfils the task) and the case of Besov spaces or traces of Sobolev spaces on d-subsets Σ of $\mathbb{R}^n$.

This problem will be, hopefully, the subject of a subsequent publication in cooperation with P. Hajłasz and P. Strzelecki.

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