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L^2-INVARIANTS AND MORSE – SMALE FLOWS ON MANIFOLDS

L^2-ІНВАРІАНТИ ТА ПОТОКИ МОРСА – СМЕЙЛА НА МНОГОВИДАХ

We study the homotopy invariants of free cochain and Hilbert complexes. These L^2-invariants are applied to the calculations of exact values of minimal numbers of closed orbits of some indexes of nonsingular Morse – Smale flows on manifolds of large dimensions.

Вивчаються гомотопічні інваріанти вільних коланцюгових та гільбертових комплексів. Ці L^2-інваріанти застосовуються при обчисленні точних значень мінімальних чисел замкнених орбіт фіксованих індексів несингулярних потоків Морса – Смейла на многовидах великих розмірностей.

1. Introduction. Let M^n be a closed smooth manifold. By a *nonsingular Morse – Smale* flow on M^n we shall mean a flow φ_t satisfying the following conditions:

1) chain-recurrent set R of φ_t consists of finitely many hyperbolic closed orbit;

2) for each pair of closed orbits of φ_t the intersection of their stable and unstable manifolds is transversal;

3) all closed orbits of φ_t are untwisted.

Notice that usually by a nonsingular Morse – Smale flow one means a flow satisfying the conditions 1) and 2) only.

Let φ_t be a nonsingular Morse – Smale flow on M^n. Denote by A_i, $i = 0, \ldots, n$, the number of closed orbits of φ_t of index i. Let also $R_i = \dim H_i(M^n; \mathbb{Q})$. Then the following inequalities hold true:

$$A_i \geq R_i - R_{i-1} + \ldots + R_0 \tag{1}$$

for all $i = 0, \ldots, n$, see [1 – 3]. Notice that they are not strict in general.

In this paper we study the following problem:

Problem. *For a manifold M^n and $i = 0, \ldots, n$ find a nonsingular Morse – Smale flow φ_t on M^n with minimal possible value A_i of* (*untwisted*!) *closed orbits of index i.*

Using numerical invariants of free cochain and Hilbert complexes of manifold M^n, see [3, 4], we give an answer to this problem for $i = 0, 1, n-2, n-1$ and $3 \leq i \leq n-4$ when $\dim M^n \geq 6$. Thus a unique unsettled case is $i = 2$ (and $n-3$ by duality).

By definition the *i-th Morse S^1-number* $\mathcal{M}_i^{S^1}(M^n)$ of a manifold M^n is the minimal number of closed orbits of index i taken over all nonsingular Morse – Smale flows on manifold M^n.

It is convenient to define the following function ρ: $\mathbb{Z} \to \mathbb{N}$ by $\rho(x) = x$ for $x \geq 0$ and $\rho(x) = 0$ for $x < 0$.

Let M^n, $n \geq 6$, be a closed manifold with zero Euler characteristic and with $\pi_1(M^n) = \pi$. Then the Morse S^1-numbers of the manifold M^n are given by the following formulas:

$$\mathcal{M}_0^{S^1}(M^n) = \mathcal{M}_{n-1}^{S^1}(M^n) = 1,$$
$$\mathcal{M}_1^{S^1}(M^n) = \mathcal{M}_{n-2}^{S^1}(M^n) = \mu(\pi) - 1,$$
$$\mathcal{M}_i^{S^1}(M^n) = \widehat{S}_{(2)}^{i+1}(M^n) + \rho\left[(-1)^i \sum_{j=0}^{i} (-1)^j \dim_{N[G]}\left(H_{(2)}^j(M^n)\right)\right]$$

for $3 \leq i \leq n-4$, where $\mu(\pi)$ is the minimal number of generators of π.

2. Stable invariants of finitely generated modules and L^2-modules. We give several definitions and results about finitely generated modules over group rings. Most of the facts are also valid for modules over a wider class of rings.

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{C}$ the field of complex numbers. Let G be a discrete countable group. Denote its integer and complex group rings by $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ respectively. Each group ring admits an augmentation epimorphism $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ ($\varepsilon\colon \mathbb{C}[G] \to \mathbb{C}$) defined by $\varepsilon\left(\sum_i \alpha_i g_i\right) = \Sigma_i \alpha_i$. Denote by $\mathbb{I}[G]$ the kernel of the epimorphism ε. The ring $\mathbb{C}[G]$ has also an involution $*\colon \mathbb{C}[G] \to \mathbb{C}[G]$ given by $\left(\sum_i \alpha_i g_i\right)^* = \sum_i \overline{\alpha}_i \overline{g}_i^{-1}$, where $\overline{\alpha}$ is the conjugate to $\alpha \in \mathbb{C}$. Define the trace tr: $\mathbb{C}[G] \to \mathbb{C}$ by tr $\left(\sum_i^k \alpha_i g_i\right) = \alpha_1$, where α_1 is the coefficient at $g_1 = e$, the unit of the group G.

The ring $\mathbb{C}[G]$ has also an inner product $\left\langle \sum_i \alpha_i g_i, \sum_i \beta_i g_i \right\rangle = \sum_i \alpha_i \overline{\beta}_i$. Then for each $r \in \mathbb{C}[G]$ its norm $|r|$ can be defined by $|r| = \mathrm{tr}(rr^*)^{1/2}$. Let $L^2(G)$ be the completion of $\mathbb{C}[G]$ with respect to this norm. Then $L^2(G)$ has a structure of a Hilbert space (with inner product given by the same formula as for the group ring $\mathbb{C}[G]$) and elements of G constitute its orthonormal basis. Notice that $\mathbb{C}[G]$ acts faithfully and continuously by left multiplication on $L^2(G)$

$$\mathbb{C}[G] \times L^2(G) \to L^2(G),$$

therefore we can regard $\mathbb{C}[G]$ as a subset of the set $\mathbf{B}(L^2(G))$ of bounded linear operators on $L^2(G)$. A a week closure of $\mathbb{C}[G]$ in $\mathbf{B}(L^2(G))$ is called the *von Neumann algebra of* G and denoted by $N[G]$. The map $N[G] \to L^2(G)$ given by $w \to w(e)$ turns out to be injective and this allows us to identify $N[G]$ with a subspace of $L^2(G)$.

Thus algebraically we have $\mathbb{C}[G] \subset N[G] \subset L^2(G)$. The involution and the trace map on $\mathbb{C}[G]$ extends to $N[G]$ by the same formulas. Moreover, the trace map can also be extended to the space $M_n(N[G])$ of $(n \times n)$-matrices over von Neumann algebra $N[G]$ by $\mathrm{tr}(W) = \sum_{i=1}^n w_{ii}$, where $W = (w_{ij})$ is a matrix with entries in $N[G]$.

Following Cohen [5] we will now define a notion of Hilbert $N[G]$-module. Let $E = \mathbb{N} \bigcup \infty$, where ∞ is the first infinite cardinal. For each $n \in E$ let $L^2(G)^n$ be the Hilbert direct sum of n copies of $L^2(G)$. Thus $L^2(G)^n$ is a Hilbert space. The von Neumann algebra $N[G]$ acts on $L^2(G)^n$ from the left, whence $L^2(G)^n$ is a left $N[G]$-module called a *free $L^2(G)$-module of range n.*

The left Hilbert $N[G]$-module M is a closed left $\mathbb{C}[G]$-submodule of $L^2(G)^n$ for some $n \in E$. If $n \in \mathbb{N}$, then Hilbert $N[G]$-module M is called *finitely generated.*

Following [5, 6] we will say that a *Hilbert $N[G]$-submodule* of M is a closed left $\mathbb{C}[G]$-submodule of M, a *Hilbert $N[G]$-ideal* is a Hilbert $N[G]$-submodule of $L^2(G)$, and a *Hilbert $N[G]$-homomorphism* $f\colon M \to N$ between Hilbert $N[G]$-modules is a continuous left $\mathbb{C}[G]$-map.

Let M be a Hilbert $N[G]$-module and let $p\colon L^2(G)^n \to L^2(G)^n$ be a orthogonal projection onto $M \subset L^2(G)^n$. Then the number

$$\dim_{N[G]}(M) = \mathrm{tr}(p) = \sum_{i=1}^n \langle p(e_i), e_i \rangle_{L^2(G)^n}$$

is called a *von Neumann dimension* of M, where $e_i = (0, \dots, 1(g), \dots, 0)$ is a standard basis in $L^2(G)^n$. It is known that $\dim_{N[G]}(V)$ is a nonnegative real number [6].

In what follows we will assume, unless otherwise stated, that Λ is an associative ring with unit e and M is a left finitely generated Λ-module. Rings for which the rank of a free module is uniquely defined are called IBN-rings. It is known that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are IBN-rings. In the present paper, we consider only IBN-rings. For a module M let $\mu(M)$ be the minimal number of its generators. If M is zero, then $\mu(M) = 0$. Evidently, $\mu(M \bigoplus F_n) \leq \mu(M) + n$, where F_n is a free module of rank n. There are examples (of stably-free modules) when this inequality is strict [3]. Recall that a Λ-module M is called *stably-free* if the direct sum of M with some free Λ-module F_k is free.

A ring Λ is said to be *Dedekind-finite* if, for any λ_1, $\lambda_2 \in \Lambda$, relation $\lambda_1 \cdot \lambda_2 = 1$ implies $\lambda_2 \cdot \lambda_1 = 1$. A ring Λ is *stably-finite* if the matrix rigs $M_n(\Lambda)$ are Dedekind-finite for all $n \in \mathbb{N}$. The terminology here follows the usage of workers in operator algebras.

Definition 1. *Let* d *be a function from the category of* Λ*-modules* M (*not necessarily over group rings*) *to the set of nonnegative integers* $\mathbb{N}_0$. *We say that this function* d *is weak additive if the following conditions holds true*:

a) $d(M) = d(N)$ *if modules* M *and* N *are isomorphic*;

b) $d(M) = 0$ *if and only if* $M = 0$;

c) $d(M \bigoplus F_n) = d(M) + n$ *for any free module* F_n *of rank* $n \in \mathbb{N}$.

Definition 2. *For a finite generated module* M *over* IBN*-ring* Λ *let us define the following function*:

$$\mu_s(M) = \lim_{n \to \infty} (\mu(M \oplus F_n) - n).$$

Lemma 1. *The function* $\mu_s(M)$ *is well defined and is weak additive for modules over stably-finite rings.*

Proof. Condition a) is obvious. Let us prove b). Suppose that $\mu_s(M) = 0$ for some non-zero module M. Then there exists $n \in \mathbb{N}$ such that for the module $N = M \bigoplus F_n$ we have $\mu(N) = n$. Therefore, there is an epimorphism f: $F_n \to N$ of a free module F_n of rank n onto the module N. In addition, there exists a canonical epimorphism p: $N = M \bigoplus F_n \to F_n$ with the kernel equal to M. Let K be the kernel of the epimorphism $p \circ f$: $F_n \to F_n$. It follows from the construction of f and p that $K \neq 0$. Moreover, $p \circ f$ is an epimorphism onto a free module, therefore it splits, whence $K \oplus F_n = F_n$. Since Λ is stably-finite, we obtain that $K = 0$. The condition c) is proved in [7].

***Corollary* 1.** *The function* $\mu_s(M)$ *is week additive for modules over the rings* $\mathbb{Z}[G]$ *and* $\mathbb{C}[G]$.

Proof. It follows from theorems of Kaplansky and Cockroft [3] that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are hopfian.

***Remark* 1.** It is clear, that for any non-zero module M we have that

$$0 < \mu_s(M) \leqslant \mu(M).$$

The difference

$$\mu(M) - \mu_s(M)$$

estimates how much times the addition a free module of rank one to the modules $M \bigoplus k\Lambda$, $k = 0, 1, \dots$, does not increase by one the number $\mu(M \bigoplus k\Lambda)$. There are also inequalities

$$\mu(M \bigoplus N) \leqslant \mu(M) + \mu(N),$$

$$\mu_s(M \bigoplus N) \leqslant \mu_s(M) + \mu_s(N).$$

It is not hard to construct examples of projective modules in which strict inequalities hold true.

Lemma 2. *For every finitely generated module M over IBN-ring Λ there exists $n \in \mathbb{N}$ such that for the module $N = M \bigoplus n\Lambda$ and for all $m \geq 0$ we have that*

$$\mu(N \bigoplus m\Lambda) = \mu(N) + m.$$

Moreover, $\mu(N)$ is additive for the module N.

Proof. An existence of such a number n is proved in [3]. It is clear, that if for a module N we have

$$\mu(N \bigoplus m\Lambda) = \mu(N) + m,$$

then $\mu(N) = d(N)$ by the virtue of the definition of the function $d(N)$.

Let N be a submodule of the free module F_k. Following H. Bass we define *f*-rank of the pair (N, F_k) to be the largest nonnegative integer r such that N contains a direct summand of F_k isomorphic to free module F_r. We shall denote this number by *f*-rank(N, F_k).

By definition *f*-rank of (N, F_k) is called *additive* if

$$f\text{-rank}(N \oplus F_m, F_k \oplus F_m) = f\text{-rank}(N, F_k) + m.$$

We note that for any submodule N of the free module F_k there exist a positive integer m_0 such that *f*-rank of $(N \oplus F_m, F_k \oplus F_m)$ is additive for all $m > m_0$, see [3] (Lemma III.7).

3. Homotopy invariants of cochain complexes. It is known that the homology (cohomology) of a free chain (cochain) complex over the ring of integers determines its homotopy type. But for a free chain (cochain) complex over arbitrary rings this is not the case, one should require the existence of a chain (cochain) map that induces homology (cohomology) isomorphisms.

Definition 3. *Let $(C_*, d_*) : C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_n$ be a free chain complex. Then the following chain complex:*

$$\big(C_*(i), d_*(i)\big)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xrightarrow{d_i} C_i$$

is called the i-th skeleton of the chain complex (C_*, d_*).

It is well known that the Euler characteristic $\chi(C_*, d_*) = \sum(-1)^i\mu(C_i)$ is an invariant of the homotopy type of the chain complex (C_*, d_*). But in general the i-th Euler characteristics of homotopy equivalent chain complexes (C_*, d_*) and (D_*, ∂_*) may differ each from other.

Definition 4. *Let $(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_n$ be a free chain complex and*

$$\chi_i(C_*, d_*) = (-1)^i\chi\big(C_*(i), d_*(i)\big).$$

The following number:

$$\chi_i^a(C_*, d_*) = \min\Big\{\chi_i(D_*, \partial_*) \mid (D_*, \partial_*) \text{ is homotopy equivalent to } (C_*, d_*)\Big\}$$

will be called the i-th Euler characteristics of the chain complex (C_*, d_*).

For cochain complexes the definitions are similar.

Theorem 1. *Let* $(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_n$ *be a free chain complex. Then* $\chi_i^a(C_*, d_*) = \chi_i(C_*, d_*)$ *if and only if* *f*-rank *of* $(d_{i+1}(C_{i+1}), C_i)$ *is additive and is equal to zero*:

$$f\text{-rank}(d_{i+1}(C_{i+1}), C_i) = 0.$$

Proof. *Necessity.* Suppose that $\chi_i^a(C_*, d_*) = \chi_i(C_*, d_*)$ but

$$f\text{-rank}(d_{i+1}(C_{i+1}), C_i) = r > 0.$$

The the module C_i can be represented in the form $C_i = \tilde{C}_i \oplus F_r$. Therefore stabilizing the boundary homomorphisms d_i and d_{i+2} via the free module F_r we can assume that the submodule $\tilde{C}_i$ is free and there is a decomposition

$$C_{i+1} \oplus F_r = \tilde{C}_{i+1} \oplus \tilde{F}_r$$

such that $d_{i+1}(0 \oplus \tilde{F}_r) = 0 \oplus F_r$. Canceling the fragment $0 \leftarrow F_r \leftarrow \tilde{F}_r \leftarrow 0$ from (C_*, d_*) we obtain the chain complex $(\tilde{C}_*, \tilde{d}_*)$ such that $\chi_i(\tilde{C}_*, \tilde{d}_*) < \chi_i(C_*, d_*)$. It follows that the chain complexes (C_*, d_*) and $(\tilde{C}_*, \tilde{d}_*)$ are homotopy equivalent but $\chi_i(\tilde{C}_*, \tilde{d}_*) < \chi_i(C_*, d_*)$ which contradicts to the definition of $\chi_i^a(C_*, d_*)$.

If *f*-rank$(d_{i+1}(C_{i+1}), C_i)$ is nonadditive the proof is similar.

Sufficiency. Suppose that there exists a chain complex

$$(\tilde{C}_*, \tilde{d}_*) : \tilde{C}_0 \xleftarrow{\tilde{d}_1} \tilde{C}_1 \xleftarrow{\tilde{d}_2} \ldots \xleftarrow{\tilde{d}_n} \tilde{C}_n$$

such that *f*-rank of $\big(\tilde{d}_{i+1}(\tilde{C}_{i+1}), \tilde{C}_i\big)$ additive and equal to zero but

$$\chi_i(\tilde{C}_*, \tilde{d}_*) > \chi_i^a(C_*, d_*).$$

Then there exists a chain complex $(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_n$ which is homotopy equivalent to $(\tilde{C}_*, \tilde{d}_*)$ and such that $\chi_i(C_*, d_*) = \chi_i^a(C_*, d_*)$. Then it follows from necessity of our theorem that *f*-rank of $\big(d_{i+1}(C_{i+1}), C_i\big)$ is additive and equal to zero.

Then by Cockroft – Swan's lemma we can stabilize the boundary homomorphisms d_j and $\tilde{d}_j$, $j = 1, 2, \ldots, n$, via some free modules F_{k_j} and $F_{\tilde{k}_j}$ respectively and obtain isomorphic chain complexes $\big(C_*^{st}, d_*^{st}\big)$ and $\big(\tilde{C}_*^{st}, \tilde{d}_*^{st}\big)$. By the construction the modules $F_{k_i} \oplus C_i \oplus F_{k_{i+1}}$ and $F_{\tilde{k}_i} \oplus \tilde{C}_i \oplus F_{\tilde{k}_{i+1}}$ are isomorphic and therefore $\chi_i(C_*^{st}, d_*^{st}) = \chi_i(\tilde{C}_*^{st}, \tilde{d}_*^{st})$. We note that

$$k_{i+1} = f\text{-rank}(d_{i+1}^{st}(C_{i+1} \oplus F_{k_{i+1}} \oplus F_{k_{i+2}}), C_i \oplus F_{k_i} \oplus F_{k_{i+1}}) =$$
$$= f\text{-rank}(\tilde{d}_{i+1}^{st}(\tilde{C}_{i+1} \oplus F_{\tilde{k}_{i+1}} \oplus F_{\tilde{k}_{i+2}}), \tilde{C}_i \oplus F_{\tilde{k}_i} \oplus F_{\tilde{k}_{i+1}}) = \tilde{k}_{i+1}.$$

Hence we get $\chi_i(\tilde{C}_*, \tilde{d}_*) = \chi_i^a(C_*, d_*)$.

Theorem 1 is proved.

***Remark* 2.** If $(C^*, d^*)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n$ is a free cochain complex then reversing arrows in Theorem 1 we obtain that $\chi_i^a(C^*, d^*) = \chi_i(C^*, d^*)$ if and only if *f*-rank of $(d^i(C^i), C^{i+1})$ is additive and equal to zero:

$$f\text{-rank}(d^i(C^i), C^{i+1}) = 0.$$

4. The value of the i-th Euler characteristic. Let $(C^*_{(2)}, d^*)\colon C^0_{(2)} \xrightarrow{d^0} C^1_{(2)} \xrightarrow{d^1} \ldots$ $\ldots \xrightarrow{d^{n-1}} C^n_{(2)}$ be a sequence of free Hilbert $N[G]$-modules and bounded $\mathbb{C}[G]$-map such that $d^{i+1} \circ d^i = 0$. Such a sequence is called a *Hilbert complex*. The *reduced cohomology of the Hilbert complex* $(C^*_{(2)}, d^*)$ is the collection of $L^2(G)$-modules $\overline{H^i}_{(2)}(C^*_{(2)}, d^*) = \operatorname{Ker} d^i / \overline{\operatorname{Im} d^{i-1}}$.

Definition 5. *Let*

$$(C^*, d^*)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n$$

be a free cochain complex over $\mathbb{Z}[G]$. *Then the following complex*:

$$\left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*,\ \operatorname{Id} \bigotimes_{\mathbb{Z}[G]} d^*\right):$$

$$L^2(G) \bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^0} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^1 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^1} \ldots \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^{n-1}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^n$$

of free Hilbert $N[G]$-modules is called a Hilbert complex generated by the $\mathbb{Z}[G]$-cochain complex (C^*, d^*).

Consider the i-th skeletons of these complexes

$$(C^*(i), d^*(i))\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{i-1}} C^i,$$

$$\left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*(i),\ \operatorname{Id} \bigotimes_{\mathbb{Z}[G]} d^*(i)\right)\colon L^2(G) \bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^0} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^1 \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^1} \ldots$$

$$\ldots \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^{i-2}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^{i-1} \xrightarrow{\operatorname{Id}\otimes_{\mathbb{Z}[G]} d^{i-1}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^i.$$

Set $\Gamma^i = C^i / d^{i-1}(C^{i-1})$. It is clear that

$$\widehat{\Gamma^i} = L^2(G) \bigotimes_{\mathbb{Z}[G]} C^i \Big/ \overline{\operatorname{Id} \bigotimes_{\mathbb{Z}[G]} d^{i-1} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^{i-1}\right)}$$

is the i-th Hilbert $N[G]$-module of the reduced cohomology of the i-th skeleton of the Hilbert complex

$$\left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*(i),\ \operatorname{Id} \bigotimes_{\mathbb{Z}[G]} d^*(i)\right).$$

For a cochain complex (C^*, d^*) over $\mathbb{Z}[G]$ set

$$\widehat{S}^i_{(2)}(C^*, d^*) = \mu_s(\Gamma^i) - \dim_{N[G]} \widehat{\Gamma^i}.$$

If (C^*, d^*) and (D^*, ∂^*) are two homotopy equivalent free cochain complexes over the group ring $\mathbb{Z}[G]$ then

$$\widehat{S}^i_{(2)}(C^*, d^*) = \widehat{S}^i_{(2)}(D^*, \partial^*).$$

The numbers $\widehat{S}^i_{(2)}(C^*, d^*)$ are nonnegative for every i.

Theorem 2. *Let* $(C^*, d^*)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n$ *be a free cochain complex over* $\mathbb{Z}[G]$*. Then*

$$\chi_i^a(C^*, d^*) =$$
$$= (-1)^i \sum_{j=0}^{i} (-1)^j \dim_{N[G]} \left(H^j_{(2)} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*,\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^* \right) \right) + \widehat{S}^{i+1}_{(2)}(C^*, d^*).$$

Proof. Suppose that the cochain complex $(C^*, d^*)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n$ is such that $\chi_i(C^*, d^*) = \chi_i^a(C^*, d^*)$. Consider the Hilbert complex

$$\left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*,\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^* \right):$$

$$L^2(G) \bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^0} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^1 \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^1} \ldots \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^{n-1}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^n$$

and let

$$\left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*(i),\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^*(i) \right):$$

$$L^2(G) \bigotimes_{\mathbb{Z}[G]} C^0 \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^0} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^1 \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^1} \ldots$$

$$\ldots \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^{i-2}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^{i-1} \xrightarrow{\mathrm{Id} \otimes_{\mathbb{Z}[G]} d^{i-1}} L^2(G) \bigotimes_{\mathbb{Z}[G]} C^i$$

be its i-th skeleton.

It is clear from additivity of $\dim N[G]$ that

$$\chi_i(C^*, d^*) =$$
$$= (-1)^i \sum_{j=0}^{i} (-1)^j \dim_{N[G]} \left(\overline{H^j}_{(2)} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*(i),\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^*(i) \right) \right) =$$
$$= \dim_{N[G]} \left(\overline{H^i}_{(2)} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*(i),\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^*(i) \right) \right) -$$
$$-(-1)^{i-1} \sum_{j=0}^{i-1} (-1)^j \dim_{N[G]} \left(\overline{H^j}_{(2)} \left(L^2(G) \bigotimes_{\mathbb{Z}[G]} C^*,\ \mathrm{Id} \bigotimes_{\mathbb{Z}[G]} d^* \right) \right).$$

Similarly to [4] one can check that

$$\dim_{N[G]}\left(\overline{H^i}_{(2)}\left(L^2(G)\bigotimes_{\mathbb{Z}[G]}C^*(i),\ \mathrm{Id}\bigotimes_{\mathbb{Z}[G]}d^*(i)\right)\right)=$$

$$=\dim_{N[G]}\left(\overline{H^i}_{(2)}\left(L^2(G)\bigotimes_{\mathbb{Z}[G]}C^*,\ \mathrm{Id}\bigotimes_{\mathbb{Z}[G]}d^*\right)\right)+\widehat{S}^{i+1}_{(2)}(C^*,d^*).$$

Theorem 2 is proved.

5. Topological applications. Let Y be a topological space endowed with some structure $K=K(Y)$ of a finite CW-complex. Denote by K_i the i-th skeleton of $K(Y)$. Let also $n(\sigma^j)$ be the total number of j-cells of $K(Y)$ and

$$\chi_i(K(Y))=(-1)^i\chi(K_i)=(-1)^i\sum_{j=0}^{i}(-1)^j n(\sigma^j).$$

Definition 6. *The cellular i-th Euler characteristics of the space Y is the minimal value of $\chi_i(K(Y))$ taken over all cellular decomposition $K(Y)$ of Y*:

$$\chi_i^c(Y)=\min\Big\{\chi_i(K(Y))\mid K(Y)\ \text{ is a cellular decomposition of }\ Y\Big\}.$$

***Remark* 3.** Let M^n be a closed (possibly only topological) manifold having a handle decomposition. Then similarly to the Definition 6 we can define the *i-th handle Euler characteristics* $\chi_i^h(M^n)$ of the manifold M^n using handle decompositions M^n.

Evidently, that if a closed manifold M^n admits a handle decomposition, then contracting each handle to its middle disk we obtain some cell decomposition of M^n. Therefore

$$\chi_i^c(M^n)\leq\chi_i^h(M^n).$$

Note that for a closed simply-connected smooth manifold $M^n (n>4)$ the following equality holds true:

$$\chi_i^c(M^n)=\chi_i^h(M^n)=\mu(H_i(M^n,\mathbb{Z}))-(-1)^{i-1}\sum_{j=0}^{i-1}(-1)^j\mu(H_j(M^n,\mathbb{Q})).$$

Now let K be a CW-complex and p: $\widetilde{K}\to K$ be the universal covering of K. Using the map p we can lift the CW-complex structure of K to $\widetilde{K}$. Then the fundamental group $\pi=\pi_1(K)$ acts free on $\widetilde{K}$ also preserving its CW-structure. This action turns each chain group $C_i(\widetilde{K},\mathbb{Z})$ into a left module over the group ring $\mathbb{Z}[\pi]$. It is evident that the resulting chain module $C_i(\widetilde{K},\mathbb{Z})$ is free. Moreover, lifting each i-cell of K to some cell of $\widetilde{K}$ we obtain a *finite* set of generators of $C_i(\widetilde{K},\mathbb{Z})$ over $\mathbb{Z}$. As a result we get a free chain complex over the ring $\mathbb{Z}[\pi]$:

$$C_*(\widetilde{K})\colon C_0(\widetilde{K},\mathbb{Z})\xleftarrow{d_1}C_1(\widetilde{K},\mathbb{Z})\xleftarrow{d_2}\ldots\xleftarrow{d_n}C_n(\widetilde{K},\mathbb{Z}).$$

Definition 7. *For a CW-complex K the following number $\chi_i^a(K)$ defined by*

$$\chi_i^a(K)=\chi_i^a(C_*(\widetilde{K})).$$

is called the i-th algebraic Euler characteristics of K.

It is well known that any two chain complexes constructed from some cellular decompositions of the same topological space K have the same homotopy type. Therefore it follows directly from the previous discussion or from [3, 8] that *the numbers* $\chi_i^a(K)$ *are invariants of the homotopy type of the cell complex* K.

It is clear that for a cell complex K we have that

$$\chi_i^a(K) \leq \chi_i^c(K).$$

For a smooth manifold M^n it is possible to define a cochain complex via Morse functions (handle decomposition). The details can be found in [3]. It is proved in [8] that the all chain complexes constructed from some Morse functions (handle decomposition) on the manifold M^n have the same homotopy type. This means that the values of i-th algebraic Euler characteristic $\chi_i^a(M^n)$ of M^n do not depend on the way of constructing a chain complex.

If the fundamental group $\pi = \pi_1(K)$ of K is non-zero then for calculation of the values of some $\chi_i^c(Y)$ one can use L^2-theory. To describe this let us recall the definition of the integers $\widehat{S}^i_{(2)}(K)$ [4].

Let $C^i(\widetilde{K}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}[G]}\big(C_i(\widetilde{K}, \mathbb{Z}), \mathbb{Z}[G]\big)$. and using involution in the ring $\mathbb{Z}[G]$ introduced the structure of left $\mathbb{Z}[G]$-module on $C^i(\widetilde{K}, \mathbb{Z})$. Consider the following cochain complex

$$C^*(\widetilde{K}) = C^0(\widetilde{K}, \mathbb{Z}) \xrightarrow{d^0} C^1(\widetilde{K}, \mathbb{Z}) \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n(\widetilde{K}, \mathbb{Z}).$$

Taking the tensor product of $C^*(\widetilde{K})$ and $L^2(G)$ as $\mathbb{Z}[G]$-module we obtain the Hilbert complex

$$C^*_{(2)}(\widetilde{K})\colon L^2(G) \bigotimes_{\mathbb{Z}[\pi]} C^0(\widetilde{K}, \mathbb{Z}) \xrightarrow{\mathrm{id}\otimes\, d^0} L^2(G) \bigotimes_{\mathbb{Z}[\pi]} C^1(\widetilde{K}, \mathbb{Z}) \xrightarrow{\mathrm{id}\otimes\, d^1} \ldots$$

$$\ldots \xrightarrow{\mathrm{id}\otimes\, d^{n-1}} L^2(G) \bigotimes_{\mathbb{Z}[\pi]} C^n(\widetilde{K}, \mathbb{Z}).$$

The $L^2(G)$-module of i-th cohomology $H^i_{(2)}(K)$ of this Hilbert complex is called $L^2(G)$-*module of* i-*th cohomology* of the space K. Therefore the following $\mathbb{Z}[\pi]$-module:

$$\widehat{\Gamma}^i(\widetilde{K}) = C^i(\widetilde{K}, \mathbb{Z})/d^{i-1}\big(C^{i-1}(\widetilde{K}, \mathbb{Z})\big),$$

can be interpreted as the *i-th cohomology module with compact support of the i-th skeleton of* $\widetilde{K}$ and $L^2(G)$-module

$$\Gamma^i(K) = L^2(G) \bigotimes_{\mathbb{Z}[\pi]} C^i(\widetilde{K}, \mathbb{Z}) / \,\mathrm{id} \bigotimes d^{i-1} \left(L^2(G) \bigotimes_{\mathbb{Z}[\pi]} C^{i-1}(\widetilde{K}, \mathbb{Z}) \right)$$

is the *i-th $L^2(G)$-module of cohomology of the i-th skeleton of K.*

Definition 8. *For a cell complex K, set*

$$\widehat{S}^i_{(2)}(K) = \widehat{S}^i_{(2)}(C^*(\widetilde{K})) = \mu_s(\widehat{\Gamma}^i(\widetilde{K})) - \dim_{N[G]}(\Gamma^i(K)),$$

From our previous discussion or from [3, 4] it follows that *the numbers* $\widehat{S}^i_{(2)}(K)$ *are invariants of the homotopy type of the cell complex* K.

Of course, for a smooth manifold M^n the values of the numbers $\widehat{S}^i_{(2)}(M^n)$ do not depend on the method of constructing a chain complex.

Theorem 3. *Let M^n, $n \geq 6$, be a closed smooth nonsimply connected manifold with $\pi_1(M^n) = \pi$. Then*

$$\chi_1^c(M^n) = \chi_1^h(M^n) = \mu(\pi) - 1,$$

$$\chi_2^c(M^n) = \chi_2^h(M^n),$$

$$\chi_i^c(M^n) = \chi_i^h(M^n) = \chi_i^a(M^n) =$$

$$= (-1)^i \sum_{j=0}^{i} (-1)^j \dim_{N[G]}(H^j_{(2)}(M^n)) + \widehat{S}^{i+1}_{(2)}(M^n)$$

for $3 \leq i \leq n-4$.

Proof. The condition $n \geq 6$ allows us to construct a handle decomposition of the manifold $M^n = \bigcup H^i_j$ such that the free chain complex

$$(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_i,$$

over $\mathbb{Z}[\pi]$ corresponding this handle decomposition satisfies following conditions:

a) $\mu(C_0) = 1$;

b) $\mu(C_1) = 1$;

c) $\chi_2^c(M^n) = \chi_2^h(M^n)$;

d) f-rank$(d_{i+1}(C_{i+1}), C_i) = 0$ and is additive for $3 \leq i \leq n-4$.

The proof follows from Theorems 1 and 2.

***Remark* 4.** For a closed smooth nonsimply connected manifold M^n, $n \geq 6$, the numbers $\chi_1^c(M^n)$, $\chi_1^h(M^n)$, $\chi_i^c(M^n)$, and $\chi_i^h(M^n)$, $3 \leq i \leq n-4$, are invariants of homotopy type of manifold.

6. Nonsingular Morse – Smale flows.

Definition 9. *A smooth flow φ_t on smooth closed manifold M^n is called nonsingular Morse – Smale if*

a) *the chain-recurrent set R of φ_t consist of finite number of hyperbolic closed orbit*;

b) *the unstable manifold of any closed orbit has transversal intersection with the stable manifold of any closed orbit.*

A vector field $\mathcal{X}$ generating a nonsingular Morse – Smale flow is also called *nonsingular Morse – Smale*. A result of K. Meyer (see [2]) says for each nonsingular Morse – Smale vector field there exists a *Lyapunov function* $f\colon M^n \longrightarrow \mathbb{R}$ for $\mathcal{X}$: that is a function satisfying the following conditions:

a) $\mathcal{X}(f)_y < 0$ for all y that are not contained in a closed orbit;

b) $df_y = 0$ if and only if y is a point on a closed orbit.

We will call f self-indexing if $f(y) = \lambda$ whenever y belongs to a closed orbit of index λ.

There are two types of closed orbit: twisted and untwisted. An untwisted closed orbit σ of index λ of a nonsingular Morse – Smale vector field $\mathcal{X}$ is said to be in the standard form if there are local coordinates $\theta \in S^1$, $x_1, \ldots, y_1, \ldots, y_{n-\lambda-1}$ on tubular neighborhood of σ such that

$$\mathcal{X} = x_1 \frac{\partial}{\partial x_1} + \ldots + x_\lambda \frac{\partial}{\partial x_\lambda} - y_1 \frac{\partial}{\partial y_1} - \ldots - y_{n-\lambda-1} \frac{\partial}{\partial y_{n-\lambda-1}}$$

on this neighborhood. If $\mathcal{X}_0$ is a nonsingular Morse – Smale vector field on M^n then there is an arc in the space of smooth vector field on M^n, $\mathcal{X}_t$, $0 \le t \le 1$, such that $\mathcal{X}_t$ is Morse – Smale for all $0 \le t \le 1$ and closed orbits of $\mathcal{X}_1$ coincide with the closed orbits of $\mathcal{X}_0$ and are in the standard form [2].

In what follow we will consider nonsingular Morse – Smale vector fields having only untwisted closed orbits.

Conversely, if a manifold M^n admits a round Morse function f: $M^n \to \mathbb{R}$, then there exists a nonsingular Morse – Smale vector field $\mathcal{X}$ on M^n, such that closed orbits of index λ of $\mathcal{X}$ coincide with singular circles of index λ of the function f. By definition a function f on M^n is said to be a *round Morse function* if its singular set $K(f)$ consists of disjoint circles and corank of the Hessian is equal to one: $\operatorname{corank}_{x \in K(f)} f = 1$ (see [3]).

It is known that under a small perturbation of a round Morse function f, each singular circle of index λ splits into two nondegenerate critical points of indexes λ and $\lambda + 1$. And conversely, if g: $M^n \to \mathbb{R}$ is a Morse function having two independent (see [3]) critical points x_1 and x_2 of g of indexes λ and $\lambda + 1$ respectively, then these points can be replaced by one singular circle of index λ. Therefore, for the construction of nonsingular Morse – Smale vector fields on a manifold M^n with zero Euler characteristics we may use Morse functions.

Definition 10. *The i-th Morse S^1-number of a manifold M^n is the minimum number of closed orbits of index i taken over all nonsingular Morse – Smale vector fields on M^n with untwisted closed orbits. This number will be denoted by $\mathcal{M}_i^{S^1}(M^n)$.*

Theorem 4. *Let M^n, $n \ge 6$, be arbitrary closed smooth manifold with zero Euler characteristic and with $\pi_1(M^n) = \pi$. Then the i-th Morse S^1-number of the manifold M^n is equal*:

$$\mathcal{M}_0^{S^1}(M^n) = \mathcal{M}_{n-1}^{S^1}(M^n) = 1,$$

$$\mathcal{M}_1^{S^1}(M^n) = \mathcal{M}_{n-2}^{S^1}(M^n) = \mu(\pi) - 1,$$

$$\mathcal{M}_i^{S^1}(M^n) = \rho(\chi_i^a(M^n)) =$$

$$= \widehat{S}_{(2)}^{i+1}(M^n) + \rho\left[(-1)^i \sum_{j=0}^{i} (-1)^j \dim_{N[G]}\left(H_{(2)}^j(M^n)\right)\right]$$

for $3 \le i \le n-4$.

Proof. Let $\mathcal{X}$ be a nonsingular Morse – Smale vector field on M^n such that all closed orbits of $\mathcal{X}$ are untwisted. Let also f: $M^n \to \mathbb{R}$ be a round Morse function corresponding to $\mathcal{X}$ and g: $M^n \to \mathbb{R}$ be an ordered Morse function obtaned by small pertrubation of f. Using g we can construct a handle decomposition of M^n and from this decomposition define the free chain complex over $\mathbb{Z}[\pi]$:

$$(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_n.$$

It is clear that

$$\chi_i^c(M^n) = \chi_i^a(C_*, d_*) \le \chi_i(C_*, d_*)$$

for $3 \le i \le n-4$. The condition $n \ge 6$ allows to construct a handle decomposition $M^n = \cup H_j^i$ of M^n such that the free complex over $\mathbb{Z}[\pi]$

$$(C_*, d_*)\colon C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} C_i$$

corresponding to this handle decomposition satisfies the following conditions:

a) $\mu(C_0) = 1$;

b) $\mu(C_1) = \mu(\pi) - 1$;

c) f-rank$(d_{i+1}(C_{i+1}, C_i) = 0$ and is additive for $3 \leq i \leq n-4$.

Using diagram technique from [3] and Theorem 3 we can construct from this handle decomposition $M^n = \cup H^i_j$ a round Morse function and therefore a nonsingular Morse – Smale vector field $\mathcal{X}$ such that the numbers of untwisted closed orbits of $\mathcal{X}$ satisfy the conditions of theorem. Homotopy invariance of the i-th Morse S^1-number of M^n for $i = 0, 1, n-1, n-2$ and $3 \leq i \leq n-4$ easily follows from previous discussions.

Theorem 4 is proved.

The calculation of $\mathcal{M}^{S^1}_2(M^n)$ and $\mathcal{M}^{S^1}_{n-3}(M^n)$ seems to be a difficult problem.

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