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Марка Григоровича Крейна
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S. Albeverio (Inst. Angewandte Math., Univ. Bonn, Germany),
S. Kuzhel, L. Nizhnik (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

SINGULARLY PERTURBED SELF-ADJOINT OPERATORS IN SCALES OF HILBERT SPACES

СИНГУЛЯРНО ЗБУРЕНІ САМОСПРЯЖЕНІ ОПЕРАТОРИ В ШКАЛАХ ГІЛЬБЕРТОВИХ ПРОСТОРІВ

Finite rank perturbations of a semi-bounded self-adjoint operator A are studied in the scale of Hilbert spaces associated with A . A concept of quasi-boundary value space is used to describe self-adjoint operator realizations of regular and singular perturbations of A by the same formula. As an application the one-dimensional Schrödinger operator with generalized zero-range potential is considered in the Sobolev space $W_2^p(\mathbb{R})$, $p \in \mathbb{N}$.

У шкалі гільбертових просторів, асоційованих з A , вивчаються скінченного рангу збурення напівобмеженого самоспряженого оператора A . Поняття квазіпростору граничних значень використовується для опису однією формулою самоспряжених операторних реалізацій як регулярних, так і сингулярних збурень оператора A . Як застосування, розглядається одновимірний оператор Шредінгера з узагальненим потенціалом нульового радіуса у просторі Соболева $W_2^p(\mathbb{R})$, $p \in \mathbb{N}$.

1. Introduction. Let A be a semibounded self-adjoint operator acting in a separable Hilbert space \mathcal{H} with inner product (\cdot, \cdot) and let $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of A , respectively. Without loss of generality, we will assume that $A \geq I$. Let

$$\mathcal{H}_s \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-s}, \quad s > 0, \quad (1.1)$$

be the standard scale of Hilbert spaces associated with A (A -scale) [1, 8]. Here, a Hilbert space \mathcal{H}_s ($s \in \mathbb{R}$) is considered as the completion of the set $\bigcap_{n \in \mathbb{N}} \mathcal{D}(A^n)$ with respect to the norm

$$\|u\|_s = \|A^{s/2}u\|, \quad u \in \bigcap_{n \in \mathbb{N}} \mathcal{D}(A^n). \quad (1.2)$$

By (1.2), the operator $A^{r/2}$ ($r \in \mathbb{R}$) can continuously be extended to an isometric mapping $A^{r/2}$ of \mathcal{H}_s onto \mathcal{H}_{s-r} (we preserve the same notation $A^{r/2}$ for this continuation). In a natural way \mathcal{H}_s and \mathcal{H}_{-s} are dual and the inner product in \mathcal{H} can be extended to a pairing

$$\langle u, \psi \rangle = (A^{s/2}u, A^{-s/2}\psi), \quad u \in \mathcal{H}_s, \quad \psi \in \mathcal{H}_{-s}, \quad (1.3)$$

such that $|\langle u, \psi \rangle| \leq \|u\|_s \|\psi\|_{-s}$.

The present paper is an extended and modified variant of [4] and its aim consists in the development of a unified approach to the study of finite rank perturbations of a self-adjoint operator A in the scale of Hilbert spaces \mathcal{H}_s .

We recall that a self-adjoint operator $\tilde{A} \neq A$ acting in \mathcal{H} is called a *finite rank perturbation* of A if the difference $(\tilde{A} - zI)^{-1} - (A - zI)^{-1}$ is a finite rank operator in \mathcal{H} for at least one point $z \in \mathbb{C} \setminus \mathbb{R}$ [16].

If \tilde{A} is a finite rank perturbation of A , then the corresponding symmetric operator¹

$$A_{\text{sym}} = A \upharpoonright_{\mathcal{D}} = \tilde{A} \upharpoonright_{\mathcal{D}}, \quad \mathcal{D} = \left\{ u \in \mathcal{D}(A) \cap \mathcal{D}(\tilde{A}) \mid Au = \tilde{A}u \right\} \quad (1.4)$$

arises naturally. This operator has finite and equal deficiency numbers.

It is important that the operator A_{sym} can be recovered uniquely by its defect subspace $N = \mathcal{H} \ominus \mathcal{R}(A_{\text{sym}})$ and the initial operator A . Namely,

$$A_{\text{sym}} = A \upharpoonright_{\mathcal{D}(A_{\text{sym}})}, \quad \mathcal{D}(A_{\text{sym}}) = \left\{ u \in \mathcal{D}(A) \mid (Au, \eta) = 0 \forall \eta \in N \right\}. \quad (1.5)$$

Moreover, the choice of an arbitrary finite dimensional subspace N of \mathcal{H} as a defect subspace allows one to determine by (1.5) a closed symmetric operator A_{sym} with finite and equal defect numbers. To underline this relation, we will use notation A_N instead of A_{sym} . Obviously, any self-adjoint extension \tilde{A} of A_N is a finite rank perturbation of A .

A finite rank perturbation \tilde{A} of A is called *regular* if $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$. Otherwise (i.e., $\mathcal{D}(A) \neq \mathcal{D}(\tilde{A})$), the operator \tilde{A} is called *singular*.

It is convenient to divide the class of singular perturbations into two subclasses. We will say that a singular perturbation \tilde{A} is *purely singular* if the symmetric operator $A_{\text{sym}} = A_N$ defined by (1.4) is densely defined (i.e., $N \cap \mathcal{D}(A) = \{0\}$) and *mixed singular* if A_N is nondensely defined (i.e., $N \cap \mathcal{D}(A) \neq \{0\}$).

Important examples of finite rank perturbations of the Schrödinger operator are given by finitely many point interactions [1, 2]. The consideration of point interactions in $L_2(\mathbb{R}^d)$ leads to purely singular perturbations and, in the case of Sobolev spaces $W_2^p(\mathbb{R}^d)$, $p \in \mathbb{N}$, mixed singular perturbations arise [5, 26]. These applications can be served as a certain motivation of the abstract results carried out in the paper.

It is well-known that finite rank regular perturbations of A can be described with the help of finite rank self-adjoint operators (potentials) acting in \mathcal{H} . Typical examples of finite rank singular perturbations are provided by the general expression

$$\tilde{A} = A + V, \quad V = \sum_{i,j=1}^n b_{ij} \langle \cdot, \psi_j \rangle \psi_i \quad (\mathcal{R}(V) \not\subset \mathcal{H}, \quad b_{ij} \in \mathbb{C}). \quad (1.6)$$

Since $\mathcal{R}(V) \not\subset \mathcal{H}$, the singular potential V is not an operator in \mathcal{H} and it acts in the spaces of A -scale. Such types of expressions appear in many areas of mathematical physics (for an extensive list of references, see [1, 2]).

In the present paper, we will study finite rank singular perturbations of A in the spaces of A -scale (1.1). *The main attention will be focused* on the description of self-adjoint extensions \tilde{A} of A_{sym} in a form that is maximally adapted for the determination of \tilde{A} with the help of additive singular perturbations (1.6) and preserves physically meaningful relations to the parameters b_{ij} of the singular potential $V = \sum_{i,j=1}^n b_{ij} \langle \cdot, \psi_j \rangle \psi_i$.

In Section 2, such a problem is solved for the case of purely singular perturbations. Precisely, since the corresponding symmetric operator $A_{\text{sym}} = A_N$ in (1.4) is densely defined, we can combine the Albeverio–Kurasov approach [2] with the boundary value spaces technique [15, 22]. The first of them allows us to involve the parameters b_{ij} of the singular potential in the determination of the corresponding self-adjoint operator realization of (1.6), the second provides convenient framework for the description of such operators. As a result, we get a simple description of self-adjoint realizations of purely

¹The symbol $A \upharpoonright_X$ means the restriction of A onto the set X .

singular perturbations (Theorem 2.2) and, moreover, we present a simple algorithm for solving an inverse problem, i.e., recovering the purely singular potential V in (1.6) by the given self-adjoint extension of A_N defined in terms of boundary value spaces.

Other approaches to the description of purely singular perturbations were recently suggested by Arlinskii and Tsekanovski [7] and Posilicano [27, 28].

The description of mixed singular perturbations of A is more complicated because the corresponding symmetric operator A_N is nondensely defined and, hence, the adjoint of A_N does not exist. To overcome this problem, a certain generalization of the concept of BVS is required. The key point here is the replacement of the adjoint operator A_N^* by a suitable object. In [13, 24], the operator A_N and its ‘adjoint’ are understood as linear relations and the description of all self-adjoint *relations* that are extensions of the graph of A_N was obtained. In [22], a pair of maximal dissipative extensions of A_N and its adjoint (maximal accumulative extension) was used instead of A_N^* . This allows one to describe self-adjoint extensions directly as operators without using linear relations technique.

The approaches mentioned above are general and they can be applied to an arbitrary nondensely defined symmetric operator. However, in the case where A_N is determined as the restriction of an initial self-adjoint operator A , it is natural to use A for the description of extensions of A_N (see [10, 11, 18]). In Section 3, developing the ideas proposed recently in [5, 26], we use A for the definition of a quasi-adjoint operator of A_N . The concept of quasi-adjoint operators allows one to generalize the definition of boundary value spaces (BVS) to the case of nondensely defined operators A_N and to preserve the simple formulas for the description of self-adjoint extensions of A_N .

One of the characteristic features of quasi-BVS extension theory that immediately follows from the definition of a quasi-BVS consists in the description of *essentially*² self-adjoint extensions of A_N . It should be noted that this property is very convenient for the description of self-adjoint differential expressions with complicated boundary conditions. Furthermore, it gives the possibility to describe finite rank regular and mixed singular perturbations of A in just the same way as purely singular perturbations.

In Section 4, the results of quasi-BVS extension theory are applied to the study of finite rank singular perturbations of A in spaces of A -scale (1.1). In recent years, such kind of problems attracted a steady interest and they naturally arise in the theory of supersingular perturbations [12, 21] and in the study of Schrödinger operators with point interactions in Sobolev spaces [5, 26].

2. The case of purely singular perturbations. 2.1. Description. In what follows we assume that $A \geq I$ is a self-adjoint operator in \mathcal{H} , N is a finite dimensional subspace of \mathcal{H} , and A_N is a symmetric operator defined by the formula

$$A_N = A \upharpoonright_{\mathcal{D}(A_N)}, \quad \mathcal{D}(A_N) = \left\{ u \in \mathcal{D}(A) \mid (Au, \eta) = 0 \ \forall \eta \in N \right\}. \quad (2.1)$$

The operator A_N is densely defined in \mathcal{H} if and only if $N \cap \mathcal{D}(A) = \{0\}$. In this case, $\mathcal{D}(A_N^*) = \mathcal{D}(A) \dot{+} N$ and

$$A_N^* f = A_N^*(u + \eta) = Au \quad \forall f = u + \eta \in \mathcal{D}(A_N^*) \ (u \in \mathcal{D}(A), \ \eta \in N). \quad (2.2)$$

If A_N is densely defined, then self-adjoint extensions of A_N admit a convenient description in terms of boundary value spaces (see [14] and references therein).

²I.e., those extensions that turn out to be self-adjoint after closure.

Definition 2.1. A triple $(\mathfrak{N}, \Gamma_0, \Gamma_1)$, where \mathfrak{N} is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(A_{\text{sym}}^*)$ into \mathfrak{N} , is called a boundary value space (BVS) of A_N if the abstract Green identity

$$(A_N^* f, g) - (f, A_N^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{N}}, \quad f, g \in \mathcal{D}(A_N^*), \quad (2.3)$$

is satisfied and the map $(\Gamma_0, \Gamma_1): \mathcal{D}(A_N^*) \rightarrow \mathfrak{N} \oplus \mathfrak{N}$ is surjective.

One of the simplest examples of BVS gives the triple³ (N, Γ_0, Γ_1) , where N is taken from (2.1), (2.2) and

$$\Gamma_0(u + \eta) = P_N A u, \quad \Gamma_1(u + \eta) = -\eta \quad (\forall u \in \mathcal{D}(A) \quad \forall \eta \in N), \quad (2.4)$$

where P_N is the orthoprojector onto N in \mathcal{H} .

The following elementary result enables one to get infinitely many BVS of A_N starting from the fixed one.

Lemma 2.1. Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a BVS of A_N and let R be an arbitrary self-adjoint operator acting in \mathfrak{N} . Then the triple $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$, where $\Gamma_0^R = \Gamma_0 - R\Gamma_1$ is also a BVS of A_N .

The next theorem provides a description of all self-adjoint extensions of A_N .

Theorem 2.1 [17]. Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a BVS of A_N . Then any self-adjoint extension \tilde{A} of A_N coincides with restriction of A_N^* to

$$\mathcal{D}(\tilde{A}) = \left\{ f \in \mathcal{D}(A_N^*) \mid (I - U)\Gamma_0 f = i(I + U)\Gamma_1 f \right\}, \quad (2.5)$$

where U is a unitary operator in \mathfrak{N} . Moreover, the correspondence $\tilde{A} \leftrightarrow U$ is a bijection between the sets of all self-adjoint extensions of A_N and all unitary operators in \mathfrak{N} .

In cases where self-adjoint extensions are described by sufficiently complicated boundary conditions (see, e.g., [19]), the representation (2.5) is not always convenient because it contains the same factor U on the both sides. To overcome this inconvenience, we outline another approach that enables one to remove one of the factors in (2.5) but, simultaneously, to preserve the description of all self-adjoint extensions of A_N . The main idea here consists in the use of a family BVS $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$ instead of a fixed BVS (see [23] for details).

Let $(\mathfrak{N}, \Gamma_0^R, \Gamma_1)$ be a family of BVS of A_N defined in Lemma 2.1. For a fixed R , Theorem 2.1 implies that the expression

$$A_{B,R} := A_N^* \upharpoonright_{\mathcal{D}(A_{B,R})}, \quad \mathcal{D}(A_{B,R}) = \left\{ f \in \mathcal{D}(A_N^*) \mid B\Gamma_0^R f = \Gamma_1 f \right\}, \quad (2.6)$$

where B is an arbitrary self-adjoint operator in \mathfrak{N} , determines a subset $\mathcal{P}_R(A_N)$ of the set $\mathcal{P}(A_N)$ of all self-adjoint extensions of A_N . More precisely, a self-adjoint extension \tilde{A} of A_N belongs to $\mathcal{P}_R(A_N)$ iff $\mathcal{D}(\tilde{A}) \cap \ker \Gamma_0^R = \mathcal{D}(A_N)$.

It is easy to verify, that the union $\bigcup_R \mathcal{P}_R(A_N)$ over all self-adjoint operators R in \mathfrak{N} coincides with $\mathcal{P}(A_N)$. Moreover, for a fixed $\tilde{A} \in \mathcal{P}(A_N)$, there exist infinitely many R such that $\tilde{A} \in \mathcal{P}_R(A_N)$. Thus formula (2.6), where R and B play a role of parameters, gives the description of all self-adjoint extensions of A_N .

³In fact, this BVS was already implicitly used in the classical works [9, 20].

2.2. Self-adjoint realizations. *2.2.1. Construction of self-adjoint realizations by additive purely singular perturbations.* Let us consider the general expression (1.6), where $\psi_j, 1 \leq j \leq n$, form a linearly independent system in \mathcal{H}_{-2} and the linear span \mathcal{X} of $\{\psi_j\}_{j=1}^n$ satisfies the condition $\mathcal{X} \cap \mathcal{H} = \{0\}$ (i.e., elements ψ_j are \mathcal{H} -independent).

Let $\{e_j\}_1^n$ be the canonical basis of \mathbb{C}^n (i.e., $e_j = (0, \dots, 1, \dots, 0)$, where 1 occurs on the j th place only). Putting $\Psi e_j := \psi_j, j = 1, \dots, n$, we define an injective linear mapping $\Psi : \mathbb{C}^n \rightarrow \mathcal{H}_{-2}$ such that $\mathcal{R}(\Psi) = \mathcal{X}$.

Let $\Psi^* : \mathcal{H}_2 \rightarrow \mathbb{C}^n$ be the adjoint operator of Ψ (in the sense $\langle u, \Psi d \rangle = \langle \Psi^* u, d \rangle_{\mathbb{C}^n} \forall u \in \mathcal{H}_2 \forall d \in \mathbb{C}^n$). It is easy to see that

$$\Psi^* u = \begin{pmatrix} \langle u, \psi_1 \rangle \\ \vdots \\ \langle u, \psi_n \rangle \end{pmatrix} \quad \forall u \in \mathcal{H}_2. \tag{2.7}$$

Using (2.7), we rewrite the singular potential $V = \sum_{i,j=1}^n b_{ij} \langle \cdot, \psi_j \rangle \psi_i$ in (1.6) as follows:

$$\sum_{i,j=1}^n b_{ij} \langle \cdot, \psi_j \rangle \psi_i = \Psi \mathbf{B} \Psi^*, \tag{2.8}$$

where the matrix $\mathbf{B} = \|b_{ij}\|_{i,j=1}^n$ consists of the coefficients b_{ij} of the potential V . In what follows we assume that \mathbf{B} is *Hermitian*, i.e., $b_{ij} = \overline{b_{ji}}$.

In order to give a meaning to $\tilde{A} = A + V$ as a self-adjoint operator in \mathcal{H} we consider a symmetric restriction A_{sym} of A

$$A_{\text{sym}} := A \upharpoonright_{\mathcal{D}(A_{\text{sym}})}, \quad \mathcal{D}(A_{\text{sym}}) = \mathcal{D}(A) \cap \ker \Psi^*. \tag{2.9}$$

By virtue of (1.3) (for $s = 2$) and (2.7), the operator A_{sym} is also defined by (2.1), where $A_{\text{sym}} = A_N$ and $N = A^{-1} \mathcal{R}(\Psi) = A^{-1} \mathcal{X}$, i.e., N is a linear span of $\{A^{-1} \psi_j\}_{j=1}^n$. Since $N \cap \mathcal{D}(A) = \{0\}$, the operator A_N is densely defined in \mathcal{H} .

Any self-adjoint extension \tilde{A} of A_N is a purely singular perturbation of A and, in general, it can be regarded as a realization of (1.6) in \mathcal{H} . In this context, there arises the natural question of whether and how one could establish a physically meaningful correspondence between the parameter \mathbf{B} of the potential $V = \Psi \mathbf{B} \Psi^*$ and self-adjoint extensions of A_N .

To do this we combine the Albeverio – Kurasov approach [2] with the BVS technique. This approach consists in the construction of some regularization

$$A_{\text{reg}} := A^+ + \Psi \mathbf{B} \Psi_{\mathbf{R}}^* = A^+ + \sum_{i,j=1}^n b_{ij} \langle \cdot, \psi_j^{\text{ex}} \rangle \psi_i, \tag{2.10}$$

of (1.6) that is well defined as an operator from $\mathcal{D}(A_N^*)$ to \mathcal{H}_{-2} . (Here, $A^+, \Psi_{\mathbf{R}}^*$, and $\langle \cdot, \psi_j^{\text{ex}} \rangle$ are extensions of A, Ψ^* , and $\langle \cdot, \psi_j \rangle$ onto $\mathcal{D}(A_N^*)$). After that, the corresponding self-adjoint realization \tilde{A} of (1.6) is determined by the formula

$$\tilde{A} = A_{\text{reg}} \upharpoonright_{\mathcal{D}(\tilde{A})}, \quad \mathcal{D}(\tilde{A}) = \left\{ f \in \mathcal{D}(A_{\text{sym}}^*) \mid A_{\text{reg}} f \in \mathcal{H} \right\}. \tag{2.11}$$

By (2.2), it is easy to see that for the definition of A^+ in (2.10) one needs to determine the action of A^+ on N . Assuming that $A^+ \upharpoonright_N$ acts as the isometric mapping A in the A -scale, we get

$$A^+f = Au + A\eta = A_N^*f + A\eta \quad \forall f = u + \eta \in \mathcal{D}(A_N^*). \quad (2.12)$$

However, the principal point in the definition of A_{reg} is the construction of $\Psi_{\mathbf{R}}^*$ or, equivalently, the definition of the functionals $\langle \cdot, \psi_j \rangle, j = 1, \dots, n$, on $\mathcal{D}(A_N^*)$.

It is clear (see (2.2)) that $\langle \cdot, \psi_j \rangle$ can be extended onto $\mathcal{D}(A_N^*)$ if we know its values on N .

Since $N = A^{-1}\mathcal{R}(\Psi)$ and $\mathcal{R}(\Psi)$ coincides with the linear span of $\psi_j, j = 1, \dots, n$, the vectors $\eta_j = A^{-1}\psi_j, j = 1, \dots, n$, form a basis of N . Using this fact and (2.2), we get that any $f \in \mathcal{D}(A_N^*)$ can be represented as $f = u + \sum_{k=1}^n \alpha_k \eta_k$ ($u \in \mathcal{D}(A), \alpha_k \in \mathbb{C}$). Thus the extended functional $\langle \cdot, \psi_j^{\text{ex}} \rangle$ is well-defined by the formula

$$\langle f, \psi_j^{\text{ex}} \rangle = \langle u, \psi_j \rangle + \sum_{k=1}^n \alpha_k r_{jk} \quad \forall f \in \mathcal{D}(A_N^*) \quad (2.13)$$

if we know the entries $r_{jk} = \langle A^{-1}\psi_k, \psi_j \rangle = \langle \eta_k, \psi_j \rangle$ of the regularization matrix $\mathbf{R} = \|r_{jk}\|_{j,k=1}^n$. In this case, by virtue of (2.7) and (2.13),

$$\Psi_{\mathbf{R}}^*f = \Psi_{\mathbf{R}}^* \left(u + \sum_{k=1}^n \alpha_k \eta_k \right) = \Psi^*u + \mathbf{R} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \langle f, \psi_1^{\text{ex}} \rangle \\ \vdots \\ \langle f, \psi_n^{\text{ex}} \rangle \end{pmatrix} \quad (2.14)$$

for any $f \in \mathcal{D}(A_N^*)$.

If $\mathcal{R}(\Psi) \subset \mathcal{H}_{-1}$, the entries r_{jk} are uniquely defined and \mathbf{R} is an Hermitian matrix. In the case where $\mathcal{R}(\Psi) \not\subset \mathcal{H}_{-1}$ the matrix \mathbf{R} is not determined uniquely [2].

In what follows we assume that \mathbf{R} is chosen as an Hermitian matrix.

Lemma 2.2. *The triple $(\mathbb{C}^n, \Gamma_0^{\mathbf{R}}, \Gamma_1)$, where the linear operators $\Gamma_i^{\mathbf{R}}: \mathcal{D}(A_N^*) \rightarrow \mathbb{C}^n$ are defined by the formulas*

$$\Gamma_0^{\mathbf{R}}f = \Psi_{\mathbf{R}}^*f, \quad \Gamma_1f = -\Psi^{-1}(A^+ - A_N^*)f = -\Psi^{-1}A\eta \quad (2.15)$$

(where $f = u + \eta, u \in \mathcal{D}(A), \eta \in N$) is a BVS of A_N .

Proof. By (1.3), $\langle u, \psi_j \rangle = (Au, \eta_j)$. Taking into account this relation and (2.2), (2.7), (2.12) it is easy to verify that the mappings

$$\Gamma_0f = \Psi^*u, \quad \Gamma_1f = -\Psi^{-1}A\eta \quad (2.16)$$

satisfy the conditions of Definition 2.1. Hence, $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ is a BVS of A_N .

It follows from (2.13), (2.7), (2.14), (2.15), and (2.16) that $\Gamma_0^{\mathbf{R}}f = \Gamma_0f - \mathbf{R}\Gamma_1f$. By Lemma 2.1 this means that $(\mathbb{C}^n, \Gamma_0^{\mathbf{R}}, \Gamma_1)$ is also a BVS of A_N .

Lemma 2.2 is proved.

Theorem 2.2. *Let \tilde{A} be a self-adjoint realization of (1.6) defined by (2.10), (2.11). Then*

$$\tilde{A} = A_{\mathbf{B}, \mathbf{R}} = A_N^* \upharpoonright_{\mathcal{D}(A_{\mathbf{B}, \mathbf{R}})}, \quad \mathcal{D}(A_{\mathbf{B}, \mathbf{R}}) = \{f \in \mathcal{D}(A_N^*) \mid \mathbf{B}\Gamma_0^{\mathbf{R}}f = \Gamma_1f\}, \quad (2.17)$$

$\Gamma_0^{\mathbf{R}}$ and Γ_1 being defined by (2.15).

Proof. Employing relations (2.10), (2.12), and (2.15), we get

$$A_{\text{reg}}f = A_N^*f + \Psi[\mathbf{B}\Gamma_0^{\mathbf{R}}f - \Gamma_1f].$$

This equality and (2.11) mean that $f \in \mathcal{D}(\tilde{A})$ if and only if $\mathbf{B}\Gamma_0^{\mathbf{R}}f - \Gamma_1f = 0$. Thus, the operator realization \tilde{A} of (1.6) coincides with the operator $A_{\mathbf{B},\mathbf{R}}$ defined by (2.6). Since \mathbf{B} is an Hermitian matrix, the operator $A_{\mathbf{B},\mathbf{R}}$ is self-adjoint.

Theorem 2.2 is proved.

Summing the results above we can state that the choice of an extension Ψ_{ex}^* of Ψ^* onto $\mathcal{D}(A_N^*)$ plays a main role and precisely this enables one to choose (see (2.15)) a more suitable⁴ BVS $(\mathbb{C}^n, \Gamma_0^{\mathbf{R}}, \Gamma_1)$ for the description of self-adjoint realizations of (1.6).

2.2.2. *Recovering purely singular potentials by a given self-adjoint extension.* Here we consider an inverse problem. Namely, for a given BVS $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ of A_N such that $\ker \Gamma_1 = \mathcal{D}(A)$ and the corresponding self-adjoint extensions

$$A_{\mathbf{B}} = A_N^* \upharpoonright_{\mathcal{D}(A_{\mathbf{B}})}, \quad \mathcal{D}(A_{\mathbf{B}}) = \left\{ f \in \mathcal{D}(A_N^*) \mid \mathbf{B}\Gamma_0f = \Gamma_1f \right\}, \quad (2.18)$$

where \mathbf{B} is an Hermitian matrix, we recover an additive purely singular perturbation $V = \Psi\mathbf{B}\Psi^*$ such that the formal expression $\tilde{A} = A + V$ possesses the self-adjoint realization $A_{\mathbf{B}}$.

We start with the definition of Ψ . Since $\ker \Gamma_1 = \mathcal{D}(A)$, the restriction $\Gamma_1 \upharpoonright_N$ determines a one-to-one correspondence between N and \mathbb{C}^n . Hence, $(\Gamma_1 \upharpoonright_N)^{-1}$ exists and $(\Gamma_1 \upharpoonright_N)^{-1}$ maps \mathbb{C}^n onto N .

Putting (cf. (2.15)) $\Psi d := -A(\Gamma_1 \upharpoonright_N)^{-1}d$, where $d \in \mathbb{C}^n$, we determine an injective linear mapping of \mathbb{C}^n to \mathcal{H}_{-2} such that $\mathcal{R}(\Psi) \cap \mathcal{H} = \{0\}$.

Set $\psi_j = \Psi e_j$, where $\{e_j\}_1^n$ is the canonical basis of \mathbb{C}^n . Putting $f = u \in \mathcal{D}(A)$, $g = A^{-1}\psi_j = A^{-1}\Psi e_j = -(\Gamma_1 \upharpoonright_N)^{-1}e_j$ in (2.3) and recalling the condition $\ker \Gamma_1 = \mathcal{D}(A)$, we establish that

$$\langle u, \psi_j \rangle = \langle Au, A^{-1}\psi_j \rangle = -(\Gamma_0u, \Gamma_1A^{-1}\psi_j)_{\mathbb{C}^n} = (\Gamma_0u, e_j)_{\mathbb{C}^n}.$$

This formula enables one to determine an extension of $\langle \cdot, \psi_j \rangle$ onto $\mathcal{D}(A_N^*)$ with the help of the boundary operator Γ_0 . Namely, $\langle f, \psi_j^{\text{ex}} \rangle := (\Gamma_0f, e_j)_{\mathbb{C}^n}$. But then, reasoning by analogy with (2.14), we conclude that $\Gamma_0f = \Psi_{\mathbf{R}}^*f$.

Now, repeating arguments of Theorem 2.2, it is easy to see that the operator $A_{\mathbf{B}}$ defined by (2.18) is a self-adjoint realization of the formal expression $A^+ + \Psi\mathbf{B}\Psi_{\mathbf{R}}^*$.

Example 2.1. *General zero-range potential in \mathbb{R} .*

A one-dimensional Schrödinger operator corresponding to a general zero-range potential at the point $x = 0$ can be given by the formal expression

$$-\frac{d^2}{dx^2} + b_{11}\langle \cdot, \delta \rangle \delta + b_{12}\langle \cdot, \delta' \rangle \delta + b_{21}\langle \cdot, \delta \rangle \delta' + b_{22}\langle \cdot, \delta' \rangle \delta', \quad (2.19)$$

where δ' is the derivative of the Dirac δ -function (with support at 0) and the coefficients b_{ij} form an Hermitian matrix.

Putting $\Psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta$ and $\Psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta'$, we get $\Psi^*u = \begin{pmatrix} \langle u, \delta \rangle \\ \langle u, \delta' \rangle \end{pmatrix}$ ($u(x) \in W_2^2(\mathbb{R})$) and, hence,

$$\Psi\mathbf{B}\Psi^* = b_{11}\langle \cdot, \delta \rangle \delta + b_{12}\langle \cdot, \delta' \rangle \delta + b_{21}\langle \cdot, \delta \rangle \delta' + b_{22}\langle \cdot, \delta' \rangle \delta'.$$

⁴From the point of view of the simplest relations between coefficients of singular potentials and parameters of BVS.

In the case under consideration, $A = -\frac{d^2}{dx^2} + I$, $\mathcal{D}(A) = W_2^2(\mathbb{R})$, where $W_2^2(\mathbb{R})$ is the Sobolev space; $A_{\text{sym}} = \left(-\frac{d^2}{dx^2} + I\right) \upharpoonright_{\{u(x) \in W_2^2(\mathbb{R}) \mid u(0)=u'(0)=0\}}$ and $A_{\text{sym}} = A_N$, where a subspace N of $L_2(\mathbb{R})$ is the linear span of functions

$$\eta_1(x) = A^{-1}\delta = \frac{1}{2}e^{-|x|}, \quad \eta_2(x) = A^{-1}\delta'(x) = -\frac{\text{sign } x}{2}e^{-|x|}. \quad (2.20)$$

Further $A_N^* f(x) = -f''(x) + f(x)$ ($f(x) \in \mathcal{D}(A_N^*) = W_2^2(\mathbb{R}) \dot{+} N = W_2^2(\mathbb{R} \setminus \{0\})$), where the symbol $f''(x)$ means the second derivative (pointwise) of $f(x)$ except the point $x = 0$.

It follows from the description of $\mathcal{D}(A_N^*)$ that any function $f \in \mathcal{D}(A_N^*)$ and its derivative f' have right(left)-side limits at the point 0. Thus, the expressions

$$g_r = \frac{g(+0) + g(-0)}{2}, \quad g_s = g(+0) - g(-0), \quad (g = f \text{ or } g = f') \quad (2.21)$$

are well-posed. To obtain a regularization of (2.19) it suffices to extend the distributions δ and δ' onto $\mathcal{D}(A_N^*)$. The most physically reasonable way, based on the extension of δ by the continuity and parity onto $W_2^2(\mathbb{R} \setminus \{0\})$ and preserving the initial homogeneity of δ' with respect to scaling transformations [2], leads to the following extensions⁵:

$$\langle f, \delta \rangle = f_r, \quad \langle f, \delta' \rangle = -f'_r \quad (f(x) \in W_2^2(\mathbb{R} \setminus \{0\})).$$

These extensions can also be determined by the general formula (2.13), if we set $\mathbf{R} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$. In this case, $\Psi_{\mathbf{R}}^* f = \begin{pmatrix} f_r \\ -f'_r \end{pmatrix}$ and the corresponding boundary operators $\Gamma_0^{\mathbf{R}}$ and Γ_1 in the BVS $(\mathbb{C}^2, \Gamma_0^{\mathbf{R}}, \Gamma_1)$ determined by (2.15) have the form

$$\Gamma_0^{\mathbf{R}} f(x) = \begin{pmatrix} f_r \\ -f'_r \end{pmatrix}, \quad \Gamma_1 f(x) = \begin{pmatrix} f'_s \\ f_s \end{pmatrix} \quad \forall f(x) \in W_2^2(\mathbb{R} \setminus \{0\}). \quad (2.22)$$

Here the operator $\Gamma_0^{\mathbf{R}} f$ turns out to be the mean value of $f(x)$ and $-f'(x)$ at the origin and Γ_1 characterizes the jumps of $f(x)$ and its derivative at the origin.

Taking into account the fact that the operator $A^+ = -\frac{d^2}{dx^2} + I$ acts on $f(x) \in W_2^2(\mathbb{R} \setminus \{0\})$ by the rule $A^+ f(x) = -\frac{d^2}{dx^2} f(x) + f(x)$, where the action of $-\frac{d^2}{dx^2} f(x)$ is understood in the distributional sense, i.e.,

$$-\frac{d^2}{dx^2} f(x) = -f''(x) - f'_s \delta(x) - f_s \delta'(x)$$

and employing Theorem 2.2 we obtain a description of self-adjoint realizations $A_{\mathbf{B}, \mathbf{R}}$ of (2.19) that are defined by the rule $A_{\mathbf{B}, \mathbf{R}} f(x) = -f''(x)$,

$$f(x) \in \mathcal{D}(A_{\mathbf{B}, \mathbf{R}}) = \left\{ f(x) \in W_2^2(\mathbb{R} \setminus \{0\}) \mid \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} f_r \\ -f'_r \end{pmatrix} = \begin{pmatrix} f'_s \\ f_s \end{pmatrix} \right\}.$$

⁵We omit index ex for such natural extensions.

Example 2.2. Point interaction in \mathbb{R}^3 .

Let us consider the self-adjoint operator $A = -\Delta + \mu^2 I$, $\mathcal{D}(A) = W_2^2(\mathbb{R}^3)$ acting in $L_2(\mathbb{R}^3)$ and its restriction $A_{\text{sym}} = -\Delta + \mu^2 I \upharpoonright_{\{u(x) \in W_2^2(\mathbb{R}^3) | u(0)=0\}}$. It is easy to see that $A_{\text{sym}} = A_N$, where N is the linear span of $\frac{e^{-\mu|x|}}{|x|}$, $\mu > 0$. The triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$, where

$$\Gamma_1 f = \lim_{|x| \rightarrow 0} |x| f(x), \quad \Gamma_0 f = \lim_{|x| \rightarrow 0} \left(f(x) - (\Gamma_1 f) \frac{e^{-\mu|x|}}{|x|} \right) \tag{2.23}$$

$(f(x) \in \mathcal{D}(A_N^*) = W_2^2(\mathbb{R}^3) \dot{+} N)$ forms a BVS of A_N . Moreover $\ker \Gamma_1 = \mathcal{D}(A)$.

It follows from (2.18) and (2.23) that the operators

$$A_b(u(x) + bu(0) \frac{e^{-\mu|x|}}{|x|}) = (-\Delta + \mu^2 I)u(x) \quad \forall u(x) \in W_2^2(\mathbb{R}^3)$$

are self-adjoint extensions of A_N . By virtue of the results of Subsection 2.2.2, the operators A_b can be considered as self-adjoint realizations of the heuristic expression $-\Delta + \mu^2 + b\langle \cdot, \delta^{\text{ex}} \rangle \delta(x)$, where $-\Delta$ is understood in the distributional sense and the extension $\delta^{\text{ex}}(x)$ of $\delta(x)$ is determined in terms of the boundary operators Γ_i as follows: $\langle f, \delta^{\text{ex}} \rangle = \Gamma_0 f$ ($f \in W_2^2(\mathbb{R}^3) \dot{+} N$).

3. The case of mixed singular perturbations. 3.1. The concept of quasi-BVS.

In the case of mixed singular perturbations, the operator A_N determined by (2.1) is nondensely defined and its adjoint operator A_N^* does not exist. Thus some modification of BVS is required to describe all self-adjoint extensions of A_N .

Let us suppose that there exists a real number $m > 1$ such that $N \cap \mathcal{D}(A^m) = \{0\}$. Then, the direct sum

$$\mathcal{L}_m := \mathcal{D}(A^m) \dot{+} N \tag{3.1}$$

is well defined and we can define on \mathcal{L}_m a quasi-adjoint operator $A_N^{(*)}$ by the rule

$$A_N^{(*)} f = A_N^{(*)}(u + \eta) = Au \quad \forall f = u + \eta \in \mathcal{L}_m \quad (u \in \mathcal{D}(A^m), \eta \in N). \tag{3.2}$$

Formula (3.2) is an analog of (2.2) for the adjoint operator A_N^* and we can use $A_N^{(*)}$ as an analog of the adjoint one.

It is easy to see that, in general, $A_N^{(*)}$ is not closable and it turns out to be closable only if A_N is densely defined.

The concept of quasi-adjoint operators allows one to modify Definition 2.1 and to extend it to the case of nondensely defined symmetric operators.

Definition 3.1. A triple $(\mathfrak{N}, \Gamma_0, \Gamma_1)$, where Γ_i are linear mappings of \mathcal{L}_m in an auxiliary Hilbert space \mathfrak{N} , is called a quasi-BVS of A_N if the abstract Green identity

$$(A_N^{(*)} f, g) - (f, A_N^{(*)} g) = (\Gamma_1 f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{N}} \quad \forall f, g \in \mathcal{L}_m \tag{3.3}$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{L}_m \rightarrow \mathfrak{N} \oplus \mathfrak{N}$ is surjective.

Proposition 3.1 [23]. The following assertions are true:

1. If A_N is densely defined, then an arbitrary BVS $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ of A_N also is a quasi-BVS of A_N .

2. If A_N is nondensely defined, then the triple $(N, \Gamma_0^R, \Gamma_1)$, where

$$\Gamma_0^R(u + \eta) = P_N Au + R\eta, \quad \Gamma_1(u + \eta) = -\eta \quad (u \in \mathcal{D}(A^m), \eta \in N), \tag{3.4}$$

is a quasi-BVS of A_N for any choice of self-adjoint operator R in N .

3. Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-BVS of A_N . Then the symmetric operator

$$A'_N = A_N^{(*)} \upharpoonright_{\mathcal{D}(A'_N)}, \quad \mathcal{D}(A'_N) = \ker \Gamma_0 \cap \ker \Gamma_1 \quad (3.5)$$

does not depend on the choice of quasi-BVS and its closure coincides with A_N .

Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-BVS of A_N . A unitary operator U acting in \mathfrak{N} is called *admissible* with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ if the equation

$$(I - U)\Gamma_0 f = i(I + U)\Gamma_1 f \quad \forall f \in \mathcal{D}(A_N) \cap \mathcal{L}_m \quad (3.6)$$

has only the trivial solution $\Gamma_0 f = \Gamma_1 f = 0$.

If A_N is densely defined, then $\mathcal{D}(A_N) \cap \mathcal{L}_m = \mathcal{D}(A_N) \cap \mathcal{D}(A^m) = \mathcal{D}(A'_N)$ and, by virtue of (3.5), any unitary operator U in \mathfrak{N} is admissible. Otherwise (A_N is nondensely defined),

$$\mathcal{D}(A_N) \cap \mathcal{L}_m = \mathcal{D}(A'_N) \dot{+} \mathcal{F}, \quad (3.7)$$

where $\dim \mathcal{F} = \dim(N \cap \mathcal{D}(A))$. Vectors $f \in \mathcal{F}$ have the form $f = u + \eta$, where η is an arbitrary element of $N \cap \mathcal{D}(A)$ and u is determined by η with the help of relation $P_N A(u + \eta) = 0$ (this determination is unique modulo $\mathcal{D}(A'_N)$).

It follows from (3.5) and (3.7) that the condition of admissibility takes away the lineal \mathcal{F} from the set of solutions of (3.6).

Theorem 3.1 (cf. Theorem 2.1). *Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a quasi-BVS of A_N . Then any self-adjoint extension \tilde{A} of A_N is the closure of the symmetric operator*

$$\tilde{A}' = A_N^{(*)} \upharpoonright_{\mathcal{D}(\tilde{A}')}, \quad \mathcal{D}(\tilde{A}') = \left\{ f \in \mathcal{D}(A_N^{(*)}) \mid (I - U)\Gamma_0 f = i(I + U)\Gamma_1 f \right\}, \quad (3.8)$$

where U is an admissible unitary operator with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$. Moreover, the correspondence $\tilde{A} \leftrightarrow U$ is a bijection between the set of all self-adjoint extensions of A_N and the set of all admissible unitary operators.

Proof. Let U be an admissible operator and let \tilde{A}' be the corresponding operator defined by (3.8). Since

$$(\Gamma_1 f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 g)_{\mathfrak{N}} = \frac{1}{2} \|(\Gamma_1 + i\Gamma_0)f\|_{\mathfrak{N}}^2 - \frac{1}{2} \|(\Gamma_1 - i\Gamma_0)g\|_{\mathfrak{N}}^2,$$

formula (3.3) implies that \tilde{A}' is a symmetric extension of A'_N . Furthermore, there exists a linear subspace \mathcal{M} of \mathcal{L}_m such that $\dim \mathcal{M} = \dim \mathfrak{N} = \dim N$ and

$$\mathcal{D}(\tilde{A}') = \mathcal{D}(A'_N) \dot{+} \mathcal{M}. \quad (3.9)$$

It follows from the property of admissibility of U and (3.9) that $\mathcal{M} \cap \mathcal{D}(A_N) = 0$. The latter relation and assertion 3 of Proposition 3.1 mean that \tilde{A}' is closable and its closure \tilde{A} is a symmetric operator defined by the formula

$$\tilde{A} = A_N^{(*)} \upharpoonright_{\mathcal{D}(\tilde{A})}, \quad \mathcal{D}(\tilde{A}) = \mathcal{D}(A_N) \dot{+} \mathcal{M}. \quad (3.10)$$

Since $\dim \mathcal{M} = \dim N$, the defect numbers of \tilde{A} in the upper (lower) half plane are equal to 0 and hence, \tilde{A} is a self-adjoint extension of A_N . Thus we show that the closure of \tilde{A}' defined by (3.8) is a self-adjoint extension of A_N .

Conversely, let \tilde{A} be a self-adjoint extension of A_N . It follows from Theorem 5.15 [22] (Chapter 1) that \tilde{A} is determined by (3.10), where $\mathcal{M} \subset \mathcal{L}_m$ and $\dim \mathcal{M} = \dim N$.

But then the symmetric operator $\tilde{A}' = \tilde{A} \upharpoonright_{\mathcal{D}(A) \cap \mathcal{L}_m}$ defined by (3.9) is an essentially self-adjoint restriction of \tilde{A} . The domain $\mathcal{D}(\tilde{A}') = \mathcal{D}(\tilde{A}) \cap \mathcal{L}_m$ admits the representation (3.8), where the admissibility of U follows from the relation $\mathcal{M} \cap \mathcal{D}(A_N) = 0$ and the unitarity of U follows from the property of \tilde{A} to be a self-adjoint operator.

Theorem 3.1 is proved.

Remark. If U is not admissible, then the domain $\mathcal{D}(\tilde{A}')$ of a symmetric operator \tilde{A}' defined by (3.8) has a nontrivial intersection with \mathcal{F} and \tilde{A}' is not closable.

By analogy with the densely defined case we can describe self-adjoint extensions of A_N as the closure of the symmetric operators

$$A'_B = A_N^{(*)} \upharpoonright_{\mathcal{D}(A'_B)}, \quad \mathcal{D}(A'_B) = \{f \in \mathcal{L}_m \mid B\Gamma_0 f = \Gamma_1 f\}, \quad (3.11)$$

where $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ is a quasi-BVS and B is a self-adjoint operator in \mathfrak{N} . In such a setting, the operator B is called *admissible* with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ if the equation $B\Gamma_0 f = \Gamma_1 f$ ($f \in \mathcal{D}(A_N) \cap \mathcal{L}_m$) has only the trivial solution $\Gamma_0 f = \Gamma_1 f = 0$.

Proposition 3.2 [23]. *If B is an admissible operator, then the closure of A'_B is a self-adjoint extension of A_N .*

A self-adjoint extension \tilde{A} of A_N can be represented as the closure of a symmetric operator A'_B defined by (3.11) if and only if $\mathcal{D}(\tilde{A}) \cap \ker \Gamma_0 = \mathcal{D}(A'_B)$.

Since (3.11) does not describe all self-adjoint extensions of A_N , a situation where any operator B is admissible in (3.11) is possible.

Proposition 3.3. *If $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ is a quasi-BVS of A_N such that $\ker \Gamma_0 \supset \mathcal{D}(A_N) \cap \mathcal{L}_m$, then the closure of A'_B defined by (3.11) is a self-adjoint extension of A_N for any self-adjoint operator B in \mathfrak{N} .*

Proof. If $\ker \Gamma_0 \supset \mathcal{D}(A_N) \cap \mathcal{L}_m$, then the equation $B\Gamma_0 f = \Gamma_1 f$ ($f \in \mathcal{D}(A_N) \cap \mathcal{L}_m$) has only the trivial solution $\Gamma_0 f = \Gamma_1 f = 0$ and hence, any self-adjoint operator B is admissible with respect to $(\mathfrak{N}, \Gamma_0, \Gamma_1)$.

Proposition 3.3 is proved.

Let us specify the obtained results and present more constructive condition of admissibility for the family of quasi-BVS $(N, \Gamma_0^R, \Gamma_1)$ determined by (3.4).

Proposition 3.4. 1. *A self-adjoint operator B acting in N is admissible with respect to $(N, \Gamma_0^R, \Gamma_1)$ if and only if the equation*

$$BP_N A \eta = (I + BR) \eta \quad \forall \eta \in N \cap \mathcal{D}(A) \quad (3.12)$$

has the unique solution $\eta = 0$.

2. *Formula (3.11) (where $\Gamma_0 = \Gamma_0^R$) determines self-adjoint extensions of A_N for any choice of B if and only if the operator R satisfies the relation $P_N A \eta = R \eta$ for all $\eta \in N \cap \mathcal{D}(A)$.*

Proof. Assertion 1 follows directly from (3.4) and the description of the elements of $\mathcal{F} \subset \mathcal{D}(A_N) \cap \mathcal{L}_m$. To establish assertion 2, it suffices to observe that $\Gamma_0^R f = P_N A u + R \eta = -P_N A \eta + R \eta$ for all elements $f = u + \eta \in \mathcal{F}$. Thus,

$$\ker \Gamma_0^R \supset \mathcal{F} \iff P_N A \eta = R \eta \quad \text{for all } \eta \in N \cap \mathcal{D}(A).$$

Employing now Proposition 3.3, we complete the proof.

Example 3.1. Let us consider a Schrödinger operator that is determined by analogy with (2.19), where δ' is replaced by a function $q \in L_2(\mathbb{R})$:

$$-\frac{d^2}{dx^2} + b_{11} \langle \cdot, \delta \rangle \delta + b_{12} \langle \cdot, q \rangle \delta + b_{21} \langle \cdot, \delta \rangle q + b_{22} \langle \cdot, q \rangle q. \quad (3.13)$$

In our case, $A = -\frac{d^2}{dx^2} + I$, $\mathcal{D}(A) = W_2^2(\mathbb{R})$ and the defect subspace $N \subset L_2(\mathbb{R})$ is the linear span of the functions $\eta_1(x) = A^{-1}\delta = \frac{1}{2}e^{-|x|}$, $\eta_2(x) = A^{-1}q(x)$.

For the sake of simplicity, we assume that the function $q(x)$ coincides with a fundamental solution $m_{2k}(x)$, $k \geq 1$, of the equation $\left(-\frac{d^2}{dx^2} + I\right)^k m_{2k}(x) = \delta$. In this case, $\eta_1 = m_2$, $\eta_2 = m_{2k+2}$.

Let us fix $m = k + 1$, then, according to (3.1), $\mathcal{L}_m = W_2^{2k+2}(\mathbb{R}) \dot{+} N \subset W_2^{2k+2}(\mathbb{R} \setminus \{0\})$. It is easy to see that an arbitrary function $f \in \mathcal{L}_m$ admits the representation

$$f(x) = u(x) - f'_s m_2(x) - f_s^{[2k+1]} m_{2k+2},$$

where $u \in W_2^{2k+2}(\mathbb{R})$ and f'_s and $f_s^{[2k+1]}$ mean the jumps of the functions $f'(x)$ and $f^{[2k+1]}(x)$ at the point $x = 0$. Here, $f^{[2k+1]}(x) := \frac{d}{dx} \left(-\frac{d^2}{dx^2} + I\right)^k f(x)$, $x \neq 0$.

By the direct verification, we get that the triple $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0 f(x) = \begin{pmatrix} f(0) \\ (f, m_2) \end{pmatrix}, \quad \Gamma_1 f(x) = \begin{pmatrix} f'_s \\ f_s^{[2k+1]} \end{pmatrix} \quad \forall f(x) \in \mathcal{L}_m$$

is a quasi-BVS of A_N .

In our case, all conditions of Proposition 3.3 are satisfied and, hence, the restriction of $A_N^{(*)}$ ($A_N^{(*)} f(x) = -f''(x) + f(x)$, $x \neq 0$) onto the collection of functions $f \in \mathcal{L}_m$ that are specified by the boundary conditions

$$f'_s = b_{11} f(0) + b_{12}(f, m_2), \quad f_s^{[2k+1]} = b_{21} f(0) + b_{22}(f, m_2)$$

is an essentially self-adjoint operator in $L_2(\mathbb{R})$. The closure of such an operator has the form $A_q + I$, where A_q is a self-adjoint realization of the heuristic expression (3.13). The operator A_q can be interpreted as the Schrödinger operator with nonlocal point interaction [6]. Its domain $\mathcal{D}(A_q)$ consists of all functions $f \in W_2^2(\mathbb{R} \setminus \{0\})$ that satisfy the boundary conditions $f_s = 0$, $f'_s = b_{11} f(0) + b_{12}(f, q)$ and the action of $A_q f$ is determined as follows:

$$A_q f = -f''(x) + b_{21} q(x) f(0) + b_{22}(f, q) q(x), \quad x \neq 0.$$

3.2. Quasi-BVS and finite rank regular perturbations. Here we are going to show that the concept of quasi-BVS enables one to describe finite rank regular perturbations of A in just the same way as finite rank purely singular perturbations. To illustrate this point, we consider the following one-dimensional regular perturbation:

$$A_\alpha = A + \alpha(\cdot, \psi)\psi, \quad \psi \in \mathcal{H}_s \setminus \mathcal{H}_{s+\epsilon} \quad (\forall \epsilon > 0). \tag{3.14}$$

The rank one operator $\alpha(\cdot, \psi)\psi$ is a bounded operator in \mathcal{H} and the operator A_α is self-adjoint on the domain $\mathcal{D}(A)$.

On the other hand, we can consider A_α and A as two self-adjoint extensions of the symmetric nondensely defined operator (cf. (2.1))

$$A_N = A \upharpoonright_{\mathcal{D}(A_N)}, \quad \mathcal{D}(A_N) = \{u \in \mathcal{D}(A) \mid (u, \psi) = (Au, A^{-1}\psi) = 0\}. \tag{3.15}$$

Here N is the linear span of $\eta = A^{-1}\psi$ (i.e., $N = \langle \eta \rangle$) and $\eta \in \mathcal{H}_{s+2} \setminus \mathcal{H}_{s+2+\epsilon}$.

Let us describe self-adjoint extensions of A_N . To do this, we fix $m > s + 2$ and consider the direct sum $\mathcal{L}_m = \mathcal{D}(A^m) \dot{+} \langle \eta \rangle$.

In what follows, without loss of generality we assume that $\|\eta\| = 1$. Then, any element $f \in \mathcal{L}_m$ admits the presentation $f = u + \beta\eta$, where $u \in \mathcal{D}(A^m)$ and $\beta \in \mathbb{C}$ and the operators Γ_0^R, Γ_1 defined by (3.4) have the form⁶

$$\Gamma_0^R(u + \beta\eta) = P_N Au + r\beta\eta = [(Au, \eta) + r\beta]\eta, \quad \Gamma_1(u + \beta\eta) = -\beta\eta,$$

where the parameter r is an arbitrary real number.

The triple $(N, \Gamma_0^R, \Gamma_1)$ is a quasi-BVS of A_N and Theorem 3.1 gives the description of all self-adjoint extensions of A_N . In particular, formula (3.11) (where $\Gamma_0 = \Gamma_0^R$) shows that the closure of operators

$$A'_b f = A'_b(u + \beta\eta) = Au, \quad \mathcal{D}(A'_b) = \{f = u + \beta\eta \mid b[(Au, \eta) + r\beta] = -\beta\} \quad (3.16)$$

are self-adjoint extensions of A_N and they coincide with operators A_α (see (3.14)) if we put

$$b = \frac{\alpha}{1 + \alpha[(A\eta, \eta) - r]}.$$

In particular, if $r = (A\eta, \eta)$, then $b = \alpha$.

4. Finite rank singular perturbations of A in spaces of A -scale. Let p be a fixed integer ($p \in \mathbb{N}$). Since \mathcal{H}_p is a Hilbert space, all known results on finite rank perturbations of A can automatically be reformulated for its image $A \upharpoonright_{\mathcal{D}(A^{p/2+1})}$ acting in \mathcal{H}_p as a self-adjoint operator. However, the specific of \mathcal{H}_p as a space of the A -scale (1.1) enables one to get a lot of new nontrivial results (see, e.g., [5, 26], where the spectral analysis of Schrödinger operators with point interactions in the Sobolev spaces $W_2^p(\mathbb{R}^d)$ was carried out). The aim of this section is to generalize the results of [5, 26] for the abstract case of a self-adjoint operator acting in \mathcal{H}_p .

4.1. Construction of BVS for powers of A_N . Let N be a finite dimensional subspace of \mathcal{H} such that $N \cap \mathcal{D}(A) = \{0\}$ and let A_N be the corresponding symmetric densely defined operator constructed by N (see (2.1)).

The following statement shows that an arbitrary power of A_N is a symmetric restriction of the same power of A defined by the special choice of a defect subspace \widetilde{M} in \mathcal{H} .

Lemma 4.1. *For any $p \in \mathbb{N}$, $A_N^{p+1} := (A_N)^{p+1}$ is a symmetric densely defined operator in \mathcal{H} and $A_N^{p+1} = (A^{p+1})_{\widetilde{M}}$, where $\widetilde{M} = N \dot{+} A^{-1}N \dot{+} \dots \dot{+} A^{-p}N$ and*

$$(A^{p+1})_{\widetilde{M}} = A^{p+1} \upharpoonright_{\mathcal{D}((A^{p+1})_{\widetilde{M}})},$$

$$\mathcal{D}((A^{p+1})_{\widetilde{M}}) = \left\{ u \in \mathcal{D}(A^{p+1}) \mid (A^{p+1}u, m) = 0 \quad \forall m \in \widetilde{M} \right\}.$$

Proof. Since $\mathcal{D}(A^{p+1}) \cap \widetilde{M} = \{0\}$, the operator $(A^{p+1})_{\widetilde{M}}$ is densely defined. To prove $A_N^{p+1} = (A^{p+1})_{\widetilde{M}}$ it suffices to observe that $\mathcal{D}(A_N^{p+1}) = \mathcal{D}((A^{p+1})_{\widetilde{M}})$.

Lemma 4.1 is proved.

The next statement gives a convenient algorithm for the construction of BVS of A_N^{p+1} starting from a fixed BVS of A_N .

⁶We use the notation r instead of R to emphasize that R is an operator multiplication by a real number r .

Theorem 4.1. Let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a BVS of A_N and let $p \in \mathbb{N}$. Then the triple $(\oplus \mathfrak{N}^{p+1}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$, where $\oplus \mathfrak{N}^{p+1} := \underbrace{\mathfrak{N} \oplus \mathfrak{N} \oplus \dots \oplus \mathfrak{N}}_{p+1 \text{ times}}$ and

$$\tilde{\Gamma}_0 f = \begin{pmatrix} \Gamma_0 f \\ \Gamma_0 A_N^* f \\ \vdots \\ \Gamma_0 (A_N^*)^p f \end{pmatrix}, \quad \tilde{\Gamma}_1 f = \begin{pmatrix} \Gamma_1 (A_N^*)^p f \\ \Gamma_1 (A_N^*)^{p-1} f \\ \vdots \\ \Gamma_1 f \end{pmatrix} \quad \forall f \in \mathcal{D}((A_N^*)^{p+1}) \quad (4.1)$$

is a BVS of A_N^{p+1} .

Proof. It follows from Lemma 4.1 that $((A_N)^{p+1})^* = (A_N^*)^{p+1}$. Hence, the operators $\tilde{\Gamma}_i$ are well defined on $\mathcal{D}(((A_N)^{p+1})^*) = \mathcal{D}((A_N^*)^{p+1})$. Furthermore employing (2.3) and (4.1) we directly verify the following equality for any $f, g \in \mathcal{D}((A_N^*)^{p+1})$:

$$\begin{aligned} & ((A_N^*)^{p+1} f, g) - (f, (A_N^*)^{p+1} g) = ((A_N^*)^{p+1} f, g) - ((A_N^*)^p f, A_N^* g) + \\ & \quad + ((A_N^*)^p f, A_N^* g) - ((A_N^*)^{p-1} f, (A_N^*)^2 g) + \dots \\ & \quad \dots + (A_N^* f, (A_N^*)^p g) - (f, (A_N^*)^{p+1} g) = \\ & = (\Gamma_1 (A_N^*)^p f, \Gamma_0 g)_{\mathfrak{N}} - (\Gamma_0 (A_N^*)^p f, \Gamma_1 g)_{\mathfrak{N}} + (\Gamma_1 (A_N^*)^{p-1} f, \Gamma_0 (A_N^*)^2 g)_{\mathfrak{N}} - \\ & - (\Gamma_0 (A_N^*)^{p-1} f, \Gamma_1 (A_N^*)^2 g)_{\mathfrak{N}} + \dots + (\Gamma_1 f, \Gamma_0 (A_N^*)^p g)_{\mathfrak{N}} - (\Gamma_0 f, \Gamma_1 (A_N^*)^p g)_{\mathfrak{N}} = \\ & = (\tilde{\Gamma}_1 f, \tilde{\Gamma}_0 g)_{\oplus \mathfrak{N}^{p+1}} - (\tilde{\Gamma}_0 f, \tilde{\Gamma}_1 g)_{\oplus \mathfrak{N}^{p+1}}. \end{aligned}$$

To prove that $(\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ maps $\mathcal{D}((A_N^*)^{p+1})$ onto $(\oplus \mathfrak{N}^{p+1}) \oplus (\oplus \mathfrak{N}^{p+1})$ some auxiliary preparations are required.

At first, the property of $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ to be a BVS of A_N and (2.2) yield

$$\mathcal{D}(A_N^{p+1}) = \ker \tilde{\Gamma}_0 \cap \ker \tilde{\Gamma}_1. \quad (4.2)$$

Further, since $\mathcal{D}(A_N^p)$ is dense in \mathcal{H} and $\dim N < \infty$, the relation $P_N \mathcal{D}(A_N^p) = N$ (P_N is the orthoprojector onto N in \mathcal{H}) holds for any $p \in \mathbb{N}$. This equality enables one to verify (with the use of (2.1)) that $A^{-1} \mathcal{D}(A_N^p) + \mathcal{D}(A_N) \supset A^{-1} N$. But then recalling that $\mathcal{D}(A) = \mathcal{D}(A_N) \dot{+} A^{-1} N$ we get

$$A^{-1} \mathcal{D}(A_N^p) + \mathcal{D}(A_N) + N = \mathcal{D}(A) \dot{+} N = \mathcal{D}(A_N^*). \quad (4.3)$$

Let us prove the surjective property of the map $(\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ for $p = 1$. To do this we present an arbitrary vectors $\tilde{F}_0, \tilde{F}_1 \in \oplus \mathfrak{N}^2 = \mathfrak{N} \oplus \mathfrak{N}$ as the vector columns $\tilde{F}_i = (F_{i0}, F_{i1})^t$ ($i = 0, 1$ and t denotes the transposition). Then equations $\tilde{\Gamma}_i f = \tilde{F}_i$ ($f \in \mathcal{D}((A_N^*)^2)$) are equivalent to the following system of equations:

$$\Gamma_i f = F_{i0}, \quad \Gamma_i A_N^* f = F_{i1}, \quad f \in \mathcal{D}((A_N^*)^2) \quad i = 0, 1. \quad (4.4)$$

Since $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ is a BVS of A_N , there exists $g' \in \mathcal{D}(A_N^*)$ such that

$$\Gamma_i g' = F_{i0}, \quad i = 0, 1. \quad (4.5)$$

It is important that such g' is not defined uniquely. Precisely, by virtue of (4.2), any $g = g' + u$, where $u \in \mathcal{D}(A_N)$ satisfy (4.5).

Let us consider the element $f = A^{-1}g + \eta = A^{-1}g' + A^{-1}u + \eta$, where $u \in \mathcal{D}(A_N)$ and $\eta \in N$ are arbitrary elements. Clearly, $f \in \mathcal{D}((A_N^*)^2)$ and, by (4.5), $\Gamma_i A_N^* f = F_{i1}$, $i = 0, 1$.

Taking into account the definition of f , we can rewrite the rest equations of (4.4) as follows:

$$\Gamma_0(A^{-1}u + \eta) = F_{00} - \Gamma_0 A^{-1}g', \quad \Gamma_1(A^{-1}u + \eta) = F_{10} - \Gamma_1 A^{-1}g',$$

where $u \in \mathcal{D}(A_N)$ and $\eta \in N$ play the role of ‘free’ variables. Employing now (4.3) for $p = 1$ and recalling the equality $\mathcal{D}(A_N) = \ker \Gamma_0 \cap \ker \Gamma_1$ we conclude that the latter two equations have a solution for a certain choice of vectors $u = u_s$ and $\eta = \eta_s$. So, we prove that $f = A^{-1}g' + A^{-1}u_s + \eta_s$ is a solution of (4.4). Hence, $(\tilde{F}_0, \tilde{F}_1)$ maps $\mathcal{D}((A_N^*)^2)$ onto $(\oplus \mathfrak{N}^2) \oplus (\oplus \mathfrak{N}^2)$.

The general case $p \in \mathbb{N}$ is verified by the induction.

Theorem 4.1 is proved.

Example 4.1. Let $\mathcal{H} = L_2(\mathbb{R})$, $A = -\frac{d^2}{dx^2} + I$, $\mathcal{D}(A) = W_2^2(\mathbb{R})$ and let A_N and $(\mathbb{C}^2, \Gamma_0^R, \Gamma_1)$ be a symmetric operator and its BVS, respectively, that are defined in Example 2.1 (see (2.22)). In this case, $\mathcal{D}(A_N^*) = W_2^2(\mathbb{R} \setminus \{0\})$ and $A_N^* f(x) = -d^2 f(x)/dx^2 + f(x)$ ($f(x) \in W_2^2(\mathbb{R} \setminus \{0\})$, $x \neq 0$).

Let $p \in \mathbb{N}$. Then $A_N^{p+1} = \left(-\frac{d^2}{dx^2} + I\right)^{p+1}$,

$$\mathcal{D}(A_N^{p+1}) = \{u(x) \in W_2^{2p+2}(\mathbb{R}) \mid u(0) = u'(0) = \dots = u^{(2p)}(0) = u^{(2p+1)}(0) = 0\}$$

and $(A_N^*)^{p+1} f(x) = \left(-\frac{d^2}{dx^2} + I\right)^{p+1} f(x)$ ($x \neq 0$) for all $f(x) \in W_2^{2p+2}(\mathbb{R} \setminus \{0\})$.

To simplify the notation we will use the following symbol for quasi-derivatives of $f(x) \in W_2^{2p+2}(\mathbb{R} \setminus \{0\})$:

$$f^{[2k]}(x) := \left(-\frac{d^2}{dx^2} + I\right)^k f(x), \quad f^{[2k+1]}(x) := \frac{d}{dx} f^{[2k]}(x), \quad k \in \mathbb{N} \cup 0.$$

Thus $(A_N^*)^{p+1} f(x) = f^{[2p+2]}(x)$.

According to Theorem 4.1 and (2.22), a triple $(\mathbb{C}^{2p+2}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$, where

$$\tilde{\Gamma}_0 f = \begin{pmatrix} f_r \\ -f_r^{[1]} \\ \vdots \\ f_r^{[2p]} \\ -f_r^{[2p+1]} \end{pmatrix}, \quad \tilde{\Gamma}_1 f = \begin{pmatrix} f_s^{[2p+1]} \\ f_s^{[2p]} \\ \vdots \\ f_s^{[1]} \\ f_s \end{pmatrix} \quad (f(x) \in W_2^{2p+2}(\mathbb{R} \setminus \{0\}))$$

is a BVS of A_N^{p+1} . Here the indexes r and s mean, respectively, the mean value and the jump at $x = 0$ of the corresponding quasi-derivative $f^{[\tau]}(x)$ (see (2.21)). The Green identity related to $(\mathbb{C}^{2p+2}, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$ has the form

$$(f^{[2p+2]}, g)_{L_2(\mathbb{R})} - (f, g^{[2p+2]})_{L_2(\mathbb{R})} = \sum_{\tau=0}^{2p+1} (-1)^\tau f_r^{[\tau]} \overline{g_s^{[2p+1-\tau]}} - \sum_{\tau=0}^{2p+1} (-1)^\tau f_s^{[2p+1-\tau]} \overline{g_r^{[\tau]}}$$

where f and g are arbitrary functions from $W_2^{2p+2}(\mathbb{R} \setminus \{0\})$ [26].

4.2. Construction of quasi-BVS for a symmetric operator A_M in \mathcal{H}_p . As was noted above, the self-adjoint operator $A_p := A \upharpoonright_{\mathcal{D}(A^{p/2+1})}$ acting in \mathcal{H}_p can be considered as an image of the initial operator $A_0 := A$ in \mathcal{H}_p . In this case $\mathcal{D}(A_p) = \mathcal{D}(A^{p/2+1})$. By analogy with (2.1), we fix a finite dimensional subspace M of \mathcal{H}_p and determine a symmetric operator

$$A_M = A_p \upharpoonright_{\mathcal{D}(A_M)}, \quad \mathcal{D}(A_M) = \{u \in \mathcal{D}(A_p) \mid (A_p u, m)_p = 0 \quad \forall m \in M\} \quad (4.6)$$

acting in \mathcal{H}_p . In this subsection, we will consider the case where

$$M = \sum_{k=0}^{p/2} \dot{+} A^{-p+k} N := A^{-p} N \dot{+} A^{-p+1} N \dot{+} \dots \dot{+} A^{-\frac{p}{2}} N. \quad (4.7)$$

Here p is assumed to be *even* and N is a finite dimensional subspace of \mathcal{H} such that $N \cap \mathcal{D}(A) = \{0\}$.

For such a choice of M the definition (4.6) of A_M can be rewritten as follows: $A_M = A \upharpoonright_{\mathcal{D}(A_M)}$,

$$\mathcal{D}(A_M) = \{u \in \mathcal{D}(A^{p/2+1}) \mid P_N A u = P_N A^2 u = \dots = P_N A^{p/2+1} u = 0\}, \quad (4.8)$$

where P_N is the orthoprojector onto N in \mathcal{H} or, that is equivalent,

$$A_M = A_N \upharpoonright_{\mathcal{D}(A_M)}, \quad \mathcal{D}(A_M) = \mathcal{D}(A_N^{p/2+1}). \quad (4.9)$$

Thus the operator A_M is closely related to A_N defined by (2.1).

It follows from (4.7) that $M \cap \mathcal{D}(A^{p/2+1}) \supset A^{-p} N \neq \{0\}$. Hence, A_M is a non-densely defined symmetric operator in \mathcal{H}_p and for it we can construct a quasi-BVS only. To do this, we chose $m = (p + 1)/(p/2 + 1)$. Then $\mathcal{D}(A_p^m) = \mathcal{D}(A^{p+1})$, the direct sum $\mathcal{L}_m = \mathcal{D}(A_p^m) \dot{+} M = \mathcal{D}(A^{p+1}) \dot{+} M$ is well posed and we can define the action of $A_M^{(*)} f$ on any element $f = u + m \in \mathcal{L}_m$ by the formula (cf. (3.2))

$$A_M^{(*)} f = A_M^{(*)} (u + m) = A_p u = Au \quad \forall u \in \mathcal{D}(A^{p+1}) \quad \forall m \in M. \quad (4.10)$$

Theorem 4.2. *Let A_N be defined by (2.1) and let $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ be a BVS of A_N such that $\ker \Gamma_1 = \mathcal{D}(A)$. Then the triple $(\oplus \mathfrak{N}^{p/2+1}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$, where $\oplus \mathfrak{N}^{p/2+1} = \underbrace{\mathfrak{N} \oplus \mathfrak{N} \oplus \dots \oplus \mathfrak{N}}_{p/2+1 \text{ times}}$ and*

$$\widehat{\Gamma}_0 f = \begin{pmatrix} \Gamma_0 f \\ \Gamma_0 A_N^* f \\ \vdots \\ \Gamma_0 (A_N^*)^{\frac{p}{2}} f \end{pmatrix}, \quad \widehat{\Gamma}_1 f = \begin{pmatrix} \Gamma_1 (A_N^*)^p f \\ \Gamma_1 (A_N^*)^{p-1} f \\ \vdots \\ \Gamma_1 (A_N^*)^{\frac{p}{2}} f \end{pmatrix} \quad \forall f \in \mathcal{L}_m \quad (4.11)$$

is a quasi-BVS of the symmetric operator A_M in \mathcal{H}_p . In particular, the Green identity

$$(A_M^{(*)}f, g)_p - (f, A_M^{(*)}g)_p = (\widehat{\Gamma}_1 f, \widehat{\Gamma}_0 g)_{\oplus \mathfrak{N}^{p/2+1}} - (\widehat{\Gamma}_0 f, \widehat{\Gamma}_1 g)_{\oplus \mathfrak{N}^{p/2+1}} \quad (4.12)$$

is true for any $f, g \in \mathcal{L}_m = \mathcal{D}(A^{p+1}) \dot{+} M$.

Proof. It follows from Lemma 4.1 that $\mathcal{D}((A_N^*)^{p+1}) = \mathcal{D}(A^{p+1}) \dot{+} \widetilde{M}$ and $(A_N^*)^{p+1}(u + \widetilde{m}) = A^{p+1}u$, where $u \in \mathcal{D}(A^{p+1})$ and $\widetilde{m} \in \widetilde{M}$. By virtue of (4.7), $M = \widetilde{M} \cap \mathcal{H}_p$. Thus, the latter relations and (4.10) imply that

$$A_M^{(*)}f = A^{-p}(A_N^*)^{p+1}f \quad (4.13)$$

for any $f \in \mathcal{L}_m = \mathcal{D}(A_M^{(*)}) = \mathcal{D}((A_N^*)^{p+1}) \cap \mathcal{H}_p$.

Using the assumption that $\ker \Gamma_1 = \mathcal{D}(A)$ and relations (4.1), (4.13), we verify the abstract Green identity for any $f, g \in \mathcal{L}_m$:

$$\begin{aligned} ((A_N^*)^{p+1}f, g) - (f, (A_N^*)^{p+1}g) &= (A^{-p}(A_N^*)^{p+1}f, g)_p - (f, A^{-p}(A_N^*)^{p+1}g)_p = \\ &= (A_M^{(*)}f, g)_p - (f, A_M^{(*)}g)_p = (\widehat{\Gamma}_1 f, \widehat{\Gamma}_0 g)_{\oplus \mathfrak{N}^{p/2+1}} - (\widehat{\Gamma}_0 f, \widehat{\Gamma}_1 g)_{\oplus \mathfrak{N}^{p/2+1}}. \end{aligned}$$

Let F_0, F_1 be an arbitrary elements from $\oplus \mathfrak{N}^{p/2+1}$. Since $\oplus \mathfrak{N}^{p/2+1}$ can be embedded into $\oplus \mathfrak{N}^{p+1}$ as a subspace $(\oplus \mathfrak{N}^{p/2+1}) \oplus \underbrace{0 \oplus \dots \oplus 0}_{p/2 \text{ times}}$, the elements F_i belong to $\oplus \mathfrak{N}^{p+1}$

and have the representations:

$$F_0 = \left(\underbrace{\eta_1^0, \eta_2^0, \dots, \eta_{p/2+1}^0}_{p+1 \text{ times}}, 0, \dots, 0 \right), \quad F_1 = \left(\underbrace{\eta_1^1, \eta_2^1, \dots, \eta_{p/2+1}^1}_{p+1 \text{ times}}, 0, \dots, 0 \right).$$

Since $(\oplus \mathfrak{N}^p, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1)$ is a BVS of A_N^{p+1} constructed in Theorem 4.1, there exists $f \in \mathcal{D}((A_N^*)^{p+1})$ such that $\widetilde{\Gamma}_0 f = F_0$ and $\widetilde{\Gamma}_1 f = F_1$. Furthermore, it follows from (4.1) and the choice of F_1 that $\Gamma_1 f = \dots = \Gamma_1 (A_N^*)^{\frac{p}{2}-1} f = 0$. These equalities and condition $\ker \Gamma_1 = \mathcal{D}(A)$ mean that $f \in \mathcal{D}(A^{p/2}) = \mathcal{H}_p$. But then, the description of \mathcal{L}_m in (4.13) implies that $f \in \mathcal{L}_m$. To complete the proof of Theorem 4.2 it suffices to observe that $\widetilde{\Gamma}_i f = \widehat{\Gamma}_i f$, where $\widehat{\Gamma}_i$ have the form (4.11).

Example 4.2 (cf. Example 4.1). Let $\mathcal{H} = L_2(\mathbb{R})$, $A = -\frac{d^2}{dx^2} + I$, $\mathcal{D}(A) = W_2^2(\mathbb{R})$ and let A_N be a symmetric operator defined in Example 2.1. In this case, \mathcal{H}_p coincides with the Sobolev space $W_2^p(\mathbb{R})$, $p \in \mathbb{N}$. Further, by (4.9), the symmetric operator A_M acting in $W_2^p(\mathbb{R})$ has the form $A_M = -\frac{d^2}{dx^2} + I$,

$$\mathcal{D}(A_M) = \left\{ u(x) \in W_2^{p+2}(\mathbb{R}) \mid u(0) = u'(0) = \dots = u^{(p)}(0) = u^{(p+1)}(0) = 0 \right\}.$$

Here, the defect subspace M is determined by (4.7) and it coincides with a linear span of fundamental solutions $m_{2j}(x)$ of the equation $\left(-\frac{d^2}{dx^2} + I\right)^j m_{2j}(x) = \delta$ and their derivatives $m_{2j-1}(x) = m'_{2j}(x)$ that belong to \mathcal{H}_p . Precisely, M is a linear span of the functions

$$m_{2j}(x) = \frac{1}{(j-1)!2^j} \sum_{r=0}^{j-1} C_{2j-2-r}^r (2j-3-2r)!! |x|^r e^{-|x|}, \quad m_{2j-1}(x) = m'_{2j}(x),$$

where index j runs the set $\{p/2 + 1, p/2 + 2, \dots, p + 1\}$.

The operator A_M is nondensely defined in $W_2^p(\mathbb{R})$. Its quasi-adjoint $A_M^{(*)}$ (see (4.10) and (4.13)) is defined on the domain

$$\mathcal{D}(A_M^{(*)}) = \mathcal{L}_m = W_2^{2p+2}(\mathbb{R}) \dot{+} M = W_2^p(\mathbb{R}) \cap W_2^{2p+2}(\mathbb{R} \setminus \{0\})$$

and acts as follows: $A_M^{(*)}f(x) = A^{-p}f^{[2p+2]}(x)$ for all $f(x) \in W_2^p(\mathbb{R}) \cap W_2^{2p+2}(\mathbb{R} \setminus \{0\})$.

Let $(\mathbb{C}^2, \Gamma_0^{\mathbf{R}}, \Gamma_1)$ be a BVS of A_N defined by (2.22). Obviously, $\ker \Gamma_1 = \mathcal{D}(A)$. According to Theorem 4.2, the triple $(\mathbb{C}^{p+2}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$, where

$$\widehat{\Gamma}_0 f = \begin{pmatrix} f_r \\ -f_r^{[1]} \\ \vdots \\ f_r^{[p]} \\ -f_r^{[p+1]} \end{pmatrix}, \quad \widehat{\Gamma}_1 f = \begin{pmatrix} f_s^{[2p+1]} \\ f_s^{[2p]} \\ \vdots \\ f_s^{[p+1]} \\ f_s^{[p]} \end{pmatrix} \tag{4.14}$$

$(f(x) \in W_2^p(\mathbb{R}) \cap W_2^{2p+2}(\mathbb{R} \setminus \{0\}))$ is a quasi-BVS of the symmetric operator A_M acting in $W_2^p(\mathbb{R})$. The corresponding Green identity has the form

$$(A_M^{(*)}f, g)_{W_2^p(\mathbb{R})} - (f, A_M^{(*)}g)_{W_2^p(\mathbb{R})} = \sum_{\tau=0}^{p+1} (-1)^\tau f_r^{[\tau]} \overline{g_s^{[2p+1-\tau]}} - \sum_{\tau=0}^{p+1} (-1)^\tau f_s^{[2p+1-\tau]} \overline{g_r^{[\tau]}}$$

where f and g are arbitrary functions from $W_2^p(\mathbb{R}) \cap W_2^{2p+2}(\mathbb{R} \setminus \{0\})$ [26].

4.3. Description of self-adjoint extensions of A_M in \mathcal{H}_p . A quasi-BVS $(\oplus \mathfrak{N}^{p/2+1}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ of A_M presented in Theorem 4.2 enables one to get a simple description of self-adjoint extensions of A_M in \mathcal{H}_p .

Lemma 4.2. *Let $\widehat{\Gamma}_0$ be determined by (4.11). Then $\ker \widehat{\Gamma}_0 \supset \mathcal{D}(A_M) \cap \mathcal{L}_m$.*

Proof. Obviously $\ker \Gamma_0 \supset \mathcal{D}(A_N)$ (since $(\mathfrak{N}, \Gamma_0, \Gamma_1)$ is a BVS of A_N). But then relations (4.9) and (4.11) give that $\widehat{\Gamma}_0 f = 0$ for any $f \in \mathcal{D}(A_M) \cap \mathcal{L}_m$.

Lemma 4.2 is proved.

By Lemma 4.2, the equation $B\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f$ ($f \in \mathcal{D}(A_M) \cap \mathcal{L}_m$) has only the trivial solution $\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f = 0$ for an arbitrary self-adjoint operator B acting in $\oplus \mathfrak{N}^{p/2+1}$. So, any B is admissible with respect to the quasi-BVS $(\oplus \mathfrak{N}^{p/2+1}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$.

The next statement is a direct consequence of Proposition 3.3.

Theorem 4.3. *For an arbitrary self-adjoint operator B in $\oplus \mathfrak{N}^{p/2+1}$ the formula*

$$A'_B = A_M^{(*)} \upharpoonright_{\mathcal{D}(A'_B)}, \quad \mathcal{D}(A'_B) = \{f \in \mathcal{L}_m \mid B\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f\}, \tag{4.15}$$

determines an essentially self-adjoint operator in \mathcal{H}_p and its closure is a self-adjoint extension of A_M in \mathcal{H}_p .

Example 4.3. Let us preserve the notation of Example 4.2 and let \mathbf{B} be an arbitrary Hermitian matrix of the order $p + 2$. Then, according to Theorem 4.3, the closure of the operator $A'_\mathbf{B}$ defined by the rule: $A'_\mathbf{B}f(x) = A^{-p}f^{[2p+2]}(x)$, where $f(x)$ belong to $W_2^p(\mathbb{R}) \cap W_2^{2p+2}(\mathbb{R} \setminus \{0\})$ and satisfy the condition

$$\mathbf{B}\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f \quad (\widehat{\Gamma}_i \text{ are defined by (4.14)})$$

is a self-adjoint extension A_B of the nondensely defined operator $A_M = -\frac{d^2}{dx^2} + I$, $\mathcal{D}(A_M) = \{u(x) \in W_2^{p+2}(\mathbb{R}) \mid u(0) = \dots = u^{(p+1)}(0) = 0\}$ acting in $W_2^p(\mathbb{R})$. The operator A_B can be interpreted as a one-dimensional Schrödinger operator with point interaction in the Sobolev space $W_2^p(\mathbb{R})$ [26].

4.4. Realization of self-adjoint extensions of A_M in \mathcal{H}_p by additive perturbations.

In mathematical physics, the self-adjoint extensions $A_{B,R}$ of A_N described in Theorem 2.2 appear naturally as self-adjoint realizations of the additive purely singular perturbations (1.6) in \mathcal{H} . Our aim is to give a similar interpretation for self-adjoint extensions A_B of A_M defined by (4.15) in the space \mathcal{H}_p . In what follows, without loss of generality, we assume that an auxiliary Hilbert space \mathfrak{N} in $(\oplus \mathfrak{N}^{p/2+1}, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ coincides with \mathbb{C}^n (here $n = \dim \mathfrak{N}$). So, $\oplus \mathfrak{N}^{p/2+1} = \mathbb{C}^{n(p/2+1)}$. In this case the operator B in (4.15) is given by an Hermitian matrix \mathbf{B} of the order $n(p/2 + 1)$.

It follows from the relation $\ker \Gamma_1 = \mathcal{D}(A)$ and equalities (2.2), (4.11) that $\ker \widehat{\Gamma}_1 = \mathcal{D}(A^{p+1})$. Hence, the restriction $\widehat{\Gamma}_1 \upharpoonright_M$ determines a one-to-one correspondence between M and $\mathbb{C}^{n(p/2+1)}$. Thus $(\widehat{\Gamma}_1 \upharpoonright_M)^{-1}$ exists and $(\widehat{\Gamma}_1 \upharpoonright_M)^{-1}$ maps $\mathbb{C}^{n(p/2+1)}$ onto M .

Putting $\Psi d := -A^{p+1}(\widehat{\Gamma}_1 \upharpoonright_M)^{-1}d$, where $d \in \mathbb{C}^{n(p/2+1)}$, we determine an injective linear mapping of $\mathbb{C}^{n(p/2+1)}$ to \mathcal{H}_{-p-2} such that $\mathcal{R}(\Psi) \cap \mathcal{H} = \{0\}$.

Let us determine its adjoint $\Psi^* : \mathcal{H}_{p+2} \rightarrow \mathbb{C}^{n(p/2+1)}$ by the formula

$$\langle u, \Psi d \rangle = (\Psi^* u, d)_{\mathbb{C}^{n(p/2+1)}} \quad \forall u \in \mathcal{H}_{p+2} = \mathcal{D}(A^{p/2+1}) \quad \forall d \in \mathbb{C}^{n(p/2+1)}. \quad (4.16)$$

To describe Ψ^* we set $\psi_j = \Psi e_j$, where $\{e_j\}_1^{n(p/2+1)}$ is the canonical basis of $\mathbb{C}^{n(p/2+1)}$. Setting $f = u \in \mathcal{D}(A^{p+1})$ and $g = A^{-p-1}\psi_j = A^{-p-1}\Psi e_j = -(\widehat{\Gamma}_1 \upharpoonright_M)^{-1}e_j$ in the Green identity (4.12), using (4.10), and recalling that $\ker \widehat{\Gamma}_1 = \mathcal{D}(A^{p+1})$, we get

$$\langle u, \psi_j \rangle = (A^{p+1}u, A^{-p-1}\psi_j) = -(\widehat{\Gamma}_0 u, \widehat{\Gamma}_1 g)_{\mathbb{C}^{n(p/2+1)}} = (\widehat{\Gamma}_0 u, e_j)_{\mathbb{C}^{n(p/2+1)}}.$$

The latter relation and (4.16) imply that

$$\Psi^* u = \begin{pmatrix} \langle u, \psi_1 \rangle \\ \vdots \\ \langle u, \psi_{n(p/2+1)} \rangle \end{pmatrix} = \widehat{\Gamma}_0 u \quad (4.17)$$

for ‘smooth’ vectors $u \in \mathcal{D}(A^{p+1}) = \mathcal{H}_{2p+2}$. The continuation of Ψ^* onto $\mathcal{D}(A^{p/2+1}) = \mathcal{H}_{p+2}$ is obtained by the closure.

Let us consider the formal expression

$$A_p + \sum_{i,j=1}^{n(p/2+1)} b_{ij} \langle \cdot, \psi_j \rangle \psi_i = A_p + \Psi \mathbf{B} \Psi^*, \quad (4.18)$$

where $\mathbf{B} = (b_{ij})_{ij}^{n(p/2+1)}$ is an Hermitian matrix of the order $n(p/2 + 1)$ and $A_p = A \upharpoonright_{\mathcal{D}(A^{p/2+1})}$ is a self-adjoint operator in \mathcal{H}_p .

In general, the singular elements ψ_j belong to \mathcal{H}_{-p-2} and hence, they are well defined on $u \in \mathcal{H}_{p+2}$. For this reason it is natural to consider the ‘potential’ $V = \Psi \mathbf{B} \Psi^*$ in (4.18) as a singular perturbation of the ‘free’ operator A_p in \mathcal{H}_p and, reasoning by

analogy with Subsection 2.2.1, to give a meaning of the formal expression (4.18) as a self-adjoint operator extension \tilde{A} of the symmetric operator (cf. (2.9))

$$A_{\text{sym}} := A_p \upharpoonright_{\mathcal{D}(A_{\text{sym}})}, \quad \mathcal{D}(A_{\text{sym}}) = \{u \in \mathcal{D}(A^{p/2+1}) \mid \Psi^*u = 0\}$$

acting in \mathcal{H}_p .

It follows from (4.8) and (4.16) that $A_{\text{sym}} = A_M$. So, in contrast to the operator $A_{\text{sym}} = A_N$ defined by (2.9), the operator $A_{\text{sym}} = A_M$ is *non-densely defined*. Therefore, a modification of the Albeverio–Kurasov approach (see Subsection 2.2.1) is required to describe self-adjoint extensions of A_M by additive mixed singular perturbation (4.18).

First of all we restrict (4.18) to the set $\mathcal{D}(A^{p+1})$ and define the action of (4.18) on vectors from the domain of definition $\mathcal{D}(A_M^{(*)}) = \mathcal{D}(A^{p+1}) \dot{+} M$ of the quasi-adjoint operator $A_M^{(*)}$ (in other words, we construct a regularization $A_p^+ + \Psi \mathbf{B} \Psi_{\mathbf{R}}^*$ of (4.18) defined on $\mathcal{D}(A^{p+1}) \dot{+} M$).

Relation (4.17) means that the extension $\Psi_{\mathbf{R}}^*$ can naturally be defined by the boundary operator $\widehat{\Gamma}_0$. Namely,

$$\Psi_{\mathbf{R}}^* f = \begin{pmatrix} \langle f, \psi_1^{\text{ex}} \rangle \\ \vdots \\ \langle f, \psi_{n(p/2+1)}^{\text{ex}} \rangle \end{pmatrix} := \widehat{\Gamma}_0 f \quad \forall f \in \mathcal{D}(A^{p+1}) \dot{+} M. \quad (4.19)$$

The extension A_p^+ of A_p can be defined by analogy with (2.12). Precisely, we only need to indicate the action of A_p^+ on M . Assuming that $A_p^+ \upharpoonright_M$ acts as the isometric mapping A^{p+1} in A -scale (see Subsection 2.2), we get

$$A_p^+ f = A_p u + A^{p+1} \mathbf{m} = A_M^{(*)} f + A^{p+1} \mathbf{m} \quad \forall f = u + \mathbf{m} \in \mathcal{D}(A_M^{(*)}). \quad (4.20)$$

After such a preparation work, the operator realization \tilde{A} of (4.18) in \mathcal{H}_p is determined by the formula (cf. (2.11))

$$\begin{aligned} \tilde{A} &= [A_p^+ + \Psi \mathbf{B} \Psi_{\mathbf{R}}^*] \upharpoonright_{\mathcal{D}(\tilde{A})}, \\ \mathcal{D}(\tilde{A}) &= \{f \in \mathcal{D}(A^{p+1}) \dot{+} M \mid A_p^+ f + \Psi \mathbf{B} \Psi_{\mathbf{R}}^* f \in \mathcal{H}_p\}. \end{aligned} \quad (4.21)$$

Theorem 4.4. *Let \mathbf{B} be an Hermitian matrix of the order $n(p/2 + 1)$. Then the operator \tilde{A} is essentially self-adjoint in \mathcal{H}_p and it can be also defined by the formula*

$$A'_{\mathbf{B}} = A_M^{(*)} \upharpoonright_{\mathcal{D}(A'_{\mathbf{B}})}, \quad \mathcal{D}(A'_{\mathbf{B}}) = \{f \in \mathcal{D}(A^{p+1}) \dot{+} M \mid \mathbf{B} \widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f\}. \quad (4.22)$$

Proof. By the definition of Ψ , $A^{p+1} \mathbf{m} = -\Psi \widehat{\Gamma}_1 \mathbf{m} = -\Psi \widehat{\Gamma}_1 f$ for any $\mathbf{m} \in M$ and $f = u + \mathbf{m}$ ($u \in \mathcal{D}(A^{p+1})$). The obtained expression, (4.19), and (4.20) yield

$$[A_p^+ + \Psi \mathbf{B} \Psi_{\mathbf{R}}^*] f = A_M^{(*)} f + \Psi [\mathbf{B} \widehat{\Gamma}_0 - \widehat{\Gamma}_1] f \quad (\forall f \in \mathcal{D}(A^{p+1}) \dot{+} M). \quad (4.23)$$

The latter equality means $f \in \mathcal{D}(\tilde{A}) \iff \mathbf{B} \widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f$ (since $\mathcal{R}(\Psi) \cap \mathcal{H} = \{0\}$ and hence, $\mathcal{R}(\Psi) \cap \mathcal{H}_p = \{0\}$). Combining this fact with (4.21) – (4.23) we conclude that \tilde{A} coincides with $A'_{\mathbf{B}}$. The property of the operator \tilde{A} to be essentially self-adjoint follows from Theorem 4.3.

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