UDC 517.42

H. Wei (Guangxi Teacher's College, Zhongshan, China),
Y. Wang (Zhongshan Univ., China)

c^*-SUPPLEMENTED SUBGROUPS AND p-NILPOTENCY OF FINITE GROUPS*

c^*-ДОПОВНЕНІ ПІДГРУПИ ТА p-НІЛЬПОТЕНТНІСТЬ СКІНЧЕННИХ ГРУП

A subgroup H of a finite group G is said to be $c^\star$-supplemented in G if there exists a subgroup K such that $G = HK$ and $H \cap K$ is permutable in G. It is proved that a finite group G which is S_4-free is p-nilpotent if $N_G(P)$ is p-nilpotent and, for all $x \in G \backslash N_G(P)$, every minimal subgroup of $P \cap P^x \cap G^{\mathcal{N}_p}$ is $c^\star$-supplemented in P and, if $p = 2$, one of the following conditions holds: (a) Every cyclic subgroup of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order 4 is $c^\star$-supplemented in P; (b) $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq Z(P \cap G^{\mathcal{N}_p})$; (c) P is quaternion-free, where P a Sylow p-subgroup of G and $G^{\mathcal{N}_p}$ the p-nilpotent residual of G. That will extend and improve some known results.

Підгрупа H скінченної групи G називається $c^\star$-доповненою в G, якщо існує підгрупа K така, що $G = HK$ та $H \cap K$ є перестановочною в G. Доведено, що скінченна група G, яка є S_4-вільною, є p-нільпотентною, якщо $N_G(P)$ p-нільпотентна і для всіх $x \in G \backslash N_G(P)$ кожна мінімальна підгрупа із $P \cap P^x \cap G^{\mathcal{N}_p}$ є $c^\star$-доповненою в P та, якщо $p = 2$, виконується одна з наступних умов: а) кожна циклічна підгрупа порядку 4 із $P \cap P^x \cap G^{\mathcal{N}_p}$ є $c^\star$-доповненою в P; b) $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq Z(P \cap \cap G^{\mathcal{N}_p})$; c) P є безкватерніонною, де P — силовська p-підгрупа групи G та $G^{\mathcal{N}_p}$ — p-нільпотентний залишок групи G. Тим самим поширено та покращено деякі відомі результати.

1. Introduction. All groups considered will be finite. For a formation $\mathcal{F}$ and a group G, there exists a smallest normal subgroup of G, called the $\mathcal{F}$-residual of G and denoted by $G^{\mathcal{F}}$, such that $G/G^{\mathcal{F}} \in \mathcal{F}$ (refer [1]). Throughout this paper, $\mathcal{N}$ and $\mathcal{N}_p$ will denote the classes of nilpotent groups and p-nilpotent groups, respectively. A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group of order 8.

General speaking, a group with a p-nilpotent normalizer of the Sylow p-subgroup need not be a p-nilpotent group. However, if one adds some embedded properties on the Sylow p-subgroup, he may obtain his desired result. For example, Wielandt proved that a group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent [2]. Ballester-Bolinches and Esteban-Romero showed that a group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent [3]. Moreover, Guo and Shum obtained a similar result by use of the permutability of some minimal subgroups of Sylow p-subgroups [4].

In the present paper, we will push further the studies. First, we introduce the $c^\star$-supplementation of subgroups which is a unify and generalization of the permutability and the c-supplementation [5, 6] of subgroups. Then, we give several sufficient conditions for a group to be p-nilpotent by using the $c^\star$-supplementation of some minimal p-subgroups. In detail, we obtain the following main theorem:

Theorem 1.1. *Let G be a group such that G is S_4-free and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if $N_G(P)$ is p-nilpotent and, for all $x \in G \backslash N_G(P)$, one of the following conditions holds*:

(a) *Every cyclic subgroup of $P \cap P^x \cap G^{\mathcal{N}_p}$ of order p or 4 (if $p = 2$) is $c^\star$-supplemented in P*;

* Project supported by NSFC (10571181), NSF of Guangxi (0447038) and Guangxi Education Department.

(b) *Every minimal subgroup of* $P \cap P^x \cap G^{\mathcal{N}_p}$ *is* $c^\star$*-supplemented in* P *and, if* $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq Z(P \cap G^{\mathcal{N}_p})$;

(c) *Every minimal subgroup of* $P \cap P^x \cap G^{\mathcal{N}_p}$ *is* $c^\star$*-supplemented in* P *and* P *is quaternion-free.*

Following the proof of Theorem 1.1, we can prove the Theorem 1.2. It can be considered as an extension of the above-mentioned result of Ballester-Bolinches and Esteban-Romero.

Theorem 1.2. *Let* P *be a Sylow* p*-subgroup of a group* G. *Then* G *is* p*-nilpotent if* $N_G(P)$ *is* p*-nilpotent and, for all* $x \in G \backslash N_G(P)$, *one of the followings holds*:

(a) *Every cyclic subgroup of* $P \cap P^x \cap G^{\mathcal{N}_p}$ *of order* p *or* 4 *(if* $p = 2$*) is permutable in* P;

(b) *Every minimal subgroup of* $P \cap P^x \cap G^{\mathcal{N}_p}$ *is permutable in* P *and, if* $p = 2$, $[\Omega_2(P \cap P^x \cap G^{\mathcal{N}_p}), P] \leq Z(P \cap G^{\mathcal{N}_p})$;

(c) *Every minimal subgroup of* $P \cap P^x \cap G^{\mathcal{N}_p}$ *is permutable in* P *and, if* $p = 2$, P *is quaternion-free.*

As an application of Theorem 1.1, we get the following theorem:

Theorem 1.3. *Let* G *be a group such that* G *is* S_4*-free and let* P *be a Sylow* p*-subgroup of* G, *where* p *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *Then* G *is* p*-nilpotent if one of the following conditions holds*:

(a) *Every cyclic subgroup of* $P \cap G^{\mathcal{N}_p}$ *of order* p *or* 4 *(if* $p = 2$*) is* $c^\star$*-supplemented in* $N_G(P)$;

(b) *Every minimal subgroup of* $P \cap G^{\mathcal{N}_p}$ *is* $c^\star$*-supplemented in* $N_G(P)$ *and, if* $p = 2$, P *is quaternion-free.*

Our results improve and extend the following theorems of Guo and Shum [7, 8].

Theorem 1.4 ([7], Main theorem)**.** *Let* G *be a group such that* G *is* S_4*-free and let* P *be a Sylow* p*-subgroup of* G, *where* p *is the smallest prime divisor of* $|G|$. *If every minimal subgroup of* $P \cap G^{\mathcal{N}}$ *is c-supplemented in* $N_G(P)$ *and, when* $p = 2$, P *is quaternion-free, then* G *is* p*-nilpotent.*

Theorem 1.5 ([8], Main theorem)**.** *Let* P *be a Sylow* p*-subgroup of a group* G, *where* p *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *If every minimal subgroup of* $P \cap G^{\mathcal{N}}$ *is permutable in* $N_G(P)$ *and, when* $p = 2$, *either every cyclic subgroup of* $P \cap G^{\mathcal{N}}$ *of order* 4 *is permutable in* $N_G(P)$ *or* P *is quaternion-free, then* G *is* p*-nilpotent.*

2. Preliminaries. Recall that a subgroup H of a group G is *permutable* (or *quasinormal*) in G if H permutes with every subgroup of G. H is *c-supplemented* in G if there exists a subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 \leq H_G = \mathrm{Core}_G(H)$ [5, 6]; in this case, if we denote $K = H_G K_1$, then $G = HK$ and $H \cap K = H_G$; of course, $H \cap K$ is permutable in G. Based on this observation, we introduce:

Definition 2.1. *A subgroup* H *of a group* G *is said to be* $c^\star$*-supplemented in* G *if there exists a subgroup* K *of* G *such that* $G = HK$ *and* $H \cap K$ *is a permutable subgroup of* G. *We say that* K *is a* $c^\star$*-supplement of* H *in* G.

It is clear from Definition 2.1 that a permutable or c-supplemented subgroup must be a $c^\star$-supplemented subgroup. But the converses are not true. For example, let $G = A_4$, the alternating group of degree 4. Then any Sylow 3-subgroup of G is c-supplemented but not permutable in G. If we take $G = \langle a, b | a^{16} = b^4 = 1, ba = a^3 b\rangle$, then $b^2(a^i b^j) = (a^i b^j)^{9+2((-1)^j - 1)} b^2$. Hence $\langle b^2 \rangle$ is permutable in G. However, $\langle b^2 \rangle$ is not c-supplemented in G as $\langle b^2 \rangle$ is in $\Phi(G)$ and not normal in G.

The following lemma on $c^{\star}$-supplemented subgroups is crucial in the sequel. The proof is a routine check, we omit its detail.

Lemma 2.1. *Let H be a subgroup of a group G. Then*:

(1) *If H is $c^{\star}$-supplemented in G, $H \leq M \leq G$, then H is $c^{\star}$-supplemented in M*;

(2) *Let $N \lhd G$ and $N \leq H$. Then H is $c^{\star}$-supplemented in G if and only if H/N is $c^{\star}$-supplemented in G/N*;

(3) *Let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. If H is $c^{\star}$-supplemented in G, then HN/N is $c^{\star}$-supplemented in G/N*;

(4) *Let $L \leq G$ and $H \leq \Phi(L)$. If H is $c^{\star}$-supplemented in G, then H is permutable in G.*

Lemma 2.2. *Let c be an element of a group G of order p, where p is a prime divisor of $|G|$. If $\langle c \rangle$ is permutable in G, then c is centralized by every element of G of order p or 4 (if $p = 2$).*

Proof. Let x be an element of G with order p or 4 (if $p = 2$). By the hypotheses, $\langle x \rangle \langle c \rangle = \langle c \rangle \langle x \rangle$. Clearly, if x is of order p, then c is centralized by c. Now assume that $p = 2$ and x is of order 4. If $[c, x] \neq 1$, then $c^{-1}xc = x^{-1}$ and $(xc)^2 = 1$. Furthermore, $|\langle x \rangle \langle c \rangle| \leq 4$, of course, $[c, x] = 1$, a contradiction. We are done.

Lemma 2.3 ([9], Lemma 2)**.** *Let $\mathcal{F}$ be a saturated formation. Assume that G is a non-$\mathcal{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = F(G)M$, where $F(G)$ is the Fitting subgroup of G. Then*:

(1) *$G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G*;

(2) *$G^{\mathcal{F}}$ is a p-group for some prime p*;

(3) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$*;

(4) *$G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

Lemma 2.4 ([10], Lemma 2.8(1))**.** *Let M be a maximal subgroup of a group G and let P be a normal p-subgroup of G such that $G = PM$, where p a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.5 ([11], Theorem 2.8)**.** *If a solvable group G has a Sylow 2-subgroup P which is quaternion-free, then $P \cap Z(G) \cap G^{\mathcal{N}} = 1$.*

Lemma 2.6. *Let G be a group and let p be a prime number dividing $|G|$ with $(|G|, p-1) = 1$. Then*:

(1) *If N is normal in G of order p, then N lies in $Z(G)$*;

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent*;

(3) *If M is a subgroup of G of index p, then M is normal in G.*

Proof. (1) Since $|\mathrm{Aut}(N)| = p - 1$ and $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, $|G/C_G(N)|$ must divide $(|G|, p-1) = 1$. It follows that $G = C_G(N)$ and $N \leq Z(G)$.

(2) Let $P \in \mathrm{Syl}_p(G)$ and $|P| = p^n$. Since P is cyclic, $|\mathrm{Aut}(P)| = p^{n-1}(p-1)$. Again, $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, so $|N_G(P)/C_G(P)|$ must divide $(|G|, p-1) = 1$. Thus $N_G(P) = C_G(P)$, and statement (2) follows by the well-known Burnside theorem.

(3) We may assume that $M_G = 1$ by induction. As everyone knows the result is true in the case where $p = 2$. So assume that $p > 2$ and consequently G is of odd order as $(|G|, p-1) = 1$. Now we know that G is solvable by the Odd Order Theorem. Let N be a minimal normal subgroup of G. Then N is an elementary abelian q-group for some prime q. It is obvious that $G = MN$ and $M \cap N$ is normal in G. Therefore $M \cap N = 1$

and $|N| = |G : M| = p$. Now $N \leq Z(G)$ by statement (1) and, of course, M is normal in G as desired.

3. Proofs of theorems.

***Proof of Theorem* 1.1.** Let G be a minimal counterexample. Then we have the following claims:

(1) M is p-nilpotent whenever $P \leq M < G$.

Since $N_M(P) \leq N_G(P)$, $N_M(P)$ is p-nilpotent. Let x be an element of $M \backslash N_M(P)$. Then, since $P \cap P^x \cap M^{\mathcal{N}_p} \leq P \cap P^x \cap G^{\mathcal{N}_p}$, every minimal subgroup of $P \cap P^x \cap M^{\mathcal{N}_p}$ is $c^\star$-supplemented in P by Lemma 2.1. If G satisfies (a), then every cyclic subgroup of $P \cap P^x \cap M^{\mathcal{N}_p}$ with order 4 is $c^\star$-supplemented in P. If G satisfies (b), then

$$[\Omega_2(P \cap P^x \cap M^{\mathcal{N}_p}), P] \leq Z(P \cap G^{\mathcal{N}_p}) \cap (P \cap M^{\mathcal{N}_p}) \leq Z(P \cap M^{\mathcal{N}_p}).$$

Now we see that M satisfies the hypotheses of the theorem. The minimality of G implies that M is p-nilpotent.

(2) $O_{p'}(G) = 1$.

If not, we consider $\overline{G} = G/N$, where $N = O_{p'}(G)$. Clearly $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$ is p-nilpotent, where $\overline{P} = PN/N$. For any $xN \in \overline{G} \backslash N_{\overline{G}}(\overline{P})$, since $\overline{G}^{\mathcal{N}_p} = G^{\mathcal{N}_p}N/N$ and $P \cap P^xN = P^{xn}$ for some $n \in N$, we have

$$\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p} = (P \cap P^{xn} \cap G^{\mathcal{N}_p}N)N/N = (P \cap P^{xn} \cap G^{\mathcal{N}_p})N/N.$$

Because $xN \in \overline{G} \backslash N_{\overline{G}}(\overline{P})$, $xn \in G \backslash N_G(P)$. Now let $\overline{P}_0 = P_0N/N$ be a minimal subgroup of $\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}$. We may assume that $P_0 = \langle y \rangle$, where y is an element of $P \cap P^{xn} \cap G^{\mathcal{N}_p}$ of order p. By the hypotheses, there exists a subgroup K_0 of P such that $P = P_0K_0$ and $P_0 \cap K_0$ is a permutable subgroup of P. It follows that $PN/N = (P_0N/N)(K_0N/N)$ and $(P_0N/N) \cap (K_0N/N) = (P_0 \cap K_0N)N/N$. If $P_0 \cap K_0N = P_0$ then $P_0 \leq P \cap K_0N = K_0$ and consequently $P_0 = P_0 \cap K_0$ is permutable in P. In this case, $\overline{P}_0$ is permutable in $\overline{P}$. If $P_0 \cap K_0N = 1$ then $\overline{P}_0$ is complemented in $\overline{P}$. Thus $\overline{P}_0$ is $c^\star$-supplemented in $\overline{P}$. Assume that G satisfies (a). Let $\overline{P}_1 = P_1N/N$ be a cyclic subgroup of $\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}$ of order 4. We may assume that $P_1 = \langle z \rangle$, where z is an element of $P \cap P^{xn} \cap G^{\mathcal{N}_p}$ of order 4. Since P_1 is $c^\star$-supplemented in P, $P = P_1K_1$ and $P_1 \cap K_1$ is permutable in P. We have $PN/N = (P_1N/N)(K_1N/N)$ and $(P_1N/N) \cap (K_1N/N) = (P_1 \cap K_1N)N/N$. If $P_1 \cap K_1N = 1$ then $\overline{P}_1$ is complemented in $\overline{P}$. If $P_1 \cap K_1N = \langle z^2 \rangle$, since $z^2 \leq \Phi(P)$ and $\langle z^2 \rangle$ is $c^\star$-supplemented in P, $\langle z^2 \rangle$ is permutable in P by Lemma 2.1. Furthermore, $\langle z^2 \rangle N/N$ is permutable in PN/N and $\overline{P}_1$ is $c^\star$-supplemented in $\overline{P}$. If $P_1 \cap K_1N = P_1$ then $P_1 = P_1 \cap K_1$ is permutable in P and $\overline{P}_1$ is permutable in $\overline{P}$. In a ward, $\overline{P}_1$ is $c^\star$-supplemented in $\overline{P}$. Now assume that G satisfies (b), then

$$\left[\Omega_2(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}), \overline{P}\right] = \left[\Omega_2(P \cap P^{xn} \cap G^{\mathcal{N}_p}), P\right] N/N \leq Z(P \cap G^{\mathcal{N}_p})N/N,$$

namely

$$\left[\Omega_2(\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathcal{N}_p}), \overline{P}\right] \leq Z(\overline{P} \cap \overline{G}^{\mathcal{N}_p}).$$

If G satisfies (c) then $\overline{P} \cong P$ is quaternion-free. Therefore $\overline{G} = G/N$ satisfies the hypotheses of the theorem. The choice of G implies that $\overline{G}$ is p-nilpotent and so is G, a contradiction.

(3) $G/O_p(G)$ is p-nilpotent and $C_G(O_p(G)) \leq O_p(G)$.

Suppose that $G/O_p(G)$ is not p-nilpotent. Then, by Frobenius' theorem (refer [12], Theorem 10.3.2), there exists a subgroup of P properly containing $O_p(G)$ such that its G-normalizer is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choice a subgroup P_1 of P such that $O_p(G) < P_1 < P$ and $N_G(P_1)$ is not p-nilpotent but $N_G(P_2)$ is p-nilpotent whenever $P_1 < P_2 \leq P$. Denote $H = N_G(P_1)$. It is obvious that $P_1 < P_0 \leq P$ for some Sylow p-subgroup P_0 of H. The choice of P_1 implies that $N_G(P_0)$ is p-nilpotent, hence $N_H(P_0)$ is also p-nilpotent. Take $x \in H \backslash N_H(P_0)$. Since $P_0 = P \cap H$, we have $x \in G \backslash N_G(P)$. Again,

$$P_0 \cap P_0^x \cap H^{\mathcal{N}_p} \leq P \cap P^x \cap G^{\mathcal{N}_p},$$

so every minimal subgroup of $P_0 \cap P_0^x \cap H^{\mathcal{N}_p}$ is $c^\star$-supplemented in P_0 by Lemma 2.1. If (a) is satisfied then every cyclic subgroup of $P_0 \cap P_0^x \cap H^{\mathcal{N}_p}$ of order 4 is $c^\star$-supplemented in P_0. If (b) is satisfied then

$$\left[\Omega_2(P_0 \cap P_0^x \cap H^{\mathcal{N}_p}), P_0\right] \leq Z(P \cap G^{\mathcal{N}_p}) \cap (P_0 \cap H^{\mathcal{N}_p}) \leq Z(P_0 \cap H^{\mathcal{N}_p}).$$

If (c) is satisfied then P_0 is quaternion-free. Therefore H satisfies the hypotheses of the theorem. The choice of G yields that H is p-nilpotent, which is contrary to the choice of P_1. Thereby $G/O_p(G)$ is p-nilpotent and G is p-solvable with $O_{p'}(G) = 1$. Consequently, we obtain $C_G(O_p(G)) \leq O_p(G)$ (refer [13], Theorem 6.3.2).

(4) $G = PQ$, where Q is an elementary abelian Sylow q-subgroup of G for a prime $q \neq p$. Moreover, P is maximal in G and $QO_p(G)/O_p(G)$ is minimal normal in $G/O_p(G)$.

For any prime divisor q of $|G|$ with $q \neq p$, since G is p-solvable, there exists a Sylow q-subgroup Q of G such that $G_0 = PQ$ is a subgroup of G [13] (Theorem 6.3.5). If $G_0 < G$, then, by (1), G_0 is p-nilpotent. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $G = PQ$ and so G is solvable. Now let $T/O_p(G)$ be a minimal normal subgroup of $G/O_p(G)$ contained in $O_{pp'}(G)/O_p(G)$. Then $T = O_p(G)(T \cap Q)$. If $T \cap Q < Q$, then $PT < G$ and therefore PT is p-nilpotent by (1). It follows that

$$1 < T \cap Q \leq C_G(O_p(G)) \leq O_p(G),$$

which is impossible. Hence $T = O_{pp'}(G)$ and $QO_p(G)/O_p(G)$ is an elementary abelian q-group complementing $P/O_p(G)$. This yields that P is maximal in G.

(5) $|P : O_p(G)| = p$.

Clearly, $O_p(G) < P$. Let P_0 be a maximal subgroup of P containing $O_p(G)$ and let $G_0 = P_0 O_{pp'}(G)$. Then P_0 is a Sylow p-subgroup of G_0. The maximality of P in G implies that either $N_G(P_0) = G$ or $N_G(P_0) = P$. If the latter holds, then $N_{G_0}(P_0) = P_0$. On the other hand, in view of (3), we have $G^{\mathcal{N}_p} \leq O_p(G)$, hence $P \cap P^x \cap G^{\mathcal{N}_p} = G^{\mathcal{N}_p}$ for every $x \in G$. Now it is easy to see that G_0 satisfies the hypotheses of the theorem. Thereby G_0 is p-nilpotent and $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $N_G(P_0) = G$ and $P_0 = O_p(G)$. This proves (5).

(6) $G = G^{\mathcal{N}_p} L$, where $L = \langle a \rangle [Q]$ is a non-abelian split extension of Q by a cyclic p-subgroup $\langle a \rangle$, $a^p \in Z(L)$ and the action of a (by conjugate) on Q is irreducible.

From (3) we see that $G^{\mathcal{N}_p} \leq O_p(G)$. Clearly, $T = G^{\mathcal{N}_p} Q \lhd G$. Let P_0 be a maximal subgroup of P containing $G^{\mathcal{N}_p}$. Then, by the maximality of P, either $N_G(P_0) = P$ or $N_G(P_0) = G$. If $N_G(P_0) = P$, then $N_M(P_0) = P_0$, where $M = P_0 T = P_0 Q$.

Evidently, $P_0 \cap P_0^x \cap M^{\mathcal{N}_p} \leq G^{\mathcal{N}_p}$ for all $x \in M \backslash N_M(P_0)$, hence M satisfies the hypotheses of the theorem. By the minimality of G, M is p-nilpotent. It follows that $T = G^{\mathcal{N}_p}Q = G^{\mathcal{N}_p} \times Q$ and so $Q \lhd G$, a contradiction. Thereby $N_G(P_0) = G$ and $P_0 \leq O_p(G)$. This infers from (5) that $O_p(G) = P_0$ and hence $P/G^{\mathcal{N}_p}$ is a cyclic group. Now applying the Frattini argument we have $G = G^{\mathcal{N}_p}N_G(Q)$. Therefore we may assume that $G = G^{\mathcal{N}_p}L$, where $L = \langle a \rangle[Q]$ is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-group $\langle a \rangle$. Noticing that $|P\colon O_p(G)| = p$ and $O_p(G) \cap N_G(Q) \lhd N_G(Q)$, we have $a^p \in Z(L)$. Also since P is maximal in G, $G^{\mathcal{N}_p}Q/G^{\mathcal{N}_p}$ is minimal normal in $G/G^{\mathcal{N}_p}$ and consequently a acts irreducibly on Q.

(7) $G^{\mathcal{N}_p}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

By Lemma 2.3 it will suffice to show that there exists a p-nilpotent maximal subgroup M of G such that $G = G^{\mathcal{N}_p}M$. In fact, let M be a maximal subgroup of G containing L. Then $M = L(M \cap G^{\mathcal{N}_p})$ and $G = G^{\mathcal{N}_p}M$. By Lemma 2.4, $M \cap G^{\mathcal{N}_p} \lhd G$, hence $M = (\langle a \rangle(M \cap G^{\mathcal{N}_p}))Q$. Write $P_0 = \langle a \rangle(M \cap G^{\mathcal{N}_p})$ and let M_0 be a maximal subgroup of M containing P_0. Then $M_0 = P_0(M_0 \cap Q)$ and $G^{\mathcal{N}_p}M_0 < G$. By applying (1) we know that $G^{\mathcal{N}_p}M_0$ is p-nilpotent, therefore

$$M_0 \cap Q \leq C_G(O_p(G)) \leq O_p(G).$$

It follows that $M_0 \cap Q = 1$ and so P_0 is maximal in M. In this case, if $P_0 \lhd M$, then $\langle a \rangle = P_0 \cap L \lhd L$, which is contrary to (6). Hence $N_M(P_0) = P_0$ and M satisfies the hypotheses of the theorem. The choice of G implies that M is p-nilpotent, as desired.

Without losing generality, we assume in the following that $P = G^{\mathcal{N}_p}\langle a \rangle$.

(8) If $G^{\mathcal{N}_p}$ has exponent p, then $G^{\mathcal{N}_p} \cap \langle a \rangle = 1$.

Assume on the contrary that $G^{\mathcal{N}_p} \cap \langle a \rangle \neq 1$ if $G^{\mathcal{N}_p}$ has exponent p. Then we can take an element c in $G^{\mathcal{N}_p} \cap \langle a \rangle$ such that c is of order p. Since P is not normal in G, $G^{\mathcal{N}_p} \cap \langle a \rangle < \langle a \rangle$. Consequently $c \in \langle a^p \rangle \leq \Phi(P)$ and $\langle c \rangle$ is permutable in P. By (6), (7) and Lemma 2.2, we see that c is centralized by both $G^{\mathcal{N}_p}$ and L, hence $c \in Z(G)$. If G satisfies (c) then, since $G^{\mathcal{N}_p} \leq G^{\mathcal{N}}$, $c = 1$ by Lemma 2.5, a contradiction. If G satisfies (a) or (b), we consider the factor group $\overline{G} = G/\langle c \rangle$. It is obvious that $N_{\overline{G}}(\overline{P}) = N_G(P)/\langle c \rangle$ is p-nilpotent, where $\overline{P} = P/\langle c \rangle$. Now let $\langle y \rangle\langle c \rangle/\langle c \rangle$ be a minimal subgroup of $G^{\mathcal{N}_p}/\langle c \rangle$, where $y \in G^{\mathcal{N}_p}$. Since y is of order p, by the hypotheses, $\langle y \rangle$ has a $c^\star$-supplement K in P. If $\langle y \rangle \cap K = 1$ then K is a maximal subgroup of P and $\langle c \rangle \leq K$. It follows that $P/\langle c \rangle = (\langle y \rangle\langle c \rangle/\langle c \rangle)(K/\langle c \rangle)$ with $\langle y \rangle\langle c \rangle/\langle c \rangle \cap K/\langle c \rangle = 1$. If $\langle y \rangle \cap K = \langle y \rangle$ then $\langle y \rangle$ is permutable in P and hence $\langle y \rangle\langle c \rangle/\langle c \rangle$ is permutable in $P/\langle c \rangle$. That is $\langle y \rangle\langle c \rangle/\langle c \rangle$ is $c^\star$-supplemented in $P/\langle c \rangle$, therefore $\overline{G}$ satisfies (a) or (b). The choice of G implies that $G/\langle c \rangle$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(9) The exponent of $G^{\mathcal{N}_p}$ is not p.

If not, $G^{\mathcal{N}_p}$ has exponent p. Let P_1 be a minimal subgroup of $G^{\mathcal{N}_p}$ not permutable in P. Then, by the hypotheses, there is a subgroup K_1 of P such that $P = P_1K_1$ and $P_1 \cap K_1 = 1$. In general, we may find minimal subgroups $P_1, P_2, \ldots, P_m$ of $G^{\mathcal{N}_p}$ and also subgroups $K_1, K_2, \ldots, K_m$ of P such that $P = P_iK_i$ and $P_i \cap K_i = 1$ for each i and every minimal subgroup of $G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m$ is permutable in P. Furthermore, we may assume that $P_i \leq K_1 \cap \ldots \cap K_{i-1}$, $i = 2, 3, \ldots, m$. Henceforth $K_1 \cap \ldots \cap K_{i-1} = P_i(K_1 \cap \ldots \cap K_i)$ for $i = 2, 3, \ldots, m$. It is easy to see that $G^{\mathcal{N}_p} \cap K_i$ is normal in P and $(G^{\mathcal{N}_p} \cap K_i)\langle a \rangle$ is a complement of P_i in P, so we may replace K_i by $(G^{\mathcal{N}_p} \cap K_i)\langle a \rangle$ and further assume that $\langle a \rangle \leq K_i$ for each i. Now, $K_1 \cap \ldots \cap K_m = (G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m)\langle a \rangle$. Since, for any $x \in G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m$, $\langle x \rangle\langle a \rangle = \langle a \rangle\langle x \rangle$, we have

$$x^a \in (G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m) \cap \langle x \rangle \langle a \rangle = \langle x \rangle.$$

This means that a induces a power automorphism of p-power order in the elementary abelian p-group $G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m$. Hence $[G^{\mathcal{N}_p} \cap K_1 \cap \ldots \cap K_m, a] = 1$ and $K_1 \cap \ldots \cap K_m$ is abelian.

Now we claim that p is even. If it is not the case, then, by [13] (Theorem 6.5.2), $K_1 \cap \ldots \cap K_m \leq O_p(G)$. Consequently, $P = G^{\mathcal{N}_p}(K_1 \cap \ldots \cap K_m) \leq O_p(G)$, a contradiction. We proceed now to consider the following two cases:

Case 1. $|\langle a \rangle| = 2^n, n > 1$.

Since $K_1 \cap \ldots \cap K_m$ is an abelian normal subgroup of P and $a \in K_1 \cap \ldots \cap K_m$, $\Phi(K_1 \cap \ldots \cap K_m) = \langle a^2 \rangle \lhd P$ and so $\Omega_1(\langle a^2 \rangle) = \langle c \rangle \leq Z(P)$, where $c = a^{2^{n-1}}$. Again, $c \in Z(L)$ by (6), so $c \in Z(G)$. If G satisfies (c) then we obtain $c = 1$ by Lemma 2.5, which is absurd. If G satisfies (a) or (b), then, with the same arguments to those used in (8), we may prove that $G/\langle c \rangle$ satisfies the hypotheses of the theorem. The minimality of G implies that $G/\langle c \rangle$ is 2-nilpotent and therefore G is also 2-nilpotent, a contradiction.

Case 2. $|\langle a \rangle| = 2$.

Since a acts irreducibly on Q, a is an involutive automorphism of Q; consequently, Q is a cyclic subgroup of order q and $b^a = b^{-1}$, where $Q = \langle b \rangle$. In this case, $G^{\mathcal{N}_2}$ is minimal normal in G. In fact, let N be a minimal normal subgroup of G contained in $G^{\mathcal{N}_2}$ and let $H = NL$. Since $N\langle a \rangle$ is maximal but not normal in H, we see that $N_H(N\langle a \rangle) = N\langle a \rangle$. Noticing that $N\langle a \rangle \cap H^{\mathcal{N}_2} \leq N$, every minimal subgroup of $N\langle a \rangle \cap$ $\cap H^{\mathcal{N}_2}$ is $c^\star$-supplemented in $N_H(N\langle a \rangle) = N\langle a \rangle$ by Lemma 2.1. If further $H < G$, then the choice of G implies that H is 2-nilpotent. Consequently, $NQ = N \times Q$ and so $1 \neq N \cap Z(P) \leq Z(G)$. The choice of N implies that $N = N \cap Z(P)$ is of order 2. This is contrary to Lemma 2.5 if G satisfies (c). Now assume that G satisfies (a) or (b). In this case, if $N \not\leq \Phi(P)$, then N has a complement to P. By applying Gaschütz Theorem [12] (I, 17.4), N also has a complement to G, say M. It follows that M is a normal subgroup of G. Furthermore, G/M is cyclic of order 2 and so $N \leq G^{\mathcal{N}_2} \leq M$, a contradiction. Hence $N \leq \Phi(P)$. Now we go to consider the factor group G/N. For any minimal subgroup $\langle y \rangle N/N$ of $(G/N)^{\mathcal{N}_2} = G^{\mathcal{N}_2}/N$, by the hypotheses, $P = \langle y \rangle K$ and $\langle y \rangle \cap K$ is permutable in P, where $y \in G^{\mathcal{N}_2}$. Since $N \leq K$, we have $P/N =$ $= (\langle y \rangle N/N)(K/N)$ and $(\langle y \rangle N/N) \cap (K/N) = (\langle y \rangle \cap K)N/N$ is permutable in P/N, so $\langle y \rangle N/N$ is $c^\star$-supplemented in P/N. This yields at once that G/N is 2-nilpotent and so is G. Hence $H = G$ and $G^{\mathcal{N}_2}$ must be a minimal normal subgroup of G; of course, $G^{\mathcal{N}_2}$ is an elementary abelian 2-group. Since $G^{\mathcal{N}_2} \cap N_G(Q) \lhd N_G(Q)$, we know that $G^{\mathcal{N}_2} \cap N_G(Q) = 1$ and so b acts fixed-point-freely on $G^{\mathcal{N}_2}$. We may assume that $N_1 =$ $= \{1, c_1, c_2, \ldots, c_q\}$ is a subgroup of $G^{\mathcal{N}_2}$ with $c_1 \in Z(P)$ and $b = (c_1, c_2, \ldots, c_q)$ is a permutation of the set $\{c_1, c_2, \ldots, c_q\}$. Noticing that $b^a = b^{-1}$ and $(c_1)^{a^{-1}ba} = (c_1)^{b^{-1}}$, $(c_2)^a = c_q$. By using $(b^i)^a = b^{-i}$ and $(c_1)^{a^{-1}b^i a} = (c_1)^{b^{-i}}$, we see that $(c_{i+1})^a =$ $= c_{q-i+1}$ for $i = 1, 2, \ldots, (q+1)/2$. Hence N_1 is normalized by both $G^{\mathcal{N}_2}$ and L and so N_1 is normal in G. The minimal normality of $G^{\mathcal{N}_2}$ implies that $G^{\mathcal{N}_2} = N_1$, thus we have $Z(P) = \{1, c_1\}$. Since $G^{\mathcal{N}_2} \cap K_1 \cap \ldots \cap K_m$ is centralized by both $G^{\mathcal{N}_2}$ and $\langle a \rangle$, we have $1 < G^{\mathcal{N}_2} \cap K_1 \cap \ldots \cap K_m \leq Z(P)$. In view of P is not abelian, we get $\Phi(P) = P' = Z(P)$, namely P is an extra-special 2-group. By applying Theorem 5.3.8 of [12], there exists some positive integer h such that $|P| = 2^{2h+1}$. Hence $|G^{\mathcal{N}_2}| = 2^{2h}$. However, $2^{2h} - 1 = (2^h + 1)(2^h - 1)$ and $q = 2^{2h} - 1$, hence $h = 1$, $q = 3$ and $|P| = 2^3$. Now we see that $L \cong S_3$ and $G^{\mathcal{N}_2}Q \cong A_4$, therefore $G \cong S_4$, which is contrary to the hypothesis on G.

(10) The final contradiction.

From (7) and (9) we see that $p = 2$ and the exponent of $G^{\mathcal{N}_2}$ is 4. By applying Lemma 2.3, $Z(G^{\mathcal{N}_2}) = \Phi(G^{\mathcal{N}_2})$ is an elementary abelian 2-group. If $\Phi(G^{\mathcal{N}_2}) \cap \langle a \rangle \neq 1$ then there exists an element c in $\Phi(G^{\mathcal{N}_2}) \cap \langle a \rangle$ such that c is of order 2. Since $\Phi(G^{\mathcal{N}_2}) \cap \langle a \rangle < \langle a \rangle$, we have $c \in \langle a^2 \rangle \leq Z(L)$. But c is also centralized by $G^{\mathcal{N}_2}$ by Lemma 2.2, so $c \in Z(G)$. If $\Phi(G^{\mathcal{N}_2}) \cap \langle a \rangle = 1$ then a induces a power automorphism of 2-power order in the elementary abelian 2-group $\Phi(G^{\mathcal{N}_2})$, hence $[\Phi(G^{\mathcal{N}_2}), a] = 1$. In view of Lemma 2.2, $\Phi(G^{\mathcal{N}_2})$ is also centralized by $G^{\mathcal{N}_2}$, hence $\Phi(G^{\mathcal{N}_2}) \leq Z(P)$. Furthermore, by the Frattini argument,

$$G = N_G(\Phi(G^{\mathcal{N}_2})) = C_G(\Phi(G^{\mathcal{N}_2}))N_G(P).$$

Noticing that $N_G(P) = P$ and $P \leq C_G(\Phi(G^{\mathcal{N}_2}))$, we get $C_G(\Phi(G^{\mathcal{N}_2})) = G$, namely $\Phi(G^{\mathcal{N}_2}) \leq Z(G)$. Thus we can also take an element c in $\Phi(G^{\mathcal{N}_2})$ such that c is of order 2 and $c \in Z(G)$. This is contrary to Lemma 2.5 if G satisfies (c). Now assume that G satisfies (a). Denote $N = \langle c \rangle$ and consider $\overline{G} = G/N$. It is clear that $N_{\overline{G}}(\overline{P}) = N_G(P)/N$ is 2-nilpotent because $N_G(P)$ is, where $\overline{P} = P/N$. For any $y \in G^{\mathcal{N}_2}$, since $\langle y \rangle$ is $c^\star$-supplemented in P, there exists a subgroup T of P such that $P = \langle y \rangle T$ and $\langle y \rangle \cap T$ is permutable in P. However, $y^2 \in \Phi(G^{\mathcal{N}_2})$, hence $\langle y^2 \rangle$ is permutable in P and $\langle y^2 \rangle T$ forms a group. Because $|P : \langle y^2 \rangle T| \leq 2$, $N \leq \langle y^2 \rangle T$. It follows that $P/N = (\langle y \rangle N/N)(\langle y^2 \rangle T/N)$ and

$$\langle y \rangle N/N \cap \langle y^2 \rangle T/N = \langle y^2 \rangle (\langle y \rangle \cap T)N/N$$

is permutable in P/N. This shows that $\overline{G}$ satisfies (a). Thereby $\overline{G}$ is 2-nilpotent and so is G, a contradiction. Finally we assume that G satisfies (b). Let M be a maximal subgroup of G containing L. Then M is 2-nilpotent by the proof of (7), hence $\Phi(G^{\mathcal{N}_2})Q$ is 2-nilpotent and $[\Phi(G^{\mathcal{N}_2}), Q] = 1$. Write $K = C_G(G^{\mathcal{N}_2}/\Phi(G^{\mathcal{N}_2}))$. Then, by the hypotheses, $P \leq K \lhd G$. The maximality of P yields that $P = K$ or $K = G$. If the former holds, then $G = N_G(P)$ is 2-nilpotent, a contradiction. If the latter holds, then $[G^{\mathcal{N}_2}, Q] \leq \Phi(G^{\mathcal{N}_2})$. This means that Q stabilizes the chain of subgroups $1 \leq \Phi(G^{\mathcal{N}_2}) \leq G^{\mathcal{N}_2}$. It follows from [13] (Theorem 5.3.2) that $[G^{\mathcal{N}_2}, Q] = 1$ and Q is normal in G, which is impossible. This completes our proof.

***Proof of Theorem* 1.3.** By applying Theorem 1.1, we only need to prove that $N_G(P)$ is p-nilpotent.

If $N_G(P)$ is not p-nilpotent, then $N_G(P)$ has a minimal non-p-nilpotent subgroup (that is, every proper subgroup of a group is p-nilpotent but itself is not p-nilpotent) H. By results of Itô [2] (IV, 5.4) and Schmidt [2] (III, 5.2), H has a normal Sylow p-subgroup H_p and a cyclic Sylow q-subgroup H_q such that $H = [H_p]H_q$. Moreover, H_p is of exponent p if $p > 2$ and of exponent at most 4 if $p = 2$. On the other hand, the minimality of H implies that $H^{\mathcal{N}_p} = H_p$. Let P_0 be a minimal subgroup of H_p and let K_0 be a $c^\star$-supplement of P_0 in H. Then $H = P_0K_0$ and $P_0 \cap K_0$ is permutable in H. If $P_0 \cap K_0 = 1$ then K_0 is maximal in H of index p. By applying Lemma 2.6 we see that K_0 is normal in H. It follows from K_0 is nilpotent that H_q is normal in H, a contradiction. If $P_0 \cap K_0 = P_0$ then P_0 is permutable in H. In this case, if $P_0H_q = H$, then $H_p = P_0$ is cyclic and H is p-nilpotent by Lemma 2.6, a contradiction. Hence $P_0H_q < H$ and $P_0H_q = P_0 \times H_q$. Thus $\Omega_1(H_p)$ is centralized by H_q. If further $C_H(\Omega_1(H_p)) < H$ then $C_H(\Omega_1(H_p))$ is nilpotent normal in H. This leads to $H_q \lhd H$, a contradiction. Therefore $\Omega_1(H_p) \leq Z(H)$. If H_p has exponent p, then $H_p = \Omega_1(H_p)$ and $H = H_p \times H_q$, again a contradiction. Thus $p = 2$ and H_2 has exponent 4. If G satisfies (b) then H_2 is quaternion-free and, by Lemma 2.5, H_q acts trivially on H_2, thus H_q is normal in H, a contradiction. Now assume that G satisfies (a). Let $P_1 = \langle x \rangle$ be a cyclic subgroup of H_2 of order 4. Since P_1 is $c^{\star}$-supplemented in H, $H = P_1K_1$ with $P_1 \cap K_1$ is permutable in H. If $|H : K_1| = 4$ then $|H \colon K_1\langle x^2 \rangle| = 2$, hence $K_1\langle x^2 \rangle \lhd H$ and so $H_q \lhd H$, a contradiction. If $|H \colon K_1| = 2$ then $K_1 \lhd H$. We still get a contradiction. Therefore $K_1 = H$ and P_1 is permutable in H. Now Lemma 2.6 implies that P_1H_q is 2-nilpotent and consequently H_q is normalized by H_2. This final contradiction completes our proof.

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Received 03.05.2006