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A NOTE ON MIXED SUMMATION INTEGRAL TYPE OPERATORS

ПРО ОПЕРАТОРИ МІШАНОГО СУМОВНО-ІНТЕГРАЛЬНОГО ТИПУ

Very recently Deo in the paper "Simultaneous approximation by Lupas operators with weighted function of Szasz operators" (J. Inequal. Pure and Appl. Math., 2004, Vol. 5, № 4) claimed to introduce the integral modifications of Lupas operators. These operators were first introduced in the year 1993 by Gupta and Srivastava. They have estimated the simultaneous approximation for these operators and termed these operators as Baskakov – Szasz operators. There are several misprints in the paper of Deo. This motivated us to study further in this direction and, in the present paper, we extend the study and estimate a saturation result in simultaneous approximation for the linear combinations of these summation integral type operators.

Нещодавно Део у роботі "Simultaneous approximation by Lupas operators with weighted function of Szasz operators" (J. Inequal. Pure and Appl. Math., 2004, Vol. 5, № 4) заявив про введення ним інтегральних модифікацій операторів Лупаса. Вперше такі оператори ввели Гупта та Шрівастава у 1993 р. Вони оцінили одночасне наближення цих операторів та назвали їх операторами Баскакова – Шаша. У роботі Део є кілька неточностей. Це спонукало авторів продовжити дослідження у згаданому напрямі. У даній статті розширено коло досліджень та отримано оцінку результату щодо насичення при одночасному наближенні для лінійних комбінацій цих операторів сумовно-інтегрального типу.

1. Introduction. Gupta and Srivastava [1] introduced the sequence of linear positive operators by combining the well-known Baskakov (Lupas) and Szasz basis functions in summation and integration respectively, to approximate Lebesgue integrable functions on the interval $[0,\infty)$ as

$$B_n(f,x) = n\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\int_0^{\infty} s_{n,\nu}(t)f(t)\,dt, \tag{1.1}$$

where $f\in C_\gamma[0,\infty)\equiv\{f\in C_\gamma[0,\infty): |f(t)|\le Me^{\gamma}\ \text{for some}\ M>0,\ \gamma>0\}$ and

$$p_{n,k}(x) = \binom{n+\nu-1}{\nu}\frac{x^\nu}{(1+x)^{n+\nu}}, \qquad s_{n,\nu}(t) = e^{-nt}\frac{(nt)^\nu}{\nu!}.$$

The norm $\|\ \|_\gamma$ is defined as $\|f\|_\gamma = \sup\limits_{0<t<\infty}|f(t)|e^{-\gamma t}$.

In [2] Deo also claimed to introduce these operators. In the same paper, Deo estimated the direct theorems in simultaneous approximation for the operators (1.1). Actually, the direct theorems in simultaneous approximation for a more general class had already been obtained by Gupta and Srivastava in [1].

It turns out that the order of approximation for the operators (1.1) is at best $O(n^{-1})$. Thus, to improve the order of approximation, Gupta and Srivastava [3] considered the linear combinations of operators (1.1), which are defined as follows:

For a fixed natural number k and arbitrary fixed distinct positive integers d_j, $j = 0,1,2,\ldots,k$, the linear conbinations $B_n(f,k,x)$ of $B_{d_jn}(f,x)$ are defined as

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k)B_{d_jn}(f,x), \tag{1.2}$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0, \quad C(0,0) = 1.$$

This type of linear combinations was first considered by May [4] to improve the order of approximation for exponential type operators. Gupta and Srivastava [3] estimated a Voronovskaja-type asymptotic formula and error estimation in simultaneous approximation for $B_n(f, k, x)$. In [5, 6], respectively, the corresponding direct estimate in terms of higher order modulus of continuity and an inverse theorem were established. Actually, a saturation result is a more curious phenomenon. The order of approximation beyond a certain limit $O(\phi(n))$, $\phi(n) \to 0$, $n \to \infty$, is possible only for a trivial subspace. The function for which $O(\phi(n))$ approximation is attained form the Favard class and those with $o(\phi(n))$ approximation forma trivial class. Thus, a saturation result consists of a determination of a saturation order $\phi(n)$, the Favard class, and the trivial class. In the present paper, we extend the study and estimate a saturation theorem in simultaneous approximation for the linear combinations of the Baskakov – Szasz operators defined by (1.2).

2. Auxiliary results. In this section, we mention certain lemmas and definitions, which are necessary to prove the saturation theorem.

Lemma 2.1 [1]. *Let the function* $\mu_{n,m}(x)$, $m \in N^0$, *be defined as*

$$\mu_{n,m}(x) = n \sum_{\nu=0}^{\infty} p_{n+r,\nu}(x) \int_0^{\infty} b_{n,\nu+r}(t)(t-x)^m \, dt.$$

Then

$$\mu_{n,0}(x) = 1, \qquad \mu_{n,1}(x) = \frac{1 + r(1+x)}{n},$$

$$\mu_{n,2}(x) = \frac{rx(1+x) + 1\,[1 + r(1+x)]^2 + nx(2+x)}{n^2},$$

and we also have the recurrence relation

$$n\mu_{n,m+1}(x) = x(1+x)\mu_{n,m}^{(1)}(x) + [(m+1) + r(1+x)]\mu_{n,m}(x) + mx(2+x)\mu_{n,m-1}(x),$$

$$m \geq 1.$$

Consequently, for each $x \in [0, \infty)$, *we have from this recurrence relation that*

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

***Remark* 2.1.** It is remarked here that the above Lemma 2.1 was not estimated correctly by Deo [2]. In the recurrence relation, he missed the last term on the right-hand side of the recurrence relation.

***Remark* 2.2.** The main result, i.e., Theorem 3.1 of [2], is not correct. Actually, the last term in the conclusion must be divided by 2.

***Remark* 2.3.** In Theorem 3.2 of [2], the existence of $f^{(r+1)}$ on $[0, \infty)$ is used while in the hypothesis its existence is assumed only on the interval $[0, a]$.

Lemma 2.2 [6]. *Let* $f \in C_\gamma[0, \infty)$. *If* $f^{(2k+r+2)}$ *exists at a fixed point* $x \in (0, \infty)$, *then*

$$\lim_{n \to \infty} n^{k+1} \{B_n^{(r)}(f, k, x) - f^{(r)}(x)\} = \sum_{i=r}^{2k+r+2} Q(i, k, r, x) f^{(i)}(x),$$

where $Q(i, k, r, x)$ *are certain polynomials in* x *of degree at most* i.

By C_0 we denote the set of continuous functions on the interval $[a, b]$ having the compact support and C_0^k the subset of C_0 of k-times continuously differentiable functions.

Lemma 2.3. *Let* $f \in C_\gamma[0, \infty)$ *and* $g \in C_0^\infty$ *with* $\operatorname{supp} g \subset (a, b)$. *Then*

$$\left| n^{k+1} \left\langle [B_{2n}^{(r)}(f, k, \cdot) - B_n^{(r)}(f, k, \cdot)],\ g \right\rangle \right| \le M \| f \|_\gamma,$$

where the constant M *is independent of* f *and* n *and* $\langle h, g \rangle = \int_0^\infty h(t) g(t) dt$.

A function f continuous in the interval $[a, b]$ is said to belong to generalized Zygmund class $\operatorname{Liz}(\alpha, k, a, b)$, $0 < \alpha < 2$, $k \in N$, if there exist a constant M such that

$$\omega_{2k}(f, \delta, a, b) \le M \delta^{\alpha k}, \qquad \delta > 0,$$

where $\omega_{2k}(f, \delta, a, b)$ denotes the modulus of continuity of $2k$-th order of f on the interval $[a, b]$. In particular, we denote by $\operatorname{Lip}^*(\alpha, a, b)$ the class $\operatorname{Liz}(\alpha, 1, a, b)$.

Theorem 2.1 [6]. *If* $0 < \alpha < 2$, $f \in C_\gamma[0, \infty)$ *and* $0 < a_1 < a_2 < a_3 < b_3 < < b_1 < b_2 < \infty$, *then the following statements* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) *hold true*:

(i) $f^{(r)}$ *exist in the interval* $[a_1, b_1]$ *and*

$$\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} = O(n^{-\alpha(k+1)/2});$$

(ii) $f^{(r)} \in \operatorname{Liz}(\alpha, k+1, a_2, b_2)$;

(iii) (a) *for* $m < \alpha(k+1) < m+1$, $m = 0, 1, 2, \ldots, 2k+1$, $f^{(r+m)}$ *exists and belongs to the class* $\operatorname{Lip}(\alpha(k+1) - r, a_2, b_2)$;

(b) *for* $\alpha(k+1) = m+1$, $m = 0, 1, 2, \ldots, 2k$, $f^{(r+m)}$ *exists and belongs to the class* $\operatorname{Lip}^*(1, a_2, b_2)$;

(iv) $\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_3, b_3]} = O(n^{-\alpha(k+1)/2})$.

3. Saturation theorem. In this section, we shall prove the following main result:

Theorem 3.1. *Let* $f \in C_\gamma[0, \infty)$ *and* $0 < a_1 < a_2 < a_3 < b_3 < b_1 < b_2 < \infty$. *Then in the following statements the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iv) $\Rightarrow$ (v) $\Rightarrow$ $\Rightarrow$ (vi) *hold true*:

(i) $f^{(r)}$ *exist in the interval* $[a_1, b_1]$ *and*

$$\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} = O(n^{-(k+1)});$$

(ii) $f^{(2k+r+1)} \in A.C.[a_2, b_2]$ *and* $f^{(2k+r+2)} \in L_\infty[a_2, b_2]$;

(iii) $\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_3, b_3]} = O(n^{-(k+1)})$;

(iv) $\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} = o(n^{-(k+1)})$;

(v) $f \in C^{2k+r+2}[a_2, b_2]$ *and* $\displaystyle\sum_{j=k+1}^{2k+r+2} Q(j, k, r, x) f^{(j)}(x) = 0, \qquad x \in [a_2, b_2]$,

where the polynomials $Q(j, k, r, x)$ *are defined in Lemma* 2.2;

(vi) $\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_3, b_3]} = o(n^{-(k+1)})$.

Proof. We first assume (i). Then in view of (i) $\Rightarrow$ (iii) of Theorem 2.1, it follows that $f^{(2k+r+1)}$ exists and is continuous on (a_1, b_1). Moreover, it is obviously seen that the statements

$$\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} = O(n^{-(k+1)}) \tag{3.1}$$

and

$$\left\| B_{2n}^{(r)}(f, k, \cdot) - B_n^{(r)}(f, k, \cdot) \right\|_{C[a_1, b_1]} = O(n^{-(k+1)}) \tag{3.2}$$

are equivalent. Obviously (3.1) $\Rightarrow$ (3.2). We just have to show that (3.2) $\Rightarrow$ (3.1). Assuming (3.2), since $\lim\limits_{n\to\infty} B_n^{(r)}(f, k, x) = f^{(r)}(x)$, we have

$$f^{(r)}(x) = B_n^{(r)}(f, k, x) + [B_{2n}^{(r)}(f, k, x) - B_n^{(r)}(f, k, x)] +$$

$$+ [B_{4n}^{(r)}(f, k, x) - B_{2n}^{(r)}(f, k, x)] + \ldots + [B_{2^m n}^{(r)}(f, k, x) - B_{2^{m-1} n}^{(r)}(f, k, x)] + \ldots,$$

$$\left\| B_n^{(r)}(f, k, \cdot) - f^{(r)} \right\|_{C[a_1, b_1]} \le \left\| B_{2n}^{(r)}(f, k, \cdot) - B_n^{(r)}(f, k, \cdot) \right\|_{C[a_1, b_1]} +$$

$$+ \left\| B_{4n}^{(r)}(f,k, \cdot) - B_{2n}^{(r)}(f,k, \cdot) \right\|_{C[a_1, b_1]} + \ldots + \left\| B_{2^m n}^{(r)}(f,k, \cdot) - B_{2^{m-1} n}^{(r)}(f,k, \cdot) \right\|_{C[a_1, b_1]} + \ldots .$$

Applying (3.2), in the above, (3.1) follows immediately.

We assume that $\{n^{k+1}(B_{2n}^{(r)}(f, k, x) - B_n^{(r)}(f, k, x))\}$ is bounded as a sequence in $C[a_1, b_1]$ and hence in $L_\infty[a_1, b_1]$. Since $L_\infty[a_1, b_1]$ is the dual space of $L_1[a_1, b_1]$, by Alaoglu's theorem there exists $h \in L_\infty[a_1, b_1]$ such that for some sequence $\{n_i\}$ of the natural numbers and for every $g \in C_0^\infty$, with $\operatorname{supp} g \subset (a_1, b_1)$

$$\lim_{n_i\to\infty} n_i^{k+1} \left\langle \left| B_{2n_i}^{(r)}(f, k, \cdot) - B_{n_i}^{(r)}(f, k, \cdot) \right|, g \right\rangle = \langle h, g \rangle. \tag{3.3}$$

Next, since $C^{2k+r+2}[a_1, b_1] \cap C_\gamma[0, \infty)$ is dense in $C_\gamma[0, \infty)$, there exists a sequence $\{f_\lambda\}_{\lambda=1}^\infty$ in $C^{2k+r+2}[a_1, b_1] \cap C_\gamma[0, \infty)$ converging to f in $\|\cdot\|_\gamma$-norm. For any $g \in C_0^\infty$ with $\operatorname{supp} g \subset (a_1, b_1)$ and each function f_λ, making use of Lemma 2.2, we get

$$\lim_{n_i\to\infty} n_i^{k+1} \left\langle \left| B_{2n_i}^{(r)}(f_\lambda, k, \cdot) - B_{n_i}^{(r)}(f_\lambda, k, \cdot) \right|, g \right\rangle =$$

$$= \left\langle -\left(1 - \frac{1}{2^{k+1}} \sum_{j=1}^{2k+r+2} Q(j, k, x) f_\lambda^{(j)}(\cdot),\ g(\cdot)\right) \right\rangle =$$

$$= \left\langle P_{2k+r+2}(D) f_\lambda(\cdot),\ g(\cdot) \right\rangle = \left\langle f_\lambda,\ P_{2k+r+2}^*(D) g(\cdot) \right\rangle, \tag{3.4}$$

where $P_{2k+2}^*(D)$ is the dual operator of $P_{2k+2}(D)$. Using Lemma 2.3, we have

$$\lim_{n_i\to\infty} n_i^{k+1} \left\langle \left| B_{2n_i}^{(r)}(f - f_\lambda, k, \cdot) - B_{n_i}^{(r)}(f - f_\lambda, k, \cdot) \right|, g \right\rangle \le M \| f - f_\lambda \|_\gamma. \tag{3.5}$$

Combining the estimates (3.3), (3.4), and (3.5), we obtain

$$\left\langle f,\ P_{2k+r+2}^*(D) g(\cdot) \right\rangle = \lim_{\lambda\to\infty} \left\langle f_\lambda,\ P_{2k+r+2}^*(D) g(\cdot) \right\rangle =$$

$$= \lim_{\lambda\to\infty} \left\{ \lim_{n_i\to\infty} \left\langle \left| B_{2n_i}^{(r)}(f - f_\lambda, k, \cdot) - B_{n_i}^{(r)}(f - f_\lambda, k, \cdot) \right|, g \right\rangle + f_\lambda P_{2k+r+2}^*(D) g(\cdot) \right\} =$$

$$= \lim_{n_i \to \infty} n_i^{k+1} \left\langle [B_{2n_i}^{(r)}(f, k, \cdot) - B_{n_i}^{(r)}(f, k, .)],\ g(\cdot) \right\rangle = \langle h, g \rangle, \tag{3.6}$$

where

$$h(x) = P_{2k+r+2}(D) f(x)$$

as a generalized function. Since $Q(2k+r+2, k, x) \neq 0$, by Lemma 2.2, regarding (3.6) as a first order linear differential equation for $f^{(2k+r+1)}$, we deduce that $f^{(2k+r+1)} \in A.C.[a_2, b_2]$ and $f^{(2k+r+2)} \in L_\infty[a_2, b_2]$. This completes the proof of (i) $\Rightarrow$ $\Rightarrow$ (ii). Next, (ii) $\Rightarrow$ (iii) follows from Lemma 2.2. The proof of (iv) $\Rightarrow$ (v) is similar to that of (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (vi) follows from Lemma 2.2.

This completes the proof of saturation theorem.

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