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WEIGHTED SHARP BOUNDEDNESS FOR MULTILINEAR COMMUTATORS

ЗВАЖЕНА ТОЧНА ОБМЕЖЕНІСТЬ ДЛЯ МУЛЬТИЛІНІЙНИХ КОМУТАТОРІВ

In this paper, the sharp estimates for some multilinear commutators related to certain sublinear integral operators are obtained. The operators include the Littlewood – Paley operator and the Marcinkiewicz operator. As application, we obtain the weighted L^p $(p > 1)$ inequalities and $L\log L$-type estimate for the multilinear commutators.

Одержано точні оцінки для деяких мультилінійних комутаторів, що пов'язані з певними сублінійними інтегральними операторами. Ці оператори включають в себе оператор Літлвуда – Палея та оператор Марцінкевича. Як застосування, отримано зважені L^p $(p > 1)$ нерівності та оцінку типу $L\log L$ для мультилінійних комутаторів.

1. Introduction. Let $b \in BMO(R^n)$ and T be the Calderón – Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$. By virtue of classical result of Coifman, Rochberg and Weiss [1], we know that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that $[b, T]$ is not bounded, in general, from $L^1(R^n)$ to $L^{1,\infty}(R^n)$. In [2], the sharp inequalities for some multilinear commutators of the Calderón – Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp estimates for some multilinear commutators related to certain sublinear integral operators. In fact, we shall establish the sharp estimates for the multilinear commutators only under certain conditions on the size of the operators. The operators include the Littlewood – Paley operator and the Marcinkiewicz operator. As the applications, we obtain the weighted norm inequalities and $L\log L$-type estimate for these multilinear commutators. In Section 2, we will give some concepts and theorems of this paper, whose proofs will appear in Section 3.

2. Preliminaries and theorems. First, let us introduce some notations (see [2 – 5]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int\limits_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well known that (see [3])

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|} \int\limits_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. For $0 < r < \infty$, we denote $f^{\#}_r$ by

$$f^{\#}_r(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy–Littlewood maximal operator, i.e., $M(f)(x) = \sup_{x\in Q} |Q|^{-1} \times \times \int_Q |f(y)|dy$; we write $M_p(f) = (M(f^p))^{1/p}$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = = M(M^{k-1}(f))(x)$ for $k \geq 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complement associated with Φ. For a function f, we denote the Φ-average by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}.$$

The main Young function to be used in this paper is $\Phi(t) = \exp(t^r) - 1$ and $\Psi(t) = = t\log^r(t+e)$, the corresponding Φ-average and maximal functions are denoted by $\|\cdot\|_{\exp L^r,Q}$, $M_{\exp L^r}$ and $\|\cdot\|_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$. We have the following inequality for any $r > 0$ and $m \in N$:

$$M(f) \leq M_{L(\log L)^r}(f), \qquad M_{L(\log L)^m}(f) \sim M^{m+1}(f).$$

For $r \geq 1$, we denote

$$\|b\|_{\mathrm{osc}_{\exp L^r}} = \sup_Q \|b - b_Q\|_{\exp L^r,Q};$$

the spaces $\mathrm{Osc}_{\exp L^r}$ is defined by

$$\mathrm{Osc}_{\exp L^r} = \left\{b \in L^1_{\log}(R^n)\colon \|b\|_{\mathrm{osc}_{\exp L^r}} < \infty\right\}.$$

It is that (see [2])

$$\|b - b_{2^kQ}\|_{\exp L^r,2^kQ} \leq Ck\|b\|_{\mathrm{Osc}_{\exp L^r}}.$$

It is obvious that $\mathrm{Osc}_{\exp L^r}$ coincides with the BMO space if $r = 1$. For $r_j > 0$ and $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$ for $j = 1,\ldots,m$, we denote $1/r = 1/r_1 + \ldots + 1/r_m$ and $\|\tilde{b}\| = \prod_{j=1}^m \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}$. Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\ldots,\sigma(j)\}$ of $\{1,\ldots,m\}$ of j different elements. For $\sigma \in C_j^m$, denote $\sigma^c = \{1,\ldots,m\} \setminus \sigma$. For $\tilde{b} = (b_1,\ldots,b_m)$ and $\sigma = \{\sigma(1),\ldots,\sigma(j)\} \in C_j^m$, denote $\tilde{b}_\sigma = (b_{\sigma(1)},\ldots,b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)}\ldots b_{\sigma(j)}$ and $\|\tilde{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} = \|b_{\sigma(1)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(1)}}}}\ldots\|b_{\sigma(j)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(j)}}}}$.

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [3]).

We are going to consider some integral operators defined below.

Let b_j, $j = 1,\ldots,m$, be the fixed locally integral functions on R^n.

Definition 1. *Let $\lambda > 3 + 2/n$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties*:

(1) $\int \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ *when* $2|y|<|x|$.
The Littlewood–Paley multilinear commutator is defined by

$$g_\lambda^{\tilde{b}}(f)(x) = \left[\int\int\limits_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^{\tilde{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x,y) = \int\limits_{R^n} \left[\prod_{j=1}^{m}(b_j(x)-b_j(z))\right] \psi_t(y-z)f(z)dz$$

and $\psi_t(x)=t^{-n}\psi(x/t)$ *for* $t>0$. *Set* $F_t(f)=\psi_t * f$. *We also define that*

$$g_\lambda(f)(x) = \left(\int\int\limits_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|F_t(f)(y)\right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood–Paley function (*see* [6]).

Let H be the Hilbert space $H = \left\{ h\colon \|h\| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty \right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be regarded as a mapping from $(0,+\infty)$ to H, and it is clear that

$$g_\lambda^{\tilde{b}}(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^{\tilde{b}}(f)(x,y) \right\|$$

and

$$g_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y) \right\|.$$

Definition 2. *Let* $\lambda>1$, $0<\gamma\le 1$ *and* Ω *be homogeneous of degree zero on* R^n *such that* $\int_{S^{n-1}} \Omega(x')d\sigma(x')=0$. *Assume that* $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, *i.e., there exists a constant* $M>0$ *such that for any* $x,\ y\in S^{n-1}$, $\left|\Omega(x)-\Omega(y)\right| \le M|x-y|^{\gamma}$. *We denote* $\Gamma(x)=\left\{(y,t)\in R_+^{n+1}\colon |x-y|<t\right\}$ *and the characteristic function of* $\Gamma(x)$ *by* $\chi_{\Gamma(x)}$. *The Marcinkiewicz multilinear commutator is defined by*

$$\mu_\lambda^{\tilde{b}}(f)(x) = \left[\int\int\limits_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^{\tilde{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x,y) = \int\limits_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[\prod_{j=1}^{m}(b_j(x)-b_j(z))\right] f(z)dz.$$

We set

$$F_t(f)(y) = \int\limits_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

We also define

$$\mu_\lambda(f)(x) = \left(\int\int\limits_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz integral (*see* [7]).

Let H be the space $H = \left\{h\colon \|h\| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_\lambda^A(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^{\tilde b}(f)(x,y) \right\|$$

and

$$\mu_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y) \right\|.$$

More generally, we define the following multilinear commutator related to certain integral operators.

Definition 3. *Let $F(x,y,t)$ be a function defined on $R^n \times R^n \times [0,+\infty)$, we denote that*

$$F_t(f)(x) = \int\limits_{R^n} F(x,y,t) f(y)dy$$

and

$$F_t^{\tilde b}(f)(x) = \int\limits_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] F(x,y,t) f(y)dy$$

for every bounded and compactly supported function f.

Let H be the Banach space $H = \{h\colon \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^{\tilde b}(f)(x)$ may be regarded as a mapping from $[0,+\infty)$ to H. Then, the multilinear commutator related to $F_t^{\tilde b}$ is defined by

$$T_{\tilde b}(f)(x) = \left\|F_t^{\tilde b}(f)(x)\right\|;$$

we also denote

$$T(f)(x) = \left\|F_t(f)(x)\right\|.$$

Note that when $b_1 = \ldots = b_m$, $T_{\tilde b}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 4, 5, 8 – 12]). Our main purpose is to establish the sharp inequalities for the multilinear commutator operators.

The following theorems are our main results:

Theorem 1. *Let* $r_j \geq 1$ *and* $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$ *for* $j = 1, \ldots, m$. *Denote* $1/r = 1/r_1 + \ldots + 1/r_m$. *Then the following statements are frue*:

(1) *For any* $0 < p < q < 1$, *there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$ *and any* $x \in R^n$,

$$(g_\lambda^{\tilde{b}}(f))_p^\#(x) \leq C \left(\|b\| M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_q(g_\lambda^{\tilde{b}_{\sigma^c}}(f)(x) \right).$$

(2) *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\left\| g_\lambda^{\tilde{b}}(f) \right\|_{L^p(w)} \leq C \|\tilde{b}\| \, \|f\|_{L^p(w)}.$$

(3) *Denote* $\Phi(t) = t \log^{1/r}(t+e)$. *If* $w \in A_1$, *then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$w\big(\big\{x \in R^n \colon g_\lambda^{\tilde{b}}(f)(x) > \lambda\big\}\big) \leq C \int\limits_{R^n} \Phi\left(\frac{\|\tilde{b}\| |f(x)|}{\lambda} \right) w(x) dx.$$

Theorem 2. *Let* $r_j \geq 1$ *and* $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$ *for* $j = 1, \ldots, m$. *Denote* $1/r = 1/r_1 + \ldots + 1/r_m$. *Then the following statements are frue*:

(1) *For any* $0 < p < q < 1$, *there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$ *and any* $x \in R^n$,

$$(\mu_\lambda^{\tilde{b}}(f))_p^\#(x) \leq C \left(\|b\| M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_q(\mu_\lambda^{\tilde{b}_{\sigma^c}}(f)(x) \right).$$

(2) *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\left\| \mu_\lambda^{\tilde{b}}(f) \right\|_{L^p(w)} \leq C \|\tilde{b}\| \|f\|_{L^p(w)}.$$

(3) *Denote* $\Phi(t) = t \log^{1/r}(t+e)$. *If* $w \in A_1$, *then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$w\big(\big\{x \in R^n \colon \mu_\lambda^{\tilde{b}}(f)(x) > \lambda\big\}\big) \leq C \int\limits_{R^n} \Phi\left(\frac{\|\tilde{b}\| |f(x)|}{\lambda} \right) w(x) dx.$$

3. Proofs of theorems. We begin with a general theorem.

Main Theorem. *Let* $r_j \geq 1$ *and* $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$ *for* $j = 1, \ldots, m$. *Denote* $1/r = 1/r_1 + \ldots + 1/r_m$. *Suppose that* T *is the same as in Definition* 1 *and such that* T *is bounded on* $L^p(w)$ *for all* $w \in A_p$, $1 < p < \infty$, *and weak* $(L^1(w), L^1(w))$ *bounded for all* $w \in A_1$. *If* T *satisfies the size condition*

$$\Big\| F_t((b_1 - (b_1)_{2Q}) \ldots (b_m - (b_m)_{2Q}) f)(x) -$$

$$- F_t((b_1 - (b_1)_{2Q}) \ldots (b_m - (b_m)_{2Q}) f)(x_0) \Big\| \leq C M_{L(\log L)^{1/r}}(f)(\tilde{x})$$

for any cube $Q = Q(x_0, d)$ *with* $\mathrm{supp}\, f \subset (2Q)^c$ *and* $x,\ \tilde{x} \in Q = Q(x_0, d)$, *then for any* $0 < p < q < 1$, *there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and any* $x \in R^n$,

$$(T_{\tilde{b}}(f))_p^{\#}(x) \le C\left(\|b\| M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f)(x))\right).$$

To prove the theorem, we need the following lemmas:

Lemma 1 (Kolmogorov, [3, p. 485])**.** *Let* $0 < p < q < \infty$ *and let* $f \ge 0$ *be an arbitrary function. We define, for* $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda\big|\{x\in R^n : f(x) > \lambda\}\big|^{1/q}, \quad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup is taken for all measurable sets E *with* $0 < |E| < \infty$. *Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \big(q/(q-p)\big)^{1/p}\|f\|_{WL^q}.$$

Lemma 2 [2]**.** *Let* $r_j \ge 1$ *for* $j = 1,\ldots,m$. *Denote* $1/r = 1/r_1 + \ldots + 1/r_m$. *Then*

$$\frac{1}{|Q|}\int\limits_Q \big|f_1(x)\ldots f_m(x)g(x)\big|dx \le \|f\|_{\exp L^{r_1},Q}\ldots\|f\|_{\exp L^{r_m},Q}\|g\|_{L(\log L)^{1/r},Q}.$$

Proof of Main Theorem. It suffices to prove that, for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int\limits_Q |T_{\tilde{b}}(f)(x) - C_0|^p dx\right)^{1/p} \le$$

$$\le C\left(\|b\| M_{L(\log L)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x}\in Q$. We first consider the case $m = 1$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{R^n\setminus 2Q}$, we write

$$F_t^{b_1}(f)(x) =$$

$$= (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x),$$

then

$$\big|T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)\big| \le \big\|F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)\big\| \le$$

$$\le \big\|(b_1(y) - (b_1)_{2Q})F_t(f)(x)\big\| + \big\|F_t((b_1 - (b_1)_{2Q})f_1)(x)\big\| +$$

$$+ \big\|F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)\big\| =$$

$$= I(x) + II(x) + III(x).$$

For $I(x)$, by Hölder's inequality for the exponent $1/l + 1/l' = 1$ with $1 < l < q/p$ and $pl = q$, we have

$$\left(\frac{1}{|Q|}\int\limits_Q |I(x)|^p dx\right)^{1/p} = \left(\frac{1}{|Q|}\int\limits_Q |b_1(x) - (b_1)_{2Q}|^p |T(f)(x)|^p dx\right)^{1/p} \le$$

$$\leq \left(\frac{C}{|2Q|}\int\limits_{2Q}|b_1(x)-(b_1)_{2Q}|^{pl'}\right)^{1/pl'}\left(\frac{1}{|Q|}\int\limits_{Q}|T(f)(x)|^{pl}dx\right)^{1/pl}\leq$$

$$\leq C\|b_1\|_{\mathrm{Osc}_{\exp L^r}}M_{pl}(T(f))(\tilde{x})\leq C\|b_1\|_{\mathrm{Osc}_{\exp L^r}}M_q(T(f))(\tilde{x}).$$

For $II(x)$, by Lemma 1 and the weak type $(1,1)$ of T, we have

$$\left(\frac{1}{|Q|}\int\limits_{Q}|B(x)|^p dx\right)^{1/p}=\left(\frac{1}{|Q|}\int\limits_{Q}\left|T((b_1-(b_1)_{2Q})f_1)(x)\right|^p dx\right)^{1/p}\leq$$

$$\leq C|2Q|^{-1}\frac{\|T((b_1-(b_1)_{2Q})f_1)\|_{L^p}}{|2Q|^{1/p-1}}\leq$$

$$\leq C|2Q|^{-1}\|T((b_1-(b_1)_{2Q})f\chi_{2Q})\|_{WL^1}\leq$$

$$\leq C|2Q|^{-1}\int\limits_{2Q}|b_1(x)-(b_1)_{2Q}\|f(x)|dx\leq$$

$$\leq C\|b_1-(b_1)_{2Q}\|_{\exp L^r,2Q}\|f\|_{L(\log L)^{1/r},2Q}\leq$$

$$\leq C\|b_1\|_{\mathrm{Osc}_{\exp L^r}}M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $III(x)$, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int\limits_{Q}|C(x)|^p dx\right)^{1/p}\leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

We now consider the case $m\geq 2$. For $b=(b_1,\ldots,b_m)$, we write

$$F_t^{\tilde{b}}(f)(x)=\int\limits_{R^n}\left[\prod_{j=1}^m(b_j(x)-b_j(y))\right]F(x,y,t)f(y)dy=$$

$$=\int\limits_{R^n}(b_1(x)-(b_1)_{2Q})-(b_1(y)-(b_1)_{2Q})\ldots(b_m(x)-$$

$$-(b_m)_{2Q})-(b_m(y)-(b_m)_{2Q})F(x,y,t)f(y)dy=$$

$$=\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int\limits_{R^n}(b(y)-(b)_{2Q})_\sigma F(x,y,t)f(y)dy=$$

$$=(b_1(x)-(b_1)_{2Q})\ldots(b_m(x)-(b_m)_{2Q})F_t(f)(x)+$$

$$+(-1)^m F_t((b_1-(b_1)_{2Q})\ldots(b_m-(b_m)_{2Q})f)(x)+$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int\limits_{R^n}(b(y)-b(x))_{\sigma^c}F(x,y,t)f(y)dy=$$

$$=(b_1(x)-(b_1)_{2Q})\ldots(b_m(x)-(b_m)_{2Q})F_t(f)(x)+$$

$$+(-1)^m F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f)(x)+$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x),$$

whence

$$\left|T_{\tilde{b}}(f)(x)-(-1)^m T((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q}))f_2)(x_0)\right|\le$$

$$\le\left\|F_{\tilde{b}}(f)(x)-(-1)^m F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q}))f_2)(x_0)\right\|\le$$

$$\le\left\|(b_1(x)-(b_1)_{2Q})\dots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\right\|+$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)\right\|+$$

$$+\left\|F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f_1)(x)\right\|+$$

$$+\left\|F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f_2)(x)-\right.$$

$$\left.-F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f_2)(x_0)\right\|=$$

$$=I_1(x)+I_2(x)+I_3(x)+I_4(x).$$

For $I_1(x)$ and $I_2(x)$, similar to the proof of the case $m=1$, we get

$$\left(\frac{1}{|Q|}\int_Q (I_1(x))^p dx\right)^{1/p}\le CM_q(T(f))(\tilde{x})$$

and

$$\left(\frac{1}{|Q|}\int_Q (I_2(x))^p dx\right)^{1/p}\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For I_3, by the weak type $(1,1)$ of T and Lemma 2, we obtain

$$\left(\frac{1}{|Q|}\int_Q (I_3(x))^p dx\right)^{1/p}\le$$

$$\le\frac{C}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|\dots|b_m(x)-(b_m)_{2Q}||f(x)|dx\le$$

$$\le C\|b_1-(b_1)_{2Q}\|_{\exp L^{r_1},2Q}\dots\|b_m-(b_m)_{2Q}\|_{\exp L^{r_m},2Q}\|f\|_{L(\log L)^{1/r},2Q}\le$$

$$\le C\|b\|M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For I_4, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int\limits_Q (I_4(x))^p dx\right)^{1/p} \leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

This completes the proof of the Main Theorem.

To prove Theorems 1 and 2, it suffices to verify that $g_\lambda^{\tilde{b}}$ and $\mu_\lambda^{\tilde{b}}$ satisfy the size condition in Main Theorem, that is

$$\left\| \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right] \times \right.$$

$$\left. \times F_t\big((b_1-(b_1)_{2Q}\big)\ldots(b_m-(b_m)_{2Q})f)(y)\right\| \leq$$

$$\leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Suppose that $\operatorname{supp} f \subset Q^c$ and $x \in Q = Q(x_0, d)$. Note that $|x_0 - z| \approx |x - z|$ for $z \in Q^c$.

For $g_\lambda^{\tilde{b}}$, by the condition of ψ and the inequality $a^{1/2} - b^{1/2} \leq (a-b)^{1/2}$ for $a \geq b > 0$, we get

$$\left\| \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right] \times \right.$$

$$\left. \times F_t((b_1-(b_1)_{2Q})\ldots(b_m-(b_m)_{2Q})f)(y)\right\| \leq$$

$$\leq \left[\int\int\limits_{R_+^{n+1}}\int\limits_{(2Q)^c}\left[\frac{t^{n\lambda/2}|x_0-x|^{1/2}}{(t+|x_0-y|)^{(n\lambda+1)/2}}\big|b_1(z)-(b_1)_{2Q}\big|\ldots\right.\right.$$

$$\left.\left.\ldots\big|b_m(z)-(b_m)_{2Q}\big|\,|f(z)|\,\big|\psi_t(y-z)\big|dz\right]^2\frac{dydt}{t^{n+1}}\right]^{1/2} \leq$$

$$\leq C\int\limits_{(2Q)^c}\big|b_1(z)-(b_1)_{2Q}\big|\ldots\big|b_m(z)-(b_m)_{2Q}\big|\,|f(z)|\times$$

$$\times\left(\int\int\limits_{R_+^{n+1}}\frac{t^{1-n+n\lambda}|x_0-x|dydt}{(t+|x_0-y|)^{n\lambda+1}(t+|y-z|)^{2n+2}}\right)^{1/2}dz;$$

noting that $2t+|y-z| \geq 2t+|x_0-z|-|x_0-y| \geq t+|x_0-z|$ for $|x_0-y| \leq t$ and $2^{k+1}t+|y-z| \geq 2^{k+1}t+|x_0-z|-|x_0-y| \geq |x_0-z|$ for $|x_0-y| \leq 2^{k+1}t$ and recalling that $\lambda > (3n+2)/n$, we get

$$t^{-n}\int\limits_{R^n}\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2}} =$$

$$= t^{-n} \int\limits_{|x_0-y|\le t} \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda} \frac{dy}{(t+|y-z|)^{2n+2}}+$$

$$+t^{-n} \sum_{k=0}^{\infty} \int\limits_{2^k t<|x_0-y|\le 2^{k+1}t} \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda} \frac{dy}{(t+|y-z|)^{2n+2}} \le$$

$$\le t^{-n} \left[\int\limits_{|x_0-y|\le t} \frac{2^{2n+2}dy}{(2t+2|y-z|)^{2n+2}} + \right.$$

$$\left. + \sum_{k=0}^{\infty} \int\limits_{|x_0-y|\le 2^{k+1}t} 2^{-kn\lambda} \frac{2^{(k+2)(2n+2)}dy}{(2^{k+2}t+2^{k+2}|y-z|)^{2n+2}} \right] \le$$

$$\le Ct^{-n} \left[\int\limits_{|x_0-y|\le t} \frac{dy}{(2t+|y-z|)^{2n+2}} + \right.$$

$$\left. + \sum_{k=0}^{\infty} \int\limits_{|x_0-y|\le 2^{k+1}t} 2^{-kn\lambda} \frac{2^{k(2n+2)}dy}{(t+2^{k+1}t+|y-z|)^{2n+2}} \right] \le$$

$$\le Ct^{-n} \left[\int\limits_{|x_0-y|\le t} \frac{dy}{(t+|x_0-z|)^{2n+2}} + \right.$$

$$\left. + \sum_{k=0}^{\infty} \int\limits_{|x_0-y|\le 2^{k+1}t} 2^{-kn\lambda} \frac{2^{k(2n+2)}dy}{(t+|x_0-z|)^{2n+2}} \right] \le$$

$$\le Ct^{-n} \left[\frac{t^n}{(t+|x_0-z|)^{2n+2}} + \sum_{k=0}^{\infty} 2^{k(3n+2-n\lambda)} \frac{t^n}{(t+|x_0-z|)^{2n+2}} \right] \le$$

$$\le \frac{C}{(t+|x_0-z|)^{2n+2}},$$

since

$$\int\limits_0^{\infty} \frac{dt}{(t+|x_0-z|)^{2n+2}} = C|x_0-z|^{-2n-1},$$

we obtain

$$\left\| \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2} \right] \times \right.$$

$$\left. \times F_t((b_1-(b_1)_{2Q})\ldots(b_m-(b_m)_{2Q})f)(y) \right\| \le$$

$$\leq C \int\limits_{(2Q)^c} \left|b_1(z) - (b_1)_{2Q}\right| \ldots \left|b_m(z) - (b_m)_{2Q}\right| |f(z)| \frac{|x_0 - x|^{1/2}}{|x_0 - z|^{n+1/2}} dz \leq$$

$$\leq C \sum_{k=1}^{\infty} \int\limits_{2^{k+1}Q \setminus 2^k Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2)} \left\| \prod_{j=1}^{m} (b_j(z) - (b_j)_{2Q}) \right\| |f(z)| dz \leq$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k/2} |2^{k+1}Q|^{-1} \int\limits_{2^{k+1}Q} \left\| \prod_{j=1}^{m} (b_j(z) - (b_j)_{2Q}) \right\| |f(z)| dz \leq$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k/2} \prod_{j=1}^{m} \|b_j - (b_j)_{2Q}\|_{\exp L^{r_j}, 2^{k+1}Q} \|f\|_{L(\log L)^{1/r}, 2^{k+1}Q} \leq$$

$$\leq C \sum_{k=1}^{\infty} k^m 2^{-k/2} \prod_{j=1}^{m} \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}} M_{L(\log L)^{1/r}}(f)(\tilde{x}) \leq$$

$$\leq C \prod_{j=1}^{m} \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}} M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $\mu_\lambda^{\tilde{b}}$, by the condition of Ω, we get

$$\left\| \left[\left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} - \left(\frac{t}{t + |x_0 - y|} \right)^{n\lambda/2} \right] \times \right.$$

$$\left. \times F_t\big((b_1 - (b_1)_{2Q}) \ldots (b_m - (b_m)_{2Q}) f\big)(y) \right\| \leq$$

$$\leq C \left[\int\int\limits_{R_+^{n+1}} \int\limits_{(2Q)^c} \left[\frac{\chi_{\Gamma(z)}(y,t) t^{n\lambda/2} |x_0 - x|^{1/2}}{(t + |x - y|)^{(n\lambda+1)/2} |y - z|^{n-1}} \times \right. \right.$$

$$\left. \left. \times \prod_{j=1}^{m} |b_j(z) - (b_j)_{2Q}| |f(z)| dz \right]^2 \frac{dydt}{t^{n+3}} \right]^{1/2} \leq$$

$$\leq C \int\limits_{(2Q)^c} \prod_{j=1}^{m} |b_j(z) - (b_j)_{2Q}| |f(z)| \left(\int\int\limits_{R_+^{n+1}} \frac{\chi_{\Gamma(z)}(y,t) t^{n\lambda - n - 3} |x_0 - x| dydt}{(t + |x - y|)^{n\lambda+1} |y - z|^{2n-2}} \right)^{1/2} dz;$$

noting that the inequalities $|x - z| \leq 2t$ and $|y - z| \geq |x - z| - t \geq |x - z| - 3t$ hold for $|x - y| \leq t$ and $|y - z| \leq t$ and the inequalities $|x - z| \leq t(1 + 2^{k+1}) \leq 2^{k+2}t$ and $|y - z| \geq |x - z| - 2^{k+3}t$ hold for $|x - y| \leq 2^{k+1}t$ and $|y - z| \leq t$, we obtain

$$\left\| \left[\left(\frac{t}{t + |x - y|} \right)^{n\lambda/2} - \left(\frac{t}{t + |x_0 - y|} \right)^{n\lambda/2} \right] \times \right.$$

$$\times F_t\big((b_1-(b_1)_{2Q})\ldots\big(b_m-(b_m)_{2Q}\big)f\big)(y)\Bigg\|\le$$

$$\le C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\,|x_0-x|^{1/2}\times$$

$$\times\left[\int\limits_0^\infty\int\limits_{|x-y|\le t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{\chi_{\Gamma(z)}(y,t)t^{-n}dydt}{(|x-z|-3t)^{2n+2}}\right]^{1/2}dz+$$

$$+\,C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\,|x_0-x|^{1/2}\times$$

$$\times\left[\int\limits_0^\infty\sum_{k=0}^\infty\int\limits_{2^kt<|x-y|\le 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{\chi_{\Gamma(z)}(y,t)t^{-n}dydt}{(|x-z|-2^{k+3}t)^{2n+2}}\right]^{1/2}dz\le$$

$$\le C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\,|x_0-x|^{1/2}\times$$

$$\times\left[\int\limits_{|x-z|/2}^\infty\frac{dt}{(|x-z|-3t)^{2n+2}}\right]^{1/2}dz\,+$$

$$+\,C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\|f(z)\|x_0-x\big|^{1/2}\times$$

$$\times\left[\sum_{k=0}^\infty\int\limits_{2^{-2-k}|x-z|}^\infty\frac{2^{-k(n\lambda+2)}(2^kt)^nt^{-n}2^kdt}{(|x-z|-2^{k+3}t)^{2n+2}}\right]^{1/2}dz\le$$

$$\le C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\,|x_0-x|^{1/2}\big|x-z\big|^{-n-1/2}dz+$$

$$+C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\,|x_0-x\big|^{1/2}\sum_{k=0}^\infty 2^{k(n-n\lambda-2)/2}|x-z|^{-n-1/2}dz\le$$

$$\le C\int\limits_{(2Q)^c}\prod_{j=1}^m\big|b_j(z)-(b_j)_{2Q}\big|\,|f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz\le$$

$$\le C\prod_{j=1}^m\|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

These yields the desired results.

By (1) and the boundedness of g_λ, μ_λ and $M_{L(\log L)^{1/r}}$, we may obtain the conclusions (2), (3) of Theorems 1 and 2. This completes the proof of Theorems 1 and 2.

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