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NON-EXPLOSION AND SOLVABILITY OF NONLINEAR DIFFUSION EQUATIONS ON NONCOMPACT MANIFOLDS*

ВІДСУТНІСТЬ ВИБУХУ ТА ІСНУВАННЯ РОЗВ'ЯЗКІВ ДЛЯ НЕЛІНІЙНИХ ДИФУЗІЙНИХ РІВНЯНЬ НА НЕКОМПАКТНИХ БАГАТОВИДАХ

We find sufficient conditions on coefficients of diffusion equation on noncompact manifold, that guarantee non-explosion of solutions in a finite time. This property leads to the existence and uniqueness of solutions for corresponding stochastic differential equation with globally non-Lipschitz coefficients.

Proposed approach is based on the estimates on diffusion generator, that weakly acts on the metric function of manifold. Such estimates enable us to single out a manifold analogue of monotonicity condition on the joint behaviour of the curvature of manifold and coefficients of equation.

Знайдено достатні умови на коефіцієнти дифузійного рівняння на некомпактному багатовиді, за яких розв'язки не вибухають у скінченний проміжок часу. Ця властивість приводить до існування та єдиності розв'язків відповідних стохастичних рівнянь з глобально неліпшицевими коефіцієнтами.

Запропонований підхід спирається на оцінки на генератор дифузії, що слабко діє на метричну функцію багатовиду. Використання таких оцінок дозволяє знайти узагальнення умови монотонності на випадок багатовиду, що поєднує поведінку кривини багатовиду та коефіцієнтів рівняння.

1. Introduction. A rigorous procedure for the construction of solutions of diffusion equations on manifold was suggested a long ago, see [1 – 3] and references therein. In comparison to the stochastic differential equation on linear space when the solution can be constructed in a global coordinate system, the main difficulty with manifold was that it does not have global coordinate system.

Let A_0, A_α be C^∞-smooth vector fields, globally defined on the oriented smooth complete connected Riemannian manifold M without boundary and δW^α denote Stratonovich differentials of independent one-dimensional Wiener processes W_t^α, $\alpha = 1, \ldots, d$,

The diffusion, written in Ito – Stratonovich form

$$\delta y_t^x = A_0(y_t^x)dt + \sum_{\alpha=1}^{d} A_\alpha(y_t^x)\delta W_t^\alpha, \quad y_0^x = x, \tag{1}$$

can be correctly defined in any local coordinate vicinity U of manifold with the use of integral equations on random intervals $t \in (\tau_{in}, \tau_{out})$

$$y^i_{t\wedge\tau_{out}}(x) = y^i_{\tau_{in}}(x) + \int\limits_{\tau_{in}}^{t\wedge\tau_{out}} A_0^i(y_s^x)ds + \sum_{\alpha=1}^{d} \int\limits_{\tau_{in}}^{t\wedge\tau_{out}} A_\alpha(y_s^x)\delta W_s^\alpha, \quad y_0^x = x. \tag{2}$$

Above τ_{in}, τ_{out} denote the times when process y_t^x enters and leaves vicinity U, i.e., $y_t^x \in U$ for all $t \in (\tau_{in}, \tau_{out})$.

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The global solution to (1) is first constructed starting from some vicinity of initial point x. Then it is extended in further domains, where comes process y_t^x, with help of (2). This procedure of gluing together the local solutions on random intervals into global solution may be correctly done for diffusion equations with locally Lipschitz smooth coefficients [1 – 4]. The resulting solution is well-defined on some random interval $[0, \tau_\infty(\omega))$, but not necessarily for all $t \geq 0$.

Since compact manifolds always have a finite covering by local coordinate vicinities, it can be shown that random time $\tau_\infty = \infty$ in compact case and the solution of diffusion equation (1) exists for all $t \geq 0$. On the contrary, because Wiener process W_t leaves any bounded ball in $\mathbb{R}^d$ with non-zero probability, in the noncompact case the explosion may occur. Depending on properties of coefficients of (1) and geometry of manifold there may exist a finite explosion time τ_∞ such that process y_t^x leaves any bounded vicinity U of manifold M at time τ_∞: $\forall U \subseteq M \; y^x_{\tau_\infty} \notin U$. Then the solution y_t^x could be correctly defined only till the *explosion time* $\tau_\infty(\omega)$.

In the case of non-explosion the probability of set $\{\omega\colon \tau_\infty(\omega) < \infty\}$ is zero and the unique global solution y_t^x of equation (1) may be defined for all $t \geq 0$. It represents a continuous adapted locally integrable process which fulfills an independent on particular coordinate vicinities variant of (1): for any function with compact support $f \in C_0^\infty(M)$

$$f(y_t^x) = f(x) + \int\limits_0^t (A_0 f)(y_s^x) ds + \sum_{\alpha=1}^d \int\limits_0^t (A_\alpha f)(y_s^x) \delta W_s^\alpha. \tag{3}$$

Since $f(y_t^x)$ and $(A.f)(y_t^x)$ are $\mathbb{R}^1$-valued processes, equation (3) represents Stratonovich equation on real line $\mathbb{R}^1$. Using functions f that coincide with the local coordinates $f(x) = x^i$ in the coordinate vicinities U of manifold, it is possible to localize equation (3) back to (2).

Moreover, in contrary to the explosion case, when there is no sense of $y_t^x(\omega)$ for ω such that $t \geq \tau_\infty(\omega)$, in non-explosion case the diffusion semigroup $P_t f(x) = \mathbf{E}\, f(y_t^x)$ has sense and the applications to the heat parabolic Cauchy problems on manifolds become possible. Due to (3) semigroup P_t is generated by the second order operator of Hörmander type

$$\mathcal{L} f = \frac{1}{2} \sum_{\alpha=1}^d A_\alpha(A_\alpha f) + A_0 f. \tag{4}$$

The known conditions for non-explosion and, hence, for the existence and uniqueness of global solutions y_t^x to problem (1), lie in the global Lipschitz assumptions on the coefficients of equation A_0, A_α and boundedness of the curvature of manifold, e.g. [1, 2, 4, 5]. Further research of non-explosion for stochastic equations on manifolds mainly concerned geometric properties of manifold, like its Brownian or martingale completeness, see, for example, [6, 7] and references therein.

However, in the case of noncompact spaces with zero curvature (like $\mathbb{R}^d$) it is also possible to prove non-explosion for a wide class of equations with nonlinear globally non-Lipschitz coefficients, that fulfill a kind of monotone conditions of dissipativity and coercitivity [8, 9]. Arises a natural question whether the global Lipschitzness assumption on the coefficients of equation can be avoided in the case of noncompact manifold.

In this article we discuss conditions on the joint behaviour of coefficients A_0, A_α of equation (1) and curvature tensor R of noncompact manifold M that guarantee non-explosion, i.e., the existence and uniqueness of solutions to (1). The found conditions generalize the dissipativity and coercitivity conditions from linear space to the manifold and permit to work with essentially nonlinear diffusion with globally non-Lipschitz coefficients.

The main idea is that the following estimate on the metric distance function ρ on process y_t^x:

$$\mathbf{E}\rho^2(o, y_t^x) \le e^{Kt}(1 + \rho^2(o, x)), \tag{5}$$

leads to the non-explosion: for all $t \ge 0$ $\rho(o, y_t^x) < \infty$ almost everywhere. Here $o \in M$ is some fixed point of manifold and $\rho(o, x)$ denotes the shortest geodesic distance between points o and x.

At the first look, estimate (5) can be obtained from the formal application of Ito–Stratonovich formula (3) to the metric function $f(y_t^x) = \rho^2(o, y_t^x)$

$$\mathbf{E}\rho^2(o, y_t^x) = \rho^2(o, x) + \int_0^t \mathbf{E}\left\{ A_0^{II} + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^{II})^2 \right\} \rho^2(o, y_s^x)ds, \tag{6}$$

where we used notation A^{II} for vector field A acting on the second variable x of function $\rho(o, x)$: $A^{II}\rho^2(o, x) = \langle A(x), \nabla_x \rangle \rho^2(o, x)$.

Then, if operator $\mathcal{L}$ fulfills the following estimate on function $\rho(o, x)$:

$$\exists\, K\colon\ \mathcal{L}^{II}\rho^2(o, x) = \left\{ A_0^{II} + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^{II})^2 \right\} \rho^2(o, x) \le K(1 + \rho^2(o, x)), \tag{7}$$

formula (6) leads to

$$\mathbf{E}\rho^2(o, y_t^x) \le \rho^2(o, x) + K\int_0^t \mathbf{E}(1 + \rho^2(o, y_s^x))ds$$

and gives non-explosion estimate (5).

However, in comparison to the Euclidean case with smooth metric $\rho^2(o, x) = \|o - x\|^2$, for the case of general manifold M function $\rho^2(o, x)$ may be non-differentiable for points $x \in N$ from some hypermanifold $N \subset M$ of lower dimension. Then operator $\left\{ A_0^{II} + \frac{1}{2}\sum_{\alpha=1}^d (A_\alpha^{II})^2 \right\}$ in the right-hand side of (6) is bad defined on it and formal reasoning (5)–(7) does not work.

One more problem in estimation of (7) is that metric $\rho^2(o, x)$ does not have a direct representation, as in the linear case $\|o - x\|^2$. It is a minimum of length functional along paths from o to x

$$\rho^2(o, x) = \inf\left\{ \int_0^1 \left|\dot\gamma(\ell)\right|^2 d\ell,\ \ \gamma(0) = o,\ \ \gamma(1) = x \right\}, \quad \text{where} \quad \dot\gamma(\ell) = \frac{\partial}{\partial \ell}\gamma(\ell), \tag{8}$$

therefore it is hard to provide the implicit representations for arbitrary differential operators, acting on it. The known approaches were mainly adapted for Laplace – Beltrami Δ or similar operators and related with the use of geodesic deviations formulas and Jacobi fields, e.g. [10, 11], survey [7] and references therein.

In [12] it is found a way to obtain upper bound (7) on the generator $\mathcal{L}$, acting on the metric function at points of its C^2-regularity. Since in general situation the metric function is not everywhere twice differentiable, results of [12] are not directly applicable to the study of non-explosion.

The article consists of two parts. First, in Lemma 1 we develop upper bounds of [12] outside of geodesic between points o and x and estimate difference approximations of second order operators. Then, in Lemma 2, we prove estimates on operator $\mathcal{L}$ that weakly acts on metric function.

These weak estimates are used in Lemma 3 to demonstrate, in analogue to arguments of [5], that process $1+\rho^2(o,y_t^x)-K\displaystyle\int_0^t\{1+\rho^2(o,y_s^x)\}ds$ represents supermartingale for sufficiently large K. This leads to moment estimate (5) and, in fact, replaces the Ito formula arguments (5) – (7).

Finally, in Theorem 2 estimates (5) are extended from $\rho^2(o,x)$ to the polynomials of metric function.

2. Main results. Let us implement the following conditions on coefficients A_0, A_α and curvature R. In particular, they generalize the classical dissipativity and coercitivity conditions [8, 9] from the linear Euclidean space to manifold:

coercitivity: $\exists\, o\in M$ such that $\forall C\in\mathbb{R}_+$ $\exists\, K_C\in\mathbb{R}^1$ such that $\forall x\in M$

$$\left\langle \widetilde{A_0}(x),\nabla^x\rho^2(o,x)\right\rangle + C\sum_{\alpha=1}^d\|A_\alpha(x)\|^2\le K_C(1+\rho^2(o,x)); \tag{9}$$

dissipativity: $\forall C,\,C'\in\mathbb{R}_+$ $\exists\, K_C\in\mathbb{R}^1$ such that $\forall x\in M$, $\forall h\in T_xM$

$$\left\langle \nabla\widetilde{A_0}(x)[h],h\right\rangle + C\sum_{\alpha=1}^d\left\|\nabla A_\alpha(x)[h]\right\|^2 -$$

$$-C'\sum_{\alpha=1}^d\langle R_x(A_\alpha(x),h)A_\alpha(x),h\rangle\le K_C\|h\|^2, \tag{10}$$

where $\widetilde{A_0}=A_0+\dfrac12\sum_{\alpha=1}^d\nabla_{A_\alpha}A_\alpha$ and $[R(A,h)A]^m=R_{p\ \ell q}^{\ m}A^pA^\ell h^q$ denotes the curvature operator, related with (1, 3) curvature tensor with components

$$R_{1\ 34}^{\ 2}=\frac{\partial\Gamma_{1\ 3}^{\ 2}}{\partial x^4}-\frac{\partial\Gamma_{1\ 4}^{\ 2}}{\partial x^3}+\Gamma_{1\ 3}^{\ j}\Gamma_{j\ 4}^{\ 2}-\Gamma_{1\ 4}^{\ j}\Gamma_{j\ 3}^{\ 2}. \tag{11}$$

For simplicity of further calculations we denoted by numbers the positions of corresponding indexes. By tradition the repeating indexes mean silent summations.

Notation $\nabla H[h]$ means the *directional covariant derivative*, defined by

$$(\nabla H(x)[h])^i=\nabla_jH^i(x)\cdot h^j. \tag{12}$$

Main result of article is the following:

Theorem 1. *Suppose that conditions* (9), (10) *are fulfilled. Then equation* (1) *has a unique solution that does not explode in a finite time and fulfills estimate* (5).

Proof. To localize equation (1) consider open set $U \subseteq M$ with compact closure $\overline{U}$ and function ζ^U with compact support such that $\sqrt{\zeta^U} \in C_0^\infty(M, [0,1])$ and $\zeta^U(z) = 1$ for $z \in \overline{U}$, $0 \le \zeta^U < 1$ outside of $\overline{U}$. Introduce operator

$$\mathcal{L}^U f = \zeta^U \mathcal{L} f = \frac{1}{2} \sum_{\alpha=1}^{d} \sqrt{\zeta^U} A_\alpha (\sqrt{\zeta^U} A_\alpha f) + \zeta^U A_0 f - \frac{1}{2} \sum_{\alpha=1}^{d} \sqrt{\zeta^U} (A_\alpha \sqrt{\zeta^U}) A_\alpha f,$$

that corresponds to the localized Stratonovich diffusion

$$\delta y_t^U(x) = \left(\zeta^U A_0 - \frac{1}{2} \sum_{\alpha=1}^{d} \sqrt{\zeta^U} (A_\alpha \sqrt{\zeta^U}) A_\alpha \right) (y_s^x) ds +$$

$$+ \sum_{\alpha=1}^{d} \sqrt{\zeta^U} (y_t^x) A_\alpha (y_t^x) \delta W_t^\alpha, \quad y_0^x = x. \tag{13}$$

Equation (13) has globally Lipschitz coefficients with all bounded derivatives, therefore it has a unique solution which is C^∞-differentiable on the initial data x [1, 2, 4, 5]. Since for initial data x outside of support ζ^U we have $y_t^U(x) = x$ for all $t \ge 0$, its diffusion semigroup $(P_t^U f)(x) = \mathbf{E} f(y_t^U(x))$ preserves the space $C_{0,+}^\infty(M)$ of non-negative continuously differentiable functions with compact support.

Now let us prepare the independent on U weak estimates on generators $\mathcal{L}^U$: $\exists K$ $\forall \zeta^U \in C_0^\infty(M, [0,1])$, $\zeta^U \big|_{\overline{U}} = 1$ $\forall \varphi \in C_{0,+}^\infty(M)$:

$$\int_M \big([\mathcal{L}^U]^* \varphi \big) \rho^2(o,x) d\sigma(x) \le K \int_M \varphi(x) (1 + \rho^2(o,x)) d\sigma(x), \tag{14}$$

where $d\sigma$ denotes the Riemannian volume on M.

As $[\mathcal{L}^U]^* = [\zeta^U \mathcal{L}]^* = \mathcal{L}^* \zeta^U$, estimate (14) follows from the weak estimate on operator $\mathcal{L}$

$$\exists K \quad \forall \psi \in C_{0,+}^\infty(M):$$

$$\int_M (\mathcal{L}^* \psi(x)) \rho^2(o,x) d\sigma(x) \le K \int_M \psi(x) (1 + \rho^2(o,x)) d\sigma(x) \tag{15}$$

if one substitutes first $\psi = \zeta^U \varphi$ and then applies $0 \le \zeta^U \le 1$. Here $\mathcal{L}^* = \frac{1}{2} \sum_{\alpha=1}^{d} [A_\alpha^*]^2 + A_0^*$ with the adjoint field X^* to vector field X defined by $X^* f = -(\operatorname{div} X) f - X f$.

To prove (15) let us first note that for any smooth vector field X in a vicinity of some point z of manifold N and smooth function f on N representations are satisfied

$$Xf(z) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_0^{\varepsilon} Xf(z^s)ds = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_0^{\varepsilon}\frac{d}{ds}f(z^s)ds = \lim_{\varepsilon\to 0}\frac{f(z^\varepsilon)-f(z)}{\varepsilon},$$

$$\tag{16}$$

$$X(Xf)(z) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_0^{\varepsilon} ds\int\limits_{-s}^{s} X(Xf)(z^\ell)d\ell =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_0^{\varepsilon} ds\int\limits_{-s}^{s}\frac{d}{d\ell}(Xf)(z^\ell)d\ell =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_0^{\varepsilon}\{(Xf)(z^s)-(Xf)(z^{-s})\}ds = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_0^{\varepsilon}\frac{d}{ds}\{f(z^s)+f(z^{-s})\}ds =$$

$$= \lim_{\varepsilon\to 0}\frac{f(z^\varepsilon)+f(z^{-\varepsilon})-2f(z)}{\varepsilon^2}.$$

Here we used notation z^ε for the differential flow along field X: $z^\varepsilon = z + \int_0^\varepsilon X(z^s)ds$.

Therefore, due to the compactness of support of function ψ in (15), the following representation of the left-hand side of (15) is valid

$$\int\limits_M (\mathcal{L}^*\psi(x))\rho^2(o,x)d\sigma(x) = \lim_{\varepsilon\to 0+}\int\limits_M \psi(x)\left\{\frac{\rho^2(o,z_0^\varepsilon(x))-\rho^2(o,z_0^0(x))}{\varepsilon}+\right.$$

$$\left.+\frac{1}{2}\sum_{\alpha=1}^{d}\frac{\rho^2(o,z_\alpha^\varepsilon(x))+\rho^2(o,z_\alpha^{-\varepsilon}(x))-2\rho^2(o,z_\alpha^0(x))}{\varepsilon^2}\right\}d\sigma(x). \tag{17}$$

Here $z_0^\varepsilon(x)$, $z_\alpha^\varepsilon(x)$ denote the shifts along vector fields A_0, A_α with initial data $z_0^0(x) = x$, $z_\alpha^0(x) = x$.

Representation (17) follows from (16) and form of adjoint field X^*, because due to the Stokes formula $\int_{\partial D} X\cdot dS = \int_D \operatorname{div} X\, d\sigma$ the increment of volume along field X is equal to $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{d\sigma(z_X^\varepsilon(x))}{d\sigma(x)} = (\operatorname{div} X)(x)$. Indeed, for $\varphi,\psi\in C_0^\infty(M)$ one has

$$\int\limits_M(\mathcal{L}^*\psi)\varphi\, d\sigma = \int\limits_M \psi(\mathcal{L}\varphi)\, d\sigma =$$

$$= \lim_{\varepsilon\to 0+}\int\limits_M \psi(x)\left\{\frac{\varphi(z_0^\varepsilon(x))-\varphi(z_0^0(x))}{\varepsilon}-\right.$$

$$\left.-\frac{1}{2}\sum_{\alpha=1}^{d}\frac{\varphi(z_\alpha^\varepsilon(x))+\varphi(z_\alpha^{-\varepsilon}(x))-2\varphi(z_\alpha^0(x))}{\varepsilon^2}\right\}d\sigma(x) =$$

$$= \lim_{\varepsilon\to 0+} \int\limits_M \left\{ \frac{1}{\varepsilon}\left[\psi(z_0^{-\varepsilon}(x))\frac{d\sigma(z_0^{-\varepsilon}(x))}{d\sigma(x)} - \psi(x)\right] + \right.$$

$$+\frac{1}{2\varepsilon^2}\sum_{\alpha=1}^{d}\left[\psi(z_\alpha^{-\varepsilon}(x))\frac{d\sigma(z_\alpha^{-\varepsilon}(x))}{d\sigma(x)} + \psi(z_\alpha^{\varepsilon}(x))\frac{d\sigma(z_\alpha^{\varepsilon}(x))}{d\sigma(x)} - \right.$$

$$\left.\left. -2\psi(z_\alpha^0(x))\frac{d\sigma(z_\alpha^0(x))}{d\sigma(x)}\right]\right\}\varphi(x)d\sigma(x), \tag{18}$$

where to get the last line with $\varphi(x)$ we shifted back along fields $\{-A_0, -A_\alpha\}$. For $\psi \in C_0^\infty(M)$ and vector fields A_0, A_α expression in figure brackets in (18) converges to $\mathcal{L}^*\psi$ uniformly on M. Due to the compactness of support of ψ we can close (18) from $\varphi(x)$ to $\rho^2(o,x)$ and, making a reverse shift along fields A_0, A_α, obtain representation (17).

Now let us estimate fractions in the right-hand side of (17).

In the vicinity of geodesic $\gamma(\ell)$, $\ell\in[0,1]$ from $\gamma(0)=o$ to $\gamma(1)=x$ that minimizes (8) consider smooth vector field H. Introduce a family of paths

$$[0,1]\times(-\delta,\delta)\ni(\ell,s)\to\gamma(\ell,s)\in M$$

such that at $s=0$ path $\gamma(\ell,s)\big|_{s=0}=\gamma(\ell)$ gives geodesic γ from o to x and parameter s corresponds to the evolution along H:

$$\frac{\partial}{\partial s}\gamma(\ell,s) = H(\gamma(\ell,s)). \tag{19}$$

Note that for $s\neq 0$ each path $\{\gamma(\ell,s),\ \ell\in[0,1]\}$ must not be geodesic, unlike in formulas for geodesic deviations. Later we will choose field H to be $H(\ell,s) = \ell^2 A_0(\gamma(\ell,s))$ or $H(\ell,s) = \ell A_\alpha(\gamma(\ell,s))$ for the first and second order differences in (17).

Lemma 1. *The following estimates on difference operators on metric function are fulfilled:*

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon)) - \rho^2(o,x)}{\varepsilon} \le$$

$$\le \int\limits_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} \big|\dot\gamma(\ell,s)\big|^2 d\ell + \int\limits_0^\varepsilon\int\limits_0^1 \left|\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,s)|^2\right| d\ell\, ds, \tag{20}$$

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon)) + \rho^2(\gamma(0,\varepsilon),\gamma(1,-\varepsilon)) - 2\rho^2(o,x)}{\varepsilon^2} \le$$

$$\le \int\limits_0^1 \frac{\partial^2}{\partial s^2}\bigg|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell + \frac{1}{2}\int\limits_0^\varepsilon\int\limits_0^1 \left|\frac{\partial^3}{\partial s^3}|\dot\gamma(\ell,s)|^2\right| d\ell\, ds, \tag{21}$$

where we used notation $\dot\gamma(\ell,s) = \dfrac{\partial}{\partial\ell}\gamma(\ell,s)$.

The right-hand side terms in (20), (21) *have the following representations in terms of field H*:

$$\frac{\partial}{\partial s}|\dot\gamma(\ell,s)|^2 = 2\langle\dot\gamma, \nabla H[\dot\gamma]\rangle, \tag{22}$$

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2 = |\nabla H[\dot\gamma]\,|^2 - \langle\dot\gamma, R(H,\dot\gamma)H\rangle + \langle\dot\gamma, \nabla(\nabla_H H)[\dot\gamma]\rangle. \tag{23}$$

The third derivative has representation $\dfrac{\partial^3}{\partial s^3}|\dot\gamma(\ell,s)|^2 = \langle\dot\gamma, \mathcal{D}[\dot\gamma]\rangle$ *with operator $\mathcal{D}$ that depends on the field H up to its third order covariant derivative and on curvature tensor and its covariant derivative.*

Proof. Let's apply (16) with $N = M\times M$, $X = H^I \otimes H^{II}$ and function $f(z) = \rho(o,x)$ for $z=(o,x)$. Using the minimal property of geodesic, i.e., that the path $\gamma(\ell,s)$ is longer than geodesic from $\gamma(0,s)$ to $\gamma(1,s)$, we can estimate terms with ε in (17) from above and obtain

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon)) - \rho^2(o,x)}{\varepsilon} \le \frac{\displaystyle\int_0^1|\dot\gamma(\ell,\varepsilon)|^2d\ell - \int_0^1|\dot\gamma(\ell,0)|^2d\ell}{\varepsilon}, \tag{24}$$

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon)) + \rho^2(\gamma(0,\varepsilon),\gamma(1,-\varepsilon)) - 2\rho^2(o,x)}{\varepsilon^2} \le$$

$$\le \frac{\displaystyle\int_0^1|\dot\gamma(\ell,\varepsilon)|^2d\ell + \int_0^1|\dot\gamma(\ell,-\varepsilon)|^2d\ell - 2\int_0^1|\dot\gamma(\ell,0)|^2d\ell}{\varepsilon^2}. \tag{25}$$

Above we actually get rid of a problem of implicit representations for operators on metric functions (8). Remark also that the only term with $s=0$, i.e., $\rho^2(o,x)$, was written exactly along geodesic $\gamma(\ell,0)$ from (8).

Let $h(s) = \displaystyle\int_0^1|\dot\gamma(\ell,s)|^2d\ell$. Then estimates

$$h(\varepsilon) - h(0) = \int_0^\varepsilon h'(s)ds = \varepsilon h'(0) + \int_0^\varepsilon [h'(s)-h'(0)]\,ds =$$

$$= \varepsilon h'(0) + \int_0^\varepsilon\left[\int_0^s h''(\tau)d\tau\right]ds = \varepsilon h'(0) + \int_0^\varepsilon\left[\int_{\varepsilon-\tau}^\varepsilon h''(\tau)ds\right]d\tau =$$

$$= \varepsilon h'(0) + \int_0^\varepsilon \tau h''(\tau)d\tau \le \varepsilon\left(h'(0) + \int_0^\varepsilon |h''(\tau)|d\tau\right),$$

$$h(s) + h(-s) - 2h(0) = \int_0^\varepsilon h'(s)ds - \int_{-\varepsilon}^0 h'(s)ds = \int_0^\varepsilon (h'(s) - h'(-s))ds =$$

$$= \int_0^\varepsilon \left[\int_{-s}^s h''(\tau) d\tau \right] ds = \varepsilon^2 h''(0) + \int_0^\varepsilon \left(\int_{-s}^s \big[h''(\tau) - h''(0) \big] d\tau \right) ds \leq$$

$$\leq \varepsilon h''(0) + \int_0^\varepsilon \left(\int_{-s}^s \left\{ \int_0^\tau |h'''(\delta)| d\delta \right\} d\tau \right) ds \leq \varepsilon^2 \left(h''(0) + \frac{1}{2} \int_0^\varepsilon |h'''(\delta)| d\delta \right),$$

lead, due to (24), (25), to the statements (20), (21).

Now let us find expressions for $\dfrac{\partial^m}{\partial s^m} \big| \dot\gamma(\ell, s) \big|^2$ in (20) and (21).

Let us use that by continuity arguments, for any ℓ and sufficiently small $\delta(\ell)$ the path $\big\{ \gamma(\ell, z) \big\}_{z \in (-\delta(\ell), \delta(\ell))}$ completely lies in some coordinate vicinity (x^i). In this coordinate system relation (19) has integral form

$$\gamma^i(\ell, s) = \gamma^i(\ell) + \int_0^s H^i(\gamma(\ell, z)) dz \tag{26}$$

with point $\gamma(\ell)$ on initial geodesic from o to x. Therefore

$$\dot\gamma^i(\ell, s) = \dot\gamma^i(\ell) + \int_0^s \partial_k H^i(\gamma(\ell, z)) \dot\gamma^k(\ell, z) dz,$$

and

$$\frac{\partial}{\partial s} \dot\gamma^i(\ell, s) = \partial_k H^i(\gamma(\ell, s)) \dot\gamma^k(\ell, s) = (\nabla_k H^i - \Gamma_{k\ h}^{\ i} H^h) \dot\gamma^k(\ell, s), \tag{27}$$

where we changed to the covariant derivatives. In particular, from above formula and (19) it follows commutation

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \ell} \gamma^i(\ell, s) = \frac{\partial}{\partial \ell} \frac{\partial}{\partial s} \gamma^i(\ell, s).$$

Relation (27) and autoparallel property of Riemannian connection

$$\partial_k g_{mn}(x) = g_{hn} \Gamma_{k\ m}^{\ h} + g_{mh} \Gamma_{k\ n}^{\ h} \tag{28}$$

lead to relation (22):

$$\frac{\partial}{\partial s} |\dot\gamma(\ell, s)|^2 = \frac{\partial}{\partial s} \Big[g_{ij}(\gamma(\ell, s)) \dot\gamma^i(\ell, s) \dot\gamma^j(\ell, s) \Big] =$$

$$= \partial_k g_{ij} \frac{\partial}{\partial s} \gamma^k \cdot \dot\gamma^i \dot\gamma^j + 2 g_{ij} \dot\gamma^i \frac{\partial}{\partial s} \dot\gamma^j =$$

$$= 2 g_{ij} \dot\gamma^i (\nabla_k H^j) \dot\gamma^k = 2 \langle \dot\gamma, \nabla H[\dot\gamma] \rangle.$$

In a similar way

$$\frac{1}{2} \frac{\partial^2}{\partial s^2} |\dot\gamma(\ell, \varepsilon)|^2 = \frac{\partial}{\partial s} \langle \dot\gamma(\ell, s), \nabla H[\dot\gamma(\ell, s)] \rangle = \frac{\partial}{\partial s} \big\{ g_{ij}(\gamma) \dot\gamma^i [\nabla_k H^j(\gamma)] \dot\gamma^k \big\} =$$

$$= \partial_m g_{ij}(\gamma) H^m \dot\gamma^i [\nabla_k H^j(\gamma)] \dot\gamma^k + g_{ij} \big\{ (\nabla_m H^i - \Gamma_{m\ h}^{\ i}) \dot\gamma^m \big\} [\nabla_k H^j(\gamma)] \dot\gamma^k +$$

$$+g_{ij}\dot\gamma^i\big[\partial_m\nabla_k H^j(\gamma)\cdot H^m(\gamma)\big]\dot\gamma^k+g_{ij}(\gamma)\dot\gamma^i[\nabla_k H^j(\gamma)]\big\{(\nabla_m H^k-\Gamma_{m\ h}^{\ k})\dot\gamma^m\big\},$$

where, after the differentiation of product, we substituted relations (19) and (27).

Using property (28), transforming partial derivative $\partial_m\nabla_k H^j$ to covariant $\nabla_m\nabla_k H^j$ and contracting the terms with connection Γ we have

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2=g_{ij}(\nabla_m H^i)\dot\gamma^m(\nabla_k H^j)\dot\gamma^k+$$

$$+\,g_{ij}\dot\gamma^i(\nabla_m\nabla_k H^j)H^m\dot\gamma^k+g_{ij}(\gamma)\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m.$$

Next commute the covariant derivatives in the second term $\nabla_m\nabla_k H^j=\nabla_k\nabla_m H^j+$ $+\,R_{h\ km}^{\ j}H^h$ to obtain

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2=|\nabla H[\dot\gamma]\,|^2+$$

$$+g_{ij}\dot\gamma^i(\nabla_k\nabla_m H^j+R_{h\ km}^{\ j}H^h)H^m\dot\gamma^k+g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m=$$

$$=|\nabla H[\dot\gamma]\,|^2-\langle\dot\gamma,R(H,\dot\gamma)H\rangle+g_{ij}\dot\gamma^i(\nabla_k\nabla_m H^j)H^m\dot\gamma^k+$$

$$+g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m$$

with curvature operator $R(H,\dot\gamma)$. Redenoting indexes $m\leftrightarrow k$ in the third term we have

$$3^{rd}+4^{th}\text{ terms}=g_{ij}\dot\gamma^i(\nabla_m\nabla_k H^j)H^k\dot\gamma^m+g_{ij}\dot\gamma^i(\nabla_k H^j)(\nabla_m H^k)\dot\gamma^m=$$

$$=g_{ij}\dot\gamma^i(\nabla_m\{H^k\nabla_k H^j\})\dot\gamma^m$$

which leads to (23).

By similar calculation the third derivative $\dfrac{\partial^3}{\partial s^3}|\dot\gamma(\ell,s)|^2$ depends on the field H and its covariant derivatives up to the third order and on the curvature tensor R and its first order covariant derivative.

The lemma is proved.

Now we apply Lemma 1 to find estimates on difference approximation (17) of operator $\mathcal{L}$.

Lemma 2. *Under coercitivity and dissipativity assumptions* (9), (10)

$$\exists K\quad\forall\psi\in C_{0,+}^\infty(M)\colon\ \int\limits_M(\mathcal{L}^*\psi(x))\rho^2(o,x)d\sigma(x)\le K\int\limits_M\psi(x)(1+\rho^2(o,x))d\sigma(x).\tag{29}$$

Proof. Let us make a particular choice $H_0(\ell,s)=\ell^2A_0(\gamma_0(\ell,s))$ and $H_\alpha(\ell,s)=$ $=\ell A_\alpha(\gamma_\alpha(\ell,s))$ in (20), (21) with $\gamma_0(\ell,s)$, $\gamma_\alpha(\ell,s)$ generated by H_0, H_α. Then due to $H(0,s)=0$ the point $\gamma(0,s)=o$ for all $s\in[-\varepsilon,\varepsilon]$ and we have from (22), (23)

$$\left\{\frac{\rho^2(o,\gamma_0(1,\varepsilon))-\rho^2(o,x)}{\varepsilon}+\right.$$

$$\left.+\,\frac{1}{2}\sum_{\alpha=1}^d\frac{\rho^2(o,\gamma_\alpha(1,\varepsilon))+\rho^2(o,\gamma_\alpha(1,-\varepsilon))-2\rho^2(o,x)}{\varepsilon^2}\right\}\le$$

$$\leq \int_0^1 I(\dot\gamma(\ell,0))d\ell + \int_0^\varepsilon \int_0^1 J(\dot\gamma(\ell,s))d\ell\, ds \tag{30}$$

where terms at $s=0$ are equal to

$$I(\dot\gamma) = 2\left\langle \nabla\left(\ell^2 A_0 + \frac{1}{2}\sum_{\alpha=1}^d \nabla_{\ell A_\alpha}[\ell A_\alpha]\right)[\dot\gamma], \dot\gamma\right\rangle +$$

$$+\sum_{\alpha=1}^d \Big\{ |\nabla(\ell A_\alpha)[\dot\gamma]\,|^2 - \langle R(\ell A_\alpha, \dot\gamma)\ell A_\alpha, \dot\gamma\rangle \Big\}$$

and rest terms have form

$$J(\dot\gamma(\ell,s)) = \left|\frac{\partial^2}{\partial s^2}|\dot\gamma_0(\ell,s)|^2\right| + \frac{1}{2}\sum_{\alpha=1}^d \left|\frac{\partial^3}{\partial s^3}|\dot\gamma(\ell,s)|^2\right|.$$

Using that $\nabla\ell[\dot\gamma] = \dfrac{\partial\ell}{\partial\ell} = 1$ and $\nabla_{A_\alpha}\ell = \dfrac{\partial\ell}{\partial s} = 0$, which leads to

$$\nabla(\nabla_{\ell A_\alpha}[\ell A_\alpha])[\dot\gamma] = \nabla_{\dot\gamma}(\ell^2\nabla_{A_\alpha}A_\alpha) = \ell^2\nabla(\nabla_{A_\alpha}A_\alpha)[\dot\gamma] + 2\ell\nabla_{A_\alpha}A_\alpha,$$

we can further rewrite term $I(\dot\gamma)$

$$I(\dot\gamma) = 2\ell^2\langle\nabla A_0[\dot\gamma], \dot\gamma\rangle + 4\ell\langle A_0, \dot\gamma\rangle +$$

$$+\sum_{\alpha=1}^d \Big\{ \ell^2|\nabla A_\alpha[\dot\gamma]\,|^2 + 2\ell\langle A_\alpha, \nabla A_\alpha[\dot\gamma]\rangle + |A_\alpha|^2 - \ell^2\langle R(A_\alpha,\dot\gamma)A_\alpha, \dot\gamma\rangle +$$

$$+\ell^2\langle\nabla(\nabla_{A_\alpha}A_\alpha)[\dot\gamma], \dot\gamma\rangle + 2\ell\langle\nabla_{A_\alpha}A_\alpha, \dot\gamma\rangle\Big\}. \tag{31}$$

Using estimate

$$\Big|\big\langle\nabla A_\alpha[\dot\gamma], A_\alpha\big\rangle\Big| \leq \frac{\ell}{2}\big|\nabla A_\alpha[\dot\gamma]\big|^2 + \frac{1}{2\ell}|A_\alpha|^2$$

we find

$$I(\dot\gamma) \leq \ell^2\left(2\big\langle\nabla\widetilde{A_0}[\dot\gamma], \dot\gamma\big\rangle + 2\sum_{\alpha=1}^d \big|\nabla A_\alpha[\dot\gamma]\big|^2 - \sum_{\alpha=1}^d \big\langle R(A_\alpha,\dot\gamma)A_\alpha, \dot\gamma\big\rangle\right) +$$

$$+4\ell\langle\widetilde{A_0}(\gamma), \dot\gamma\rangle + 2\sum_{\alpha=1}^d |A_\alpha|^2. \tag{32}$$

Using that

$$\nabla^{\gamma(\ell)}\rho^2(o,\gamma(\ell)) = 2\rho(o,\gamma(\ell))\nabla^{\gamma(\ell)}\rho(o,\gamma(\ell)) = 2\ell\rho(o,x)\frac{\dot\gamma(\ell)}{\rho(o,x)} = 2\ell\dot\gamma,$$

we have

$$2\ell\langle\widetilde{A_0}(\gamma), \dot\gamma\rangle = \Big\langle\widetilde{A_0}(\gamma), \nabla^{\gamma(\ell)}\rho^2(o,\gamma(\ell))\Big\rangle.$$

Finally, applying the coercitivity and dissipativity assumptions (9), (10) to (32), we conclude

$$\int_0^1 I(\dot\gamma)d\ell \le \int_0^1 \Big\{2K_C\ell^2|\dot\gamma|^2 + K_{C'}(1+\rho^2(o,\gamma(\ell)))\Big\}d\ell \le K(1+\rho^2(o,x)), \quad (33)$$

where we also used that $\ell \le 1$ and path $\gamma(\ell,0)=\gamma(\ell)$ realizes the geodesic between o and x.

Due to (22), (23) and analogous representation of the third derivative, the rest terms $J(\dot\gamma)$ in (30) are estimated by

$$J(\dot\gamma) \le T_0\,|\dot\gamma_0(\ell,s)|^2 + \sum_{\alpha=1}^d T_\alpha\,|\dot\gamma_\alpha(\ell,s)|^2$$

with some functions T_0, T_α depending on the coefficients of equation and curvature tensor and their covariant derivatives up to the third order. Since in (17) the support of ψ is compact and the limits are taken in some δ-vicinity of point x, the possible paths $\gamma(\ell,s)$ belong to the bounded set

$$Z_{\psi,o,\delta} = \Big\{y\in M\colon\ y \text{ lies on some geodesics from } o \text{ to } x\in B(\operatorname{supp}\psi,\delta)\Big\}.$$

Therefore

$$\int_0^1 J(\dot\gamma)d\ell \le \sup_{z\in Z_{\psi,o,\delta}}\big|\{T_0,T_\alpha\}(z)\big|\cdot\left(\int_0^1|\dot\gamma_0(\ell,s)|^2d\ell + \sum_{\alpha=1}^d\int_0^1|\dot\gamma_\alpha(\ell,s)|^2d\ell\right).$$

Due to (22) the integrals $v_s = \int_0^1 |\dot\gamma(\ell,s)|^2d\ell$ are estimated in the following way

$$v_s \le v_0 + \int_0^s v_s' ds = \rho^2(o,x) + \int_0^s\left(\int_0^1 \big\langle\nabla H[\dot\gamma(\ell,s)],\dot\gamma(\ell,s)\big\rangle d\ell\right)ds \le$$

$$\le \rho^2(o,x) + \sup_{y\in Z_{\psi,o,\delta}}|\nabla H(y)|\cdot\int_0^s h(s)ds$$

which gives

$$v_s \le \rho^2(o,x)\exp\left\{s\sup_{y\in Z_{\psi,o,\delta}}|\nabla H|(y)\right\}.$$

We come to

$$\int_0^\varepsilon\int_0^1 J(\dot\gamma)d\ell\,ds \le$$

$$\le \varepsilon\rho^2(o,x)\cdot\sup_{y\in Z_{\psi,o,\delta}} M(A_0,\nabla A_0,\nabla^2A_0,A_\alpha,\nabla A_\alpha,\nabla^2A_\alpha,\nabla^3A_\alpha,R,\nabla R)$$

with a finite resulting constant $\sup M$ due to the compactness of set $Z_{\psi,o,\delta}$.

Combining the above estimate with (33) and (17), and taking limit $\lim\limits_{\varepsilon\to 0+}$ we have (29). The lemma is proved.

Let us recall that we developed weak estimates on $\mathcal{L}$ because metric $\rho^2(o,x)$ may be non-differentiable at all points x and Ito formula arguments were not applicable. Now, similar to [5], we replace the Ito formula approach to non-explosion estimate (5) – (7) by weak estimates (29) and a statement that some process on manifold represents a supermartingale. By definition, process X_t is **supermartingale** with respect to the flow of σ-algebras $\mathcal{F}_t$ if for all $0 \le s \le t$ it is satisfied $\mathbf{E}(X_t|\mathcal{F}_s) \le X_s$. Here $\mathbf{E}(\cdot|\mathcal{F}_s)$ denotes the conditional expectation with respect to σ-algebra $\mathcal{F}_s$.

Lemma 3. *Under coercitivity and dissipativity conditions* (9), (10) *there is an independent on sets $U\subseteq M$ constant K such that process*

$$[1+\rho^2(o,y_t^U(x))] - K\int\limits_0^t \big[1+\rho^2(o,y_s^U(x))\big]ds \tag{34}$$

is an integrable supermartingale with respect to the canonical flow of σ-algebras $\mathcal{F}_t$, related with d-dimensional Wiener process $W_t^\alpha,\ \alpha=1,\ldots,d$, in (1).

Proof. First recall, that semigroup P_t^U, generated by localized process $y_t^U(x)$ (13), preserves the space $C^\infty_{0,+}(M)$ of nonnegative continuously differentiable functions with compact support. Therefore the integrals below are finite and weak estimate (29) implies

$$\forall\varphi\in C^\infty_{0,+}(M)\colon\ \frac{d}{dt}\int\limits_M \varphi(x)\left\{P_t^U(1+\rho^2(o,\cdot))\right\}(x)d\sigma(x) =$$

$$= \frac{d}{dt}\int\limits_M \left\{[P_t^U]^*\varphi\right\}(x)(1+\rho^2(o,x))d\sigma(x) =$$

$$= \int\limits_M [\mathcal{L}^U]^*\left\{[P_t^U]^*\varphi\right\}(x)\cdot(1+\rho^2(o,x))d\sigma(x) =$$

$$= \int\limits_M [\mathcal{L}]^*\left(\zeta^U(x)\left\{[P_t^U]^*\varphi\right\}(x)\right)\cdot(1+\rho^2(o,x))d\sigma(x) \le$$

$$\le K\int\limits_M \big\{[P_t^U]^*\varphi\big\}(x)\cdot(1+\rho^2(o,x))d\sigma(x) = K\int\limits_M \varphi(x)\big\{P_t^U(1+\rho^2(o,\cdot))\big\}(x)d\sigma(x),$$

where we used that due to the compactness of support of function $\zeta^U\ge 0$ the integrand $\psi=\zeta^U(x)\left\{[P_t^U]^*\varphi\right\}\in C^\infty_{0,+}(M)$, then applied (29) and property $\zeta^U\le 1$. To come to the last line we also applied that $\mathcal{L}1=0$.

Therefore for all $\varphi\in C^\infty_{0,+}(M)$ we have estimate

$$\int\limits_M \varphi(x)\cdot\left\{P_t^U(1+\rho^2(o,\cdot))\right\}(x)d\sigma(x) \le$$

$$\le \int\limits_M \varphi(x)\cdot\left((1+\rho^2(o,x)) + K\int\limits_0^t \left\{P_s^U(1+\rho^2(o,\cdot))\right\}(x)ds\right)d\sigma(x)$$

and its pointwise consequence

$$\left\{P_t^U(1+\rho^2(o,\cdot))\right\}(x) \le (1+\rho^2(o,x)) + K\int\limits_0^t \left\{P_s^U(1+\rho^2(o,\cdot))\right\}(x)ds. \qquad (35)$$

Next we use the Markov property of process $y_t^U(x)$. In particular, for semigroup P_t^U it gives

$$(P_t^U f)(y_s^U(x)) = \mathbf{E}(f(y_{t+s}^U(x)) \,|\, \mathcal{F}_s), \quad t,s \ge 0, \qquad (36)$$

which permits to substitute process y_t^x instead of initial data x. Property (36) can be checked by taking $q_\tau = \mathbf{E}([P_{t-\tau}^U f](y_{s+\tau}^U) \,|\, \mathcal{F}_s)$ and using Ito formula for depending on time functions to get $q_s' = 0$, $s \in [0,t]$. Therefore $q_0 = q_t$ and (36) is true. After that (36) should be closed from C^2 to continuous functions.

Let us substitute instead of x initial data $y_\tau^U(x)$ in (35) to obtain from (36) for function $h(x) = 1+\rho^2(o,x)$ that

$$\mathbf{E}(h(y_{t+\tau}^U(x))|\mathcal{F}_\tau) = (P_t^U h)(y_\tau^U(x)) \le h(y_\tau^U(x)) + K\int\limits_0^t \left\{P_s^U h\right\}(y_\tau^U(x))ds =$$

$$= h(y_\tau^U(x)) + K\,\mathbf{E}\left(\int\limits_\tau^{t+\tau} h(y_s^U(x))ds \,|\mathcal{F}_\tau\right). \qquad (37)$$

Inequality (37) actually means that the process (34) is supermartingale. Indeed, the supermartingale property

$$\mathbf{E}\left(h(y_{t+\tau}^U(x)) - K\int\limits_0^{t+\tau} h(y_s^U(x))ds \,\middle|\, \mathcal{F}_\tau \right) \le h(y_\tau^U(x)) - K\int\limits_0^\tau h(y_s^U(x))ds$$

coincides with (37). The integrability of process (34) follows from the compactness of the closure of set $\{x : \zeta^U(x) > 0\}$.

The lemma is proved.

***End of proof of Theorem* 1.** Suppose that initial data $x \in U$. Introduce stopping time

$$\tau^U(\omega) = \inf\{t \ge 0 \colon y_t^x \notin U\}.$$

The Doob – Meyer free choice theorem, e.g. [13], permits to substitute any finite stopping times $0 \le S \le T$ into the supermartingale property $\mathbf{E}(X_T|\mathcal{F}_S) \le X_S$. Let's apply it with $S = 0$ and $T = t \wedge \tau^U$ to supermartingale (34). Due to $\mathbf{E}(\cdot|\mathcal{F}_0) = \mathbf{E}(\cdot)$ we have

$$m_t = \mathbf{E}(1+\rho^2(o, y_{t\wedge\tau^U}^U(x))) \le (1+\rho^2(o,x)) + K\mathbf{E}\int\limits_0^{t\wedge\tau^U} (1+\rho^2(o,y_s^U(x)))ds \le$$

$$\le m_0 + K\mathbf{E}\int\limits_0^t (1+\rho^2(o,y_{s\wedge\tau^U}^U(x)))ds = m_0 + K\int\limits_0^t m_s ds,$$

where $y^U_{s\wedge\tau^U}(x) = y^U_{\tau^U}(x)$ for $s \geq \tau^U$ is a stopped process on the boundary of U and we enlarged the upper limit of integral.

Gronwall – Bellmann inequality implies that

$$\mathbf{E}(1+\rho^2(o, y^U_{t\wedge\tau^U}(x))) \leq e^{Kt}(1+\rho^2(o,x)). \tag{38}$$

Choose now a sequence of balls $U_n = \{z \in M : \ \rho(o,z) < n\}$, then after number n_0 such that $\rho(o,x) > n_0$, the sequence of stopping times τ^{U_n} is monotone increasing. Due to (38)

$$\mathbf{E}\, 1_{\{\omega: t\geq \tau^{U_n}(\omega)\}} \cdot (1+\rho^2(o, y^U_{t\wedge\tau^U}(x))) \leq$$

$$\leq \mathbf{E}(1+\rho^2(o, y^U_{t\wedge\tau^U}(x))) \leq e^{Kt}(1+\rho^2(o,x))$$

with characteristic function 1_A of set A.

Since for $t \geq \tau^{U_n}$ $\rho(o, y^x_t) = n$, we have

$$\mathbf{E}\, 1_{\{\omega: t\geq \tau^{U_n}(\omega)\}} \leq \frac{e^{Kt}(1+\rho^2(o,x))}{1+n^2} \to 0, \quad n \to \infty,$$

and almost everywhere

$$\tau_\infty = \lim_{n\to\infty} \tau^{U_n} = \infty. \tag{39}$$

As $\zeta^U \big|_{\overline{U}} = 1$, the processes $y^{U_n}_t(x)$ and $y^{U_m}_t(x)$ coincide till the first exit time from vicinity $U_{n\wedge m}$. Therefore the unique solution y^x_t to problem (1) equals to solutions $y^{U_n}_t(x)$ till the first exit time $t \leq \tau^{U_n}$.

Property (39) implies that for coercitive and dissipative coefficients in (1) the limit process $y^x_t = \lim\limits_{n\to\infty} y^{U_n}_t(x)$ is correctly defined for all $t \geq 0$ as a unique solution to (1). In particular it does not explode in a finite time.

The theorem is proved.

In next theorem we generalize statement of Lemma 3 to the polynomials of metric function. Remark that the convex function of supermartingale should not be a supermartingale again, therefore the application of coercitivity and dissipativity conditions (9), (10) is necessary to find appropriate constant K_P in (41).

Theorem 2. *Let P be a positive monotone polynomial function on half-line $\mathbb{R}_+$ such that*

$$\exists C \quad \forall z \geq 0\colon \ (1+z)P'(z) \leq C\, P(z), \quad (1+z)|P''(z)| \leq C\, P'(z). \tag{40}$$

Under coercitivity and dissipativity assumptions (9), (10) *there is constant K_P such that for any vicinity U the process*

$$P(\rho^2(o, y^U_t(x))) - K_P \int\limits_0^t P(\rho^2(o, y^U_s(x)))ds \tag{41}$$

is integrable supermartingale.

Moreover, a unique solution y^x_t to problem (1) *fulfills*

$$\mathbf{E}P(\rho^2(o, y_t^x)) \leq e^{K_P t} P(\rho^2(o, x)) \tag{42}$$

and process

$$P(\rho^2(o, y_t^x)) - K_P \int_0^t P(\rho^2(o, y_s^x))ds \tag{43}$$

represents supermartingale.

Proof. This statement is verified like in the previous theorem, the only difference is that due to the monotonicity of P the first order estimate (20) transforms to

$$\frac{P(\rho^2(\gamma(0,\varepsilon), \gamma(1,\varepsilon))) - P(\rho^2(o,x))}{\varepsilon} \leq$$

$$\leq \frac{P\left(\int_0^1 |\dot\gamma(\ell,\varepsilon)|^2 d\ell\right) - P\left(\int_0^1 |\dot\gamma(\ell,0)|^2 d\ell\right)}{\varepsilon} \leq$$

$$\leq \frac{\partial}{\partial s}\Big|_{s=0} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right) + \int_0^\varepsilon \left|\frac{\partial^2}{\partial s^2} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right)\right| ds =$$

$$= P'(\rho^2(o,x)) \int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell + \int_0^\varepsilon J_P(\dot\gamma_s) ds \tag{44}$$

with $J_P(\dot\gamma_s) = \left|\frac{\partial^2}{\partial s^2} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right)\right|$.

Similarly, the second order estimate (21) becomes

$$\frac{P(\rho^2(\gamma(0,\varepsilon), \gamma(1,\varepsilon))) + P(\rho^2(\gamma(0,\varepsilon), \gamma(1,-\varepsilon))) - 2P(\rho^2(o,x))}{\varepsilon^2} \leq$$

$$\leq \frac{P\left(\int_0^1 |\dot\gamma(\ell,\varepsilon)|^2 d\ell\right) + P\left(\int_0^1 |\dot\gamma(\ell,-\varepsilon)|^2 d\ell\right) - 2P\left(\int_0^1 |\dot\gamma(\ell,0)|^2 d\ell\right)}{\varepsilon^2} \leq$$

$$\leq \frac{\partial^2}{\partial s^2}\Big|_{s=0} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right) + \frac{1}{2}\int_0^\varepsilon \left|\frac{\partial^3}{\partial s^3} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right)\right| ds =$$

$$= P'(\rho^2(o,x)) \int_0^1 \frac{\partial^2}{\partial s^2}\Big|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell +$$

$$+ P''(\rho^2(o,x)) \left[\int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell\right]^2 + \int_0^\varepsilon N_P(\dot\gamma_s) ds, \tag{45}$$

with $N_p(\dot\gamma_s) = \frac{1}{2}\left|\frac{\partial^3}{\partial s^3} P\left(\int_0^1 |\dot\gamma(\ell,s)|^2 d\ell\right)\right|$.

Therefore we have additional term with P'' in comparison to (21). Its multiple is treated in a similar way

$$\int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell = 2\int_0^1 \langle \nabla(\ell A_\alpha)[\dot\gamma], \dot\gamma\rangle d\ell =$$

$$= 2\int_0^1 \langle A_\alpha + \ell\nabla A_\alpha[\dot\gamma], \dot\gamma\rangle d\ell \le 2\left(\int_0^1 \big|A_\alpha + \ell\nabla A_\alpha[\dot\gamma]\big|^2 d\ell\right)^{1/2}\left(\int_0^1 |\dot\gamma|^2 d\ell\right)^{1/2}.$$

Due to $\int_0^1 |\dot\gamma(\ell,o)|^2 d\ell = \rho^2(o,x)$ we have

$$P''(\rho^2(o,x))\left[\int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} |\dot\gamma(\ell,s)|^2 d\ell\right]^2 \le$$

$$\le 4|P''(\rho^2(o,x))|\rho^2(o,x)\int_0^1 \big|A_\alpha + \ell\nabla A_\alpha[\dot\gamma]\big|^2 d\ell \le$$

$$\le 8C\,P'(\rho^2(o,x))\int_0^1 \big(|A_\alpha|^2 + \ell^2|\nabla A_\alpha[\dot\gamma]|^2\big) d\ell.$$

This leads to additional terms in the right-hand side of (31) and, due to the coercitivity and dissipativity assumptions (9), (10), gives estimate on all terms in lines (44) and (45)

$$\text{line (44)} + \text{line (45)} \le P'(\rho^2(o,x))\cdot K(1+\rho^2(o,x)) + \int_0^\varepsilon \big\{J_P(\dot\gamma_s) + N_P(\dot\gamma_s)\big\} ds \le$$

$$\le KC\,P(\rho^2(o,x)) + \int_0^\varepsilon \big\{J_P(\dot\gamma_s) + N_P(\dot\gamma_s)\big\} ds.$$

Therefore (33) transforms to

$$\left\{\frac{P(\rho^2(o,\gamma_0(1,\varepsilon))) - P(\rho^2(o,x))}{\varepsilon} + \right.$$

$$\left. + \frac{1}{2}\sum_{\alpha=1}^d \frac{P(\rho^2(o,\gamma_\alpha(1,\varepsilon))) + P(\rho^2(o,\gamma_\alpha(1,-\varepsilon))) - 2P(\rho^2(o,x))}{\varepsilon^2}\right\} \le$$

$$\le KC\,P(\rho^2(o,x)) + \varepsilon \sup_{\gamma_s\subset Z_{\psi,o,\delta}} \big\{J_P(\dot\gamma_s) + N_P(\dot\gamma_s)\big\}. \tag{46}$$

Like in the proof of Lemma 2, the rest terms with J_P, N_P vanish for $\varepsilon \to 0+$. Therefore (29) adopts form

$$\int\limits_M (\mathcal{L}^*\psi(x))P(\rho^2(o,x))d\sigma(x) \le K_P \int\limits_M \psi(x)P(\rho^2(o,x))d\sigma(x). \tag{47}$$

Proceeding further like in Lemma 3, we obtain that (41) is a supermartingale.

In particular, an analogue of estimate (38) is true

$$\mathbf{E}P(\rho^2(o, y^U_{t\wedge\tau^U}(x))) \le e^{K_P t}P(\rho^2(o,x)). \tag{48}$$

Next consider measurable random set $V_n(t) = \{\omega\colon \forall s \in [0,t]\ y^x_t(\omega) \in U_n\}$ that corresponds to paths of process $y^x_t(\omega)$, staying inside of set U_n till time t. Then $y^x_t(\omega) = y^{U_n}_{t\wedge\tau^{U_n}}(x,\omega)$ for all $\omega \in V_n(t)$ and (48) leads to

$$\mathbf{E}\, 1_{V_n(t)} P\big(\rho^2(o, y^x_t(x))\big) \le \mathbf{E}P\big(\rho^2(o, y^{U_n}_{t\wedge\tau^{U_n}}(x))\big) \le e^{K_P t}P(\rho^2(o,x)). \tag{49}$$

Due to non-explosion $\lim\limits_{n\to\infty} \tau^{U_n}(\omega) = \infty$, each path $y^x_t(\omega)$ completely lies in some U_n for sufficiently large n. Therefore sequence $V_n(t)$ is increasing to the full probability space and lower limit $\varliminf\limits_{n\to\infty} 1_{V_n(t)}(\omega) = 1$ a.e. The application of Fatoux lemma $\Big($i.e., that for $f_n \ge 0$ the lower limits fulfill $\displaystyle\int \varliminf_{n\to\infty} f_n d\mu \le \varliminf_{n\to\infty} \int f_n d\mu\Big)$ to the left-hand side of (49):

$$\mathbf{E}\, P\big(\rho^2(o, y^x_t(x))\big) \le \varliminf_{n\to\infty} \mathbf{E}P\big(\rho^2(o, y^{U_n}_{t\wedge\tau^{U_n}}(x))\big) \le e^{K_P t}P(\rho^2(o,x))$$

leads to the statement (42).

To check that (43) represents supermartingale, let us apply Doob – Meyer free choice theorem with $S = s\wedge\tau^U$ and $T = t\wedge\tau^U$ to supermartingales (41). It follows that processes

$$\theta^n_t = P\big(\rho^2(o, y^{U_n}_{t\wedge\tau^{U_n}})\big) - K_P \int\limits_0^{t\wedge\tau^{U_n}} P\big(\rho^2(o, y^{U_n}_{s\wedge\tau^{U_n}})\big)ds$$

represent supermartingales, i.e., for all $0 \le s \le t$ and $A \in \mathcal{F}_s$

$$\mathbf{E}\theta^n_t 1_A \le \mathbf{E}\theta^n_s 1_A. \tag{50}$$

Since for all $\omega \in V_n(t)$ and $s \in [0,t]$ process θ^n_t coincides with the limit process

$$\theta^n_s(\omega) = \theta^\infty_s(\omega) \overset{\mathrm{df}}{=} P(\rho^2(o, y^x_t(\omega))) - K_P \int\limits_0^s P(\rho^2(o, y^x_s(\omega)))ds,$$

we can replace θ^n_s by θ^∞_s, $s \in [0,t]$, on set $V_n(t)$ in the calculation below

$$\mathbf{E}\left(1_{V_n(t)}\theta^\infty_t + (1 - 1_{V_n(t)})\theta^n_t\right)1_A =$$

$$= \mathbf{E}\theta^n_t 1_A \le \mathbf{E}\theta^n_s 1_A = \mathbf{E}\left(1_{V_n(t)}\theta^\infty_s + (1 - 1_{V_n(t)})\theta^n_s\right)1_A. \tag{51}$$

Here we also applied property (50).

Then notice that the independent on n estimate is true

$$\sup_{n\geq 1}\sup_{s\in[0,t]}\mathbf{E}[\theta_s^n]^2<\infty$$

due to (48) applied to function P^2 instead of P (this function again fulfills (40)). Then, because $(1-1_{V_n(t)})1_A\to 1_A$ a.e. for $n\to\infty$, the terms with $(1-1_{V_n(t)})$ in (51) tend to zero.

Moreover, due to estimate (42), process $|\theta_s^\infty|$ is integrable $\sup_{s\in[0,t]}\mathbf{E}|\theta_s^\infty|<\infty$ and gives an integrable majorant for $1_{V_n(t)}\theta_s^\infty 1_A$ for $s\in[0,t]$. Using that $1_{V_n(t)}\to 1$ a.e. one takes a limit in (51) to get the supermartingale property $\forall\, 0\leq s\leq t\ \mathbf{E}\theta_t^\infty 1_A\leq$ $\leq\mathbf{E}\theta_s^\infty 1_A$ for process θ_t^∞ (43).

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