UDC 517.5

A. Sandikçi, A. T. Gürkanli (Ondokuz Mayis Univ., Turkey)

THE SPACE $\Omega_m^p(R^d)$ AND SOME PROPERTIES
ПРОСТІР $\Omega_m^p(R^d)$ ТА ДЕЯКІ ВЛАСТИВОСТІ

Let m be a v-moderate function defined on R^d and let $g \in L^2(R^d)$. In this work, we define $\Omega_m^p(R^d)$ to be the vector space of $f \in L_m^2(R^d)$ such that the Gabor transform $V_g f$ belongs to $L^p(R^{2d})$, where $1 \leq p < \infty$. We endowe it with a norm and show that it is a Banach space with this norm. We also study some preliminary properties of $\Omega_m^p(R^d)$. Later we discuss inclusion properties and obtain the dual space of $\Omega_m^p(R^d)$. At the end of this work, we study multipliers from $L_w^1(R^d)$ into $\Omega_w^p(R^d)$ and from $\Omega_w^p(R^d)$ into $L_{w^{-1}}^\infty(R^d)$, where w is Beurling's weight function.

Нехай m є v-помірною функцією, що визначена на R^d, і $g \in L^2(R^d)$. У даній роботі $\Omega_m^p(R^d)$ визначено як векторний простір елементів $f \in L_m^2(R^d)$ таких, що перетворення Габора $V_g f$ належить до $L^p(R^{2d})$, де $1 \leq p < \infty$. Цей простір оснащено нормою і показано, що він є банаховим із цією нормою. Також вивчено деякі попередні властивості $\Omega_m^p(R^d)$. Розглянуто властивості включення, одержано дуальний до $\Omega_m^p(R^d)$ простір. Насамкінець вивчено мультиплікатори з $L_w^1(R^d)$ до $\Omega_w^p(R^d)$ та з $\Omega_w^p(R^d)$ до $L_{w^{-1}}^\infty(R^d)$, де w є ваговою функцією Берлінга.

1. Introduction. Throughout this paper, $C_c(R^d)$ and $C_0(R^d)$ denote the space of complex-valued continuous functions on R^d with compact support and the space of complex-valued continuous functions on R^d vanishing at infinity, respectively. For $1 \leq p \leq \infty$, we consider the Lebesgue spaces $\left(L^p(R^d), \|\cdot\|_p\right)$. For any function $f : R^d \to C$, the translation and modulation operator are defined as $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i wt} f(t)$ for $x, w \in R^d$, respectively. It is easy to see that $T_x M_t = e^{-2\pi i xt} M_t T_x$ and $\|T_x M_t f\|_p = \|f\|_p$ [1]. A weight is a positive locally integrable function $m : R^d \to (0, \infty)$. A weight v is called submultiplicative if $v(x+y) \leq v(x) v(y)$ for all $x, y \in R^d$. A weight w is right moderate (or simply v-moderate) if there exists a submultiplicative function v such that $w(x+y) \leq w(x) v(y)$ for all $x, y \in R^d$. Especially any continuous submultiplicative function satisfying $w(x) \geq 1$ is called Beurling's weight function. For $1 \leq p < \infty$, we set

$$L_w^p(R^d) = \left\{ f \mid fw \in L^p(R^d) \right\},$$

$$\|f\|_{p,w} = \left\{ \int\limits_{R^d} |f(x)|^p w^p(x)\, dx \right\}^{\frac{1}{p}}.$$

This is a Banach space with the norm.

Particularly, $L_w^1(R^d)$ is a Banach convolution algebra. It is called a Beurling algebra. Let $L_{w^{-1}}^\infty(R^d)$ be the algebra of all measurable functions f on R^d for which

$$\|f\|_{\infty, w^{-1}} = \operatorname*{ess\,sup}_{x \in R^d} \left| \frac{f(x)}{w(x)} \right| < \infty.$$

Under the norm $\|\cdot\|_{\infty, w^{-1}}$, $L_{w^{-1}}^\infty(R^d)$ is a Banach algebra, which is the dual space of $L_w^1(R^d)$ [2]. It is also known that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the dual of $L_w^p(R^d)$ is the space $L_{w^{-1}}^q(R^d)$ [2–4].

Let w_1 and w_2 be two weight functions. We say that $w_2 < w_1$ if and only if there exists $c > 0$ such that $w_2(x) < cw_1(x)$ for all $x \in R^d$. Two weights w_1 and w_2 are equivalent, denoted $w_1 \approx w_2$, if there exist contants $A, B > 0$ such that $Aw_1(x) \leq \leq w_2(x) \leq Bw_1(x)$.

Let $\langle x,t \rangle = \sum_{i=1}^{d} x_i t_i$ be the usual scalar product on R^d. For $f \in L^1(R^d)$, the Fourier transform $\hat{f}$ (or Ff) is given by the relation

$$\hat{f}(t) = \int\limits_{R^d} f(x)\, e^{-2\pi i \langle x,t \rangle} dx.$$

It is known that $\hat{f} \in C_0(R^d)$.

In engineering, t is a frequency and $\hat{f}(t)$ is the amplitude of the frequency t. In the physics, t is the momentum variable. To obtain information about local properties of f and about some local frequency spectrum, we restrict f to an interval and take the Fourier transform. Therefore, given any fixed function $g \neq 0$ (called the window function), the Short-Time Fourier transform (STFT) or Gabor transform, of a function f with respect to g is defined by

$$V_g f(x,w) = \int\limits_{R^d} f(t)\, \overline{g(t-x)} e^{-2\pi i t w} dt$$

for $x, w \in R^d$. It is known that if $f, g \in L^2(R^d)$, then $V_g f \in L^2\left(R^d \times R^d\right)$ and $V_g f$ is uniformly continuous. Moreover,

$$V_g\left(T_u M_\eta f\right)(x,w) = e^{-2\pi i u w} V_g f\left(x-u, w-\eta\right)$$

for all $x, w, u, \eta \in R^d$ [1]. A very important inequality for STFT was proved by E. Lieb [5]. That is if $f, g \in L^2(R^d)$ and $2 \leq p < \infty$, then

$$\iint\limits_{R^{2d}} \left|V_g f(x,w)\right|^p dxdw \leq \left(\frac{2}{p}\right)^d \left(\|f\|_2 \|g\|_2\right)^p.$$

If $1 \leq p \leq 2$ and $f, g \in L^2(R^d)$, then

$$\iint\limits_{R^{2d}} \left|V_g f(x,w)\right|^p dxdw \geq \left(\frac{2}{p}\right)^d \left(\|f\|_2 \|g\|_2\right)^p.$$

The equality holds if and only if $p > 1$ and f, g are certain Gaussians.

For two Banach modules B_1 and B_2 over a Banach algebra A, we write $M_A(B_1, B_2)$ or $\mathrm{Hom}_A(B_1, B_2)$ for the space of all bounded linear operators satisfying $T(ab) = = aT(b)$ for all $a \in A, b \in B_1$. This operators are called multiplier (right) or module homomorphism from B_1 into B_2.

2. The space $\Omega_m^p(R^d)$.

Definition 1. *Let v be a weight and m be a v-moderate function on R^d. For $1 \leq \leq p < \infty$ and $g \in L^2(R^d)$, define*

$$\Omega_m^p(R^d) = \left\{ f \in L_m^2(R^d) \colon V_g f \in L^p\left(R^{2d}\right) \right\}.$$

It is easy to see that $\|f\|_\Omega = \|f\|_{2,m} + \|V_g f\|_p$ is a norm on the vector space $\Omega_m^p(R^d)$.

Theorem 1. *Let* $1 \leq p < \infty$. *Then the following assertions are true*:

a) $\left(\Omega_m^p(R^d), \|\cdot\|_\Omega\right)$ *is a Banach space*;

b) *if* $v(z) \geq 1$ *is a submultiplicative function, then* $\Omega_m^p(R^d)$ *is a translation invariant and the function* $z \to T_z f$ *is continuous from* R^d *into* $\Omega_m^p(R^d)$;

Proof. a) Suppose that $(f_n)_{n\in N}$ is a Cauchy sequence in $\Omega_m^p(R^d)$. Clearly, $(f_n)_{n\in N}$ and $(V_g f_n)_{n\in N}$ are Cauchy sequences in $L_m^2(R^d)$ and $L^p\left(R^{2d}\right)$, respectively. Since $L_m^2(R^d)$ and $L^p\left(R^{2d}\right)$ are Banach spaces, there exists $f \in L_m^2(R^d)$ and $h \in L^p\left(R^{2d}\right)$ such that $\|f_n - f\|_{2,m} \to 0$, $\|V_g f_n - h\|_p \to 0$. Moreover, using the subsequence property, we obtain $V_g f = h$. Thus, $\|f_n - f\|_\Omega \to 0$ and $f \in \Omega_m^p(R^d)$. Hence, $\Omega_m^p(R^d)$ is a Banach space.

b) Let $f \in \Omega_m^p(R^d)$ be given. Then we write $f \in L_m^2(R^d)$ and $V_g f \in L^p\left(R^{2d}\right)$. It is easy to see that $\|T_z f\|_{2,m} \leq v(z)\|f\|_{2,m}$ and $T_z f \in L_m^2(R^d)$ for all $z \in R^d$. Using the properties of Gabor transform, we obtain

$$V_g\left(T_z f\right)(x,w) = V_g f\left(x - z, w\right) \tag{1}$$

and

$$\|V_g\left(T_z f\right)\|_p = \|V_g f\|_p \,.$$

Thus, we have

$$\|T_z f\|_\Omega \leq v(z)\|f\|_\Omega < \infty$$

and $T_z f \in \Omega_m^p(R^d)$. This means that $\Omega_m^p(R^d)$ is a translation invariant. From equality (1) we have

$$|V_g\left(T_z f\right)(x,w)| = |V_g f\left(x - z, w\right)|$$

and

$$\|V_g\left(T_z f\right) - V_g f\|_p = \left\|T_{(z,0)}\left(V_g f\right) - V_g f\right\|_p .$$

It is known that the function $z \to T_z f$ and $(z,u) \to T_{(z,u)} f$ are continuous from R^d into $L_m^2(R^d)$ and from R^{2d} into $L^p\left(R^{2d}\right)$, respectively, by Lemma 1.6 in [6]. By using these properties, the proof is completed.

Theorem 2. $\Omega_m^p(R^d)$ *is an essential Banach module over* $L_v^1(R^d)$.

Proof. It is known that $\Omega_m^p(R^d)$ is a Banach space by Theorem 1. Let $f \in \Omega_m^p(R^d)$ and $h \in L_v^1(R^d)$. Since $L_m^p(R^d)$ is a Banach module over $L_v^1(R^d)$, we have $f * h \in$ $\in L_m^2(R^d)$ and $\|f * h\|_{2,m} \leq \|f\|_{2,m}\|h\|_{1,v}$ [7]. Moreover, using the equality $V_g f(x, w) = e^{-2\pi i x w}\left(f * M_w g^*\right)(x)$, we obtain

$$\begin{aligned}\|V_g\left(f * h\right)\|_p &= \left\|e^{-2\pi i x w}\left(\left(f * h\right) * M_w g^*\right)\right\|_p \leq \\ &\leq \|h\|_1\|f * M_w g^*\|_p \leq \|h\|_{1,v}\|V_g f\|_p < \infty.\end{aligned} \tag{2}$$

Thus, $V_g\left(f * h\right) \in L^p\left(R^{2d}\right)$. From (2) we write

$$\|f*h\|_{\Omega} = \|f*h\|_{2,m} + \|V_g(f*h)\|_p \leq$$
$$\leq \|f\|_{2,m}\|h\|_{1,v} + \|V_g f\|_p \|h\|_{1,v} = \|h\|_{1,v}\|f\|_{\Omega}.$$

Hence, $\Omega_m^p(R^d)$ is a Banach module over $L_v^1(R^d)$.

It is known that $L_v^1(R^d)$ has a bounded approximate identity [8]. To show that $\Omega_m^p(R^d)$ is an essential module in $L_v^1(R^d)$, it suffices to prove that $L_v^1(R^d) * \Omega_m^p(R^d)$ is dense in $\Omega_m^p(R^d)$ by Module Factorization Theorem. Take any $h \in \Omega_m^p(R^d)$. Since the map $z \to T_z h$ is contiuonus from R^d into $\Omega_m^p(R^d)$ by Theorem 1, for any given $\varepsilon > 0$ there exists a compact neighbourhood U of the unit element of R^d such that $\|T_z h - h\|_{\Omega} < \varepsilon$ for all $z \in U$. Let f be a continuous function on R^d for which $f \geq 0$, $\int_U f(x)dx = 1$ and the support of f is contained in U. Then

$$\|f*h - f\|_{\Omega} = \left\| \int_{R^d} f(z)h(y-z)\,dz - \int_{R^d} f(z)h(y)\,dz \right\|_{\Omega} =$$
$$= \left\| \int_U f(z)\left(h(y-z) - h(y)\right)dz \right\|_{\Omega} \leq$$
$$\leq \int_U f(z)\|T_z h - h\|_{\Omega}\,dz = \|T_z h - h\|_{\Omega} \int_U f(z)dz =$$
$$= \|T_z h - h\|_{\Omega} < \varepsilon.$$

Thus, $L_v^1(R^d) * \Omega_m^p(R^d)$ is dense in $\Omega_m^p(R^d)$ and the proof is completed.

***Corollary* 1.** *Let $(e_\alpha)_{\alpha \in I}$ be a bounded approximate identity in $L_v^1(R^d)$. Since $\Omega_m^p(R^d)$ is an essential Banach module over $L_v^1(R^d)$, we have $\lim_{\alpha} e_\alpha * f = f$ for all $f \in \Omega_m^p(R^d)$ by Corollary* 15.3 *in* [9].

Proposition 1. *If $2 \leq p < \infty$, then the spaces $\Omega_m^p(R^d)$ and $L_m^2(R^d)$ are algebrically isomorphic and topologically homeomorphic.*

Proof. Take any $f \in \Omega_m^p(R^d)$. Then we write $f \in L_m^2(R^d)$ and $\|f\|_2 \leq \|f\|_{2,m} < \infty$. Conversely, let $f \in L_m^2(R^d)$. By the Lieb Uncertainty Principle, we have

$$\|V_g f\|_p \leq \left(\frac{2}{p}\right)^{\frac{d}{p}} \|f\|_2 \|g\|_2 < \infty.$$

Thus, $f \in \Omega_m^p(R^d)$ and consequently, we obtain $\Omega_m^p(R^d) = L_m^2(R^d)$. Moreover, it is easy to see that the norms $\|\cdot\|_{2,m}$, $\|\cdot\|_{\Omega}$ are equivalent.

Proposition 2. *Let $2 \leq p < \infty$. Then $C_c(R^d)$ is dense in $\Omega_m^p(R^d)$.*

Proof. It is easy to see the inclusion $C_c(R^d) \subset \Omega_m^p(R^d)$. Take any $f \in \Omega_m^p(R^d)$. Let $C_0 = \max\left\{ \left(\frac{2}{p}\right)^{\frac{d}{p}} \|g\|_2, 1 \right\}$. Since $C_c(R^d)$ is dense in $L_m^2(R^d)$, for any given $\varepsilon > 0$ there exists $h \in C_c(R^d)$ such that

$$\|f-h\|_2 \leq \|f-h\|_{2,m} < \frac{\varepsilon}{2C_0} < \frac{\varepsilon}{2}. \tag{3}$$

By the Lieb Uncertainty Principle and (3), we write

$$\|f-h\|_\Omega = \|f-h\|_{2,m} + \|V_g(f-h)\|_p \leq$$
$$\leq \|f-h\|_{2,m} + \left(\frac{2}{p}\right)^{\frac{d}{p}} \|g\|_2 \|f-h\|_2 <$$
$$< \frac{\varepsilon}{2C_0} + \left(\frac{2}{p}\right)^{\frac{d}{p}} \|g\|_2 \frac{\varepsilon}{2C_0} < \frac{\varepsilon}{2} + C_0\frac{\varepsilon}{2C_0} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof.

Lemma 1. *Let w be Beurling's weight function. Then for every $f \in \Omega_w^p(R^d)$, $f \neq 0$, there exists $c(f) > 0$ such that*

$$c(f)\,w(z) \leq \|T_z f\|_\Omega \leq w(z)\|f\|_\Omega.$$

Proof. Let $f \in \Omega_w^p(R^d)$. By Theorem 1.9 in [6], there exists $c(f) > 0$ such that

$$c(f)\,w(z) \leq \|T_z f\|_{2,w} \leq w(z)\|f\|_{2,w}.$$

Moreover, it is known that $\|V_g(T_z f)\|_p = \|V_g f\|_p$ by Theorem 1. Hence,

$$c(f)\,w(z) \leq \|T_z f\|_{2,w} + \|V_g(T_z f)\|_p \leq w(z)\|f\|_{2,w} + \|V_g f\|_p \leq$$
$$\leq w(z)\|f\|_{2,w} + w(z)\,\|V_g f\|_p =$$
$$= w(z)\left(\|f\|_{2,w} + \|V_g f\|_p\right) = w(z)\|f\|_\Omega$$

for all $f \in \Omega_w^p(R^d)$. Consequently, we obtain

$$c(f)\,w(z) \leq \|T_z f\|_\Omega \leq w(z)\|f\|_\Omega.$$

It is easy to prove the following lemma.

Lemma 2. *Let w_1 and w_2 be Beurling's weight functions and $\Omega_{w_1}^p(R^d) \subset \Omega_{w_2}^p(R^d)$. Then $\Omega_{w_1}^p(R^d)$ is a Banach space under the norm $\|f\|_{\Omega_w^p} = \|f\|_{\Omega_{w_1}^p} + \|f\|_{\Omega_{w_2}^p}$.*

Theorem 3. *If w_1 and w_2 are Beurling's weight functions, then $\Omega_{w_1}^p(R^d) \subset \Omega_{w_2}^p(R^d)$ if and only if $w_2 < w_1$.*

Proof. Suppose $w_2 < w_1$. Then there exists $c > 0$ such that $w_2(z) \leq cw_1(z)$ for all $z \in R^d$. Let $f \in \Omega_{w_1}^p(R^d)$. Then we write $\|fw_2\|_2 \leq c\,\|fw_1\|_2$. Furthermore, since $\|V_g f\|_p < \infty$, we have

$$\|f\|_{\Omega_{w_2}^p} = \|f\|_{2,w_2} + \|V_g f\|_p \leq c\|f\|_{2,w_1} + c\,\|V_g f\|_p = c\|f\|_{\Omega_{w_1}^p} < \infty$$

and $\Omega_{w_1}^p(R^d) \subset \Omega_{w_2}^p(R^d)$.

Conversely, assume that $\Omega_{w_1}^p(R^d) \subset \Omega_{w_2}^p(R^d)$. For given $f \in \Omega_{w_1}^p(R^d)$ we have $f \in \Omega_{w_2}^p(R^d)$. By the Lemma 1, the function $z \to \|T_z f\|_{\Omega_{w_1}^p}$ is equivalent to the weight function w_1 and the function $z \to \|T_z f\|_{\Omega_{w_2}^p}$ is equivalent to the weight function w_2. Hence, there are costants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 w_1(z) \leq \|T_z f\|_{\Omega_{w_1}^p} \leq c_2 w_1(z), \tag{4}$$

$$c_3 w_2(z) \leq \|T_z f\|_{\Omega_{w_2}^p} \leq c_4 w_2(z) \tag{5}$$

for every $z \in R^d$. By Lemma 2, the space $\Omega^p_{w_1}(R^d)$ is a Banach space under the norm $\|f\|_{\Omega^p_w} = \|f\|_{\Omega^p_{w_1}} + \|f\|_{\Omega^p_{w_2}}$, $f \in \Omega^p_{w_1}(R^d)$. Thus, by closed graph mapping theorem the norms $\|\cdot\|_{\Omega^p_{w_1}}$ and $\|\cdot\|_{\Omega^p_w}$ are equivalent. Hence, there exists $c > 0$ such that

$$\|f\|_{\Omega^p_{w_2}} \le c\|f\|_{\Omega^p_{w_1}} \tag{6}$$

for all $f \in \Omega^p_{w_1}(R^d)$. Moreover, we also have $T_z f \in \Omega^p_{w_2}(R^d)$ and

$$\|T_z f\|_{\Omega^p_{w_2}} \le c\,\|T_z f\|_{\Omega^p_{w_1}}\,.$$

If we combine (4), (5) and (6) find $w_2(z) \le \dfrac{cc_2}{c_3} w_1(z)$. If we take $k = \dfrac{cc_2}{c_3}$, then we have $w_2(z) \le k w_1(z)$ for all $z \in R^d$.

The theorem is proved.

Let $\Phi_p : \Omega^p_m(R^d) \to L^2_{m^{-1}}(R^d) \times L^q\left(R^{2d}\right)$, $\Phi_p\,(f) = (f, V_g f)$ be a function and $H = \Phi_p\left(\Omega^p_m(R^d)\right)$. Then

$$\|\Phi_p\,(f)\| = \|(f, V_g f)\| = \|f\|_{2,m} + \|V_g f\|_p$$

is a norm on H and Φ_p is an isometry.

Theorem 4. *If* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *then the dual of the space* $\Omega^p_m(R^d)$ *is the space*

$$L^2_{m^{-1}}(R^d) \times L^q\left(R^{2d}\right)/K,$$

where

$$K = \left\{ \begin{array}{c} (\Phi, \Psi) \in L^2_{m^{-1}}(R^d) \times L^q\left(R^{2d}\right) \left| \int\limits_{R^d} f\,(x)\,\Phi\,(x)\,dx + \right. \\ + \iint\limits_{R^{2d}} V_g f\,(y,w)\,\Psi\,(y,w)\,dydw = 0, \quad (f, V_g f) \in H \end{array} \right\}.$$

Proof. Φ_p is an isometry. Since $\Omega^p_m(R^d)$ is a Banach space, $H = \Phi_p\left(\Omega^p_m(R^d)\right)$ is closed. By the Duality Theorem in [10], we have

$$H^* \cong L^2_{m^{-1}}(R^d) \times L^q\left(R^{2d}\right)/K, \tag{7}$$

where H^* is the dual of H. Also, since Φ_p is an isometry, from (7) we obtain

$$\left(\Omega^p_m(R^d)\right)^* \cong L^2_{m^{-1}}(R^d) \times L^q\left(R^{2d}\right)/K.$$

3. Multiplier from $L^1_w(R^d)$ into $\Omega^p_w(R^d)$ and from $\Omega^p_w(R^d)$ into $L^\infty_{w^{-1}}(R^d)$. Let w be Beurling's weight function on R^d and $(e_\alpha)_{\alpha \in I}$ be bounded approximate identity in the weighted space $L^1_w(R^d)$. The relative completion $\widetilde{\Omega^p_w}(R^d)$ of $\Omega^p_w(R^d)$ is defined by

$$\widetilde{\Omega^p_w}(R^d) = \left\{ f \in L^1_w(R^d) \,\middle|\, f * e_\alpha \in \Omega^p_w(R^d) \right.$$
$$\left. \text{for all} \quad \alpha \in I \quad \text{and} \quad \sup_{\alpha \in I} \|f * e_\alpha\|_\Omega < \infty \right\}.$$

It is known that $\widetilde{\Omega_w^p}(R^d)$ is a Banach space with the norm

$$\|f\|_\Omega = \sup_{\alpha\in I} \|f * e_\alpha\|_\Omega .$$

It is also known that this does not depend on the approximate identity $(e_\alpha)_{\alpha\in I}$ [11].

Theorem 5. *If $g \in L^2(R^d)$, then the space $M\left(L_w^1(R^d), \Omega_w^p(R^d)\right)$ and $\widetilde{\Omega_w^p}(R^d)$ are algebrically isomorphic and homeomorphic.*

Proof. It is known that $\Omega_w^p(R^d)$ is an essential Banach module over $L_w^1(R^d)$ by Theorem 3. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity of $L_w^1(R^d)$. Hence,

$$\|f * e_\alpha - f\|_\Omega \to 0$$

for all $f \in \Omega_w^p(R^d)$ by Corollary 1. We also have $\|f\|_{2,w} \le \|f\|_\Omega$. Thus, by Theorem 3.8 in [11], we have

$$M\left(L_w^1(R^d), \qquad \Omega_w^p(R^d)\right) \cong \widetilde{\Omega_w^p}(R^d).$$

Theorem 6. *If $g \in L^2(R^d)$, then the space $\mathrm{Hom}_{L_w^1}\left(\Omega_w^p(R^d), L_{w^{-1}}^\infty(R^d)\right)$ and $L_{w^{-1}}^2(R^d) \times L^q\left(R^{2d}\right)/K$ are algebrically isomorphic and homeomorphic.*

Proof. It is known that $\Omega_w^p(R^d)$ is an essential Banach module over $L_w^1(R^d)$ by Theorem 2 and $\left(\Omega_w^p(R^d)\right)^* \cong L_{w^{-1}}^2(R^d) \times L^q\left(R^{2d}\right)/K$ by Theorem 4. If we use Corollary 2.13 in [4], we obtain

$$\mathrm{Hom}_{L_w^1}\left(\Omega_w^p(R^d), L_{w^{-1}}^\infty(R^d)\right) = \mathrm{Hom}_{L_w^1}\left(\Omega_w^p(R^d), \left(L_w^1(R^d)\right)^*\right) =$$
$$= \left(\Omega_w^p(R^d) * L_w^1(R^d)\right)^* =$$
$$= \left(\Omega_w^p(R^d)\right)^* \cong L_{w^{-1}}^2(R^d) \times L^q\left(R^{2d}\right)/K.$$

1. *Gröchenig K.* Foundations of time-freuquency analysis. – Boston: Birkhauser, 2001.
2. *Rieter H.* Classical harmonic analysis and local compact groups. – Oxford: Clarendan Press, 1968.
3. *Murthy G. N. K., Unni K. R.* Multipliers on weighted spaces, functional analysis and its applications // Int. Conf. Modras. Lect. Notes Math. – 1973. – **399**.
4. *Rieffel M. A.* Induced Banach representation of Banach algebras and locally compact groups // J. Funct. Anal. – 1967. – P. 443 – 491.
5. *Lieb E. H.* Integral bounds for radar ambiguity functions and Wigner distributions // J. Math. Phys. – 1990. – **31**, № 3. – P. 594 – 599.
6. *Fischer R. H., Gürkanli A. T., Liu T. S.* On family of weighted spaces // Math. Slovaca. – 1996. – **46**, № 1. – P. 71 – 82.
7. *Feichtinger H.* Generalized amalgams, with applications to Fourier transform // Can. J. Math. – 1990. – **42**, № 3. – P. 395 – 409.
8. *Gaudry G. I.* Multipliers of weighted Lebesgue and measure spaces // Proc. London Math. Soc. – 1969. – **3**, № 19.
9. *Doran R. S., Wichman J.* Approximate identities and factorization in Banach modules // Lect. Notes Math. – 1979. – **768**.
10. *Liu T., Van Rooij A.* Sums and intersections of normed linear spaces // Math. Nachr. – 1969. – **42**. – S. 29 – 42.
11. *Duyar C., Gürkanli A. T.* Multipliers and relative completion in weighted Lorentz spaces // Acta Math. Sci. Ser. B. – 2003. – **23**, № 4. – P. 467 – 476.

Received 16.05.2005