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SIGN CHANGES IN RATIONAL L_w^1-APPROXIMATION
ЗНАКОЗМІНИ У РАЦІОНАЛЬНОМУ L_w^1-НАБЛИЖЕННІ

Let $f \in L_w^1[-1,1]$, let $r_{n,m}(f)$ be a best rational L_w^1-approximation for f with respect to real rational functions of degree at most n in the numerator and of degree at most m in the denominator, let $m = m(n)$, and let $\lim_{n\to\infty}(n - m(n)) = \infty$. Then we show that the counting measures of certain subsets of sign changes of $f - r_{n,m}(f)$ converge weakly to the equilibrium measure on $[-1,1]$ as $n \to \infty$. Moreover, we prove discrepancy estimates between these counting measures and the equilibrium measure.

Нехай $f \in L_w^1[-1,1]$ і $r_{n,m}(f)$ — найкраще L_w^1-наближення для f відносно дійсних раціональних функцій степеня не більше ніж n у чисельнику та степеня не більше ніж m у знаменнику, $m = m(n)$ і $\lim_{n\to\infty}(n - m(n)) = \infty$. У цьому випадку продемонстровано, що лічильні міри певних підмножин знакозмін $f - r_{n,m}(f)$ слабко збігаються до рівноважної міри на $[-1,1]$ при $n \to \infty$. Також доведено оцінки відхилення цих лічильних мір від рівноважної міри.

1. Introduction. Let w be a weight function on $I = [-1,1]$ positive a.e. on $[-1,1]$ in the sense of Lebesgue. Let $L_w^1[-1,1]$ denote the class of all real-valued measurable functions f on $[-1,1]$ such that $f(t)w(t)$ is Lebesgue-integrable on $[-1,1]$ and let $\|f\|_{1,w}$ be the weighted L^1-norm in $L_w^1[-1,1]$, i.e.,

$$\|f\|_{1,w} := \int_{-1}^{1} |f(t)|w(t)dt = \int |f|w.$$

Let

$$\mathcal{R}_{n,m} = \left\{ r = \frac{p}{q} : \deg p \le n, \ \ \deg q \le m \right\}$$

be the class of all real-valued rational functions with numerator in $\mathcal{P}_n$ and denominator in $\mathcal{P}_m$. Here, $\mathcal{P}_k$ denotes the class of all algebraic polynomials of degree not exceeding k. For given $f \in L_w^1[-1,1]$, denote by $r_{n,m}(f)$ a best L_w^1-approximation of f with respect to $\mathcal{R}_{n,m}$.

We say that a function $g \in L_w^1[-1,1]$ does not change its sign at $x_0 \in [-1,1]$ if $f > 0$ (or $f < 0$) a.e. in some neighborhood U of x_0 in $[-1,1]$. All other points of $[-1,1]$ are called *sign changes* of f.

We denote by μ the equilibrium measure of $[-1,1]$.

In [1], Króo and Peherstorfer have proved the following result. Let $n, m \in \mathbb{N}_0$, $0 \le$ $\le m < n+1$, and $f \in L_w^1[-1,1]$ with $w(x) \equiv 1$. If $f - r_{n,m}(f)$ has no sign change on $(\alpha, \beta) \subset [-1,1]$, then

$$\mu[\alpha,\beta] = \frac{\arccos\alpha - \arccos\beta}{\pi} \le \frac{1}{n-m+2}$$

and this upper bound is, in general, best possible. Our aim is to make more precise statements about the distribution of sign changes of the error function, and to generalize to general weights.

To formulate our results, we need the notation of counting measures. Let A be a finite point set of k points. Then we define the discrete measure ν_A that associates the mass $\dfrac{1}{k}$ to every point of A. For specified subsets A of the set of sign changes of $f - r_{n,m}(f)$, the normalized counting measure ν_A will be compared with the equilibrium measure μ of $[-1, 1]$. In other words, we are interested in upper bounds for the discrepancy

$$D[\nu_A - \mu] := \sup_{-1 \le \alpha \le \beta \le 1} \big|(\nu_A - \mu)([\alpha, \beta])\big|$$

between ν_A and μ. We obtain distribution estimates for the sign changes of $f - r_{n,m}(f)$ on $[-1, 1]$ that generalize our results for polynomial approximation in [2].

2. Distribution of sign changes. In the following, c, c_i, $i = 1, 2, \ldots$, will denote positive constants independent of n. Let w be positive a.e. and measurable on $[-1, 1]$,

$$\int w = 1,$$

and, for $0 < \varepsilon \le 1$, let us define

$$\varphi(w, \varepsilon) := \inf \left\{ \int_A w : A \subset [-1, 1], \mu(A) \ge \varepsilon \right\}.$$

By $\varepsilon_n(w)$, we denote the unique solution of the equation

$$\varphi(w, \varepsilon) = e^{-n\varepsilon}.$$

Then $\varepsilon_n(w) \to 0$ as $n \to \infty$ and

$$\varepsilon_n(w) \ge c_1 \frac{\log n}{n}, \quad n = 1, 2, \ldots$$

(cf. [3], Lemma 2.3). Concerning the rate of $\varepsilon_n(w)$ as $n \to \infty$, we have $\varepsilon_n(w) = O\left(\log \dfrac{n}{n}\right)$ for Jacobi weights and $\varepsilon_n(w) = O(n^{-1/(1+\alpha)})$ for weights of type $w(x) = e^{-|x|^\alpha}$ with $\alpha > 0$.

The best approximation $r = r_{n,m}(f) = \dfrac{p_0}{q_0}$ is well-known to be characterized by the fact that, for every $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$, we have

$$\int \frac{pq_0 - qp_0}{q_0^2} \operatorname{sgn}(f - r) w \le \int_{Z(f-r)} \frac{|pq_0 - qp_0|}{q_0^2} w \tag{1}$$

(cf. [1], Lemma 1), where we denote

$$Z(g) := \{x : g(x) = 0\}$$

for functions $g \in L^1_w[-1, 1]$ and

$$\operatorname{sgn} g(x) = \begin{cases} 1 & \text{if } g(x) > 0, \\ 0 & \text{if } g(x) = 0, \\ -1 & \text{if } g(x) < 0. \end{cases}$$

It can easily be seen from (1) that the error function $f-r$ changes its sign at least $n+m+1$ times on $[-1,1]$ if $\mu(Z(f-r))=0$ and $r \notin \mathcal{R}_{n-1,m-1}$.

The function $\varphi \in L^1_w[-1,1]$ is called *sign function* if $\varphi^2 = 1$ on $[-1,1]$. φ is said to be orthogonal to $\mathcal{P}_n$ (written $\varphi \perp \mathcal{P}_n$) if

$$\int \varphi P w = 0 \quad \text{for all} \quad P \in \mathcal{P}_n.$$

Two sign functions φ and ψ are called *equivalent* if $\varphi = \psi$ or $\varphi = -\psi$ a.e. on $[-1,1]$.

Let $r = r_n(f) = \dfrac{p_0}{q_0}$ with no common factors, then the *defect* of r is defined as

$$d_{n,m} = d_{n,m}(r) = \min(n - \text{ grad } p_0, \ m - \text{ grad } q_0).$$

Then (1) is equivalent to

$$\int_{-1}^{1} \frac{p}{q_0^2} \operatorname{sgn}(f-r) w \le \int_{Z(f-r)} \frac{|p|}{q_0^2} w \quad \text{for all} \quad p \in \mathcal{P}_{n+m-d_{n,m}}, \tag{2}$$

i.e., 0 is the best approximation of $f-r$ in the subspace $q_0^{-2}\mathcal{P}_{n+m-d_{n,m}}$.

According to Proposition 20 of Cheney and Wulbert [4], there exists a sign function ψ such that $\psi(x) = \operatorname{sgn}(f-r)(x)$ for all $x \in [-1,1] \setminus Z(f-r)$ and

$$\int \frac{p}{q_0^2} \psi w = 0 \quad \text{for all} \quad p \in \mathcal{P}_{n+m-d_{n,m}}.$$

Then the sign changes of $f-r$ are identical with the sign changes of ψ. Therefore, ψ and, consequently, $f-r$ has at least $n+m-d_{m,n}$ sign changes in $(-1,1)$.

Concerning the distribution of these sign changes, we are able to prove estimates for a subset of $n-m+d_{n,m}+1$ sign changes. To this end, we use the weaker condition

$$\int p \operatorname{sgn}(f-r) w \le \int_{Z(f-r)} |p| w \quad \text{for all} \quad p \in \mathcal{P}_{n-m+d_{n,m}}, \tag{3}$$

which holds because of (2).

Theorem 1. *Let $f \in L^1_w[-1,1]$ and let $r = r_{n,m}(f)$ be a best L^1_w-approximation to f on $\mathcal{R}_{n,m}$, $m < n+1$. Then there exist $n-m+d_{n,m}+1$ sign changes of $f-r$ at points*

$$-1 < \xi_0^{(n,m)} < \ldots < \xi_{n-m+d_{n,m}}^{(n,m)} < 1 \tag{4}$$

such that the normalized counting measure $\nu_{n,m}$ of the these points satisfies

$$D[\nu_{n,m} - \mu] \le c\sqrt{\varepsilon_{n-m+d_{n,m}}(w)},$$

where c is an absolute constant, not depending on n, m or f.

***Corollary* 1.** *Let $m = m(n)$ and let*

$$n - m(n) \to \infty \quad \textit{as} \quad n \to \infty,$$

then the normalized counting measures $\nu_{n,m}$ of the point sets in (4) *converge weakly to μ for $n \to \infty$.*

For special weights w, the numbers $\varepsilon_n(w)$ are well-known and one gets explicit discrepancy estimates between $\nu_{n,m}$ and μ.

As a special example, we consider generalized Jacobi weights where, as in the polynomial case, the estimate of Theorem 1 can be sharpened.

Theorem 2. *Let $-1 = t_1 < t_2 < \ldots < t_k = 1$ be fixed points, $a_1, \ldots, a_k > -1$ be fixed numbers and w a weight function that satisfies*

$$w(x) \geq c_1 \prod_{i=1}^{k} |x - t_i|^{a_i}$$

with $c_1 > 0$. Then the normalized counting measure $\nu_{n,m}$ of (4) satisfy

$$D[\nu_{n,m} - \mu] \leq c \frac{(\log(n - m + d_{n,m}))^2}{n - m + d_{n,m}}$$

for all $n \geq m + 2$, where c is an absolute constant.

3. Proofs. In the following, we use a lemma of Króo and Peherstorfer [5].

Lemma 1 [5]. *Let φ be a sign function, $\varphi \perp \mathcal{P}_n$, and assume that φ has exactly $n+1$ sign changes at the point $y_1, \ldots, y_{n+1}$ with*

$$y_0 = -1 < y_1 < y_2 < \ldots < y_{n+1} < 1 = y_{n+2}.$$

If the sign function ψ is not equivalent to φ and $\psi \perp \mathcal{P}_n$, then ψ has a sign change in each interval (y_i, y_{i+1}), $0 \leq i \leq n + 1$.

Denote by $U_{n,w}(x)$ the Chebychev polynomial of second kind with respect to w; i.e., $U_{n,w}(x)$ is a monic polynomial in $\mathcal{P}_n$ and

$$\|U_{n,w}\|_{1,w} = \min\{\|P\|_{1,w} : P \text{ monic in } \mathcal{P}_n\},$$

$U_{n,w}$ is unique and characterized by exactly n sign changes (or zeros) at points

$$-1 < y_1^{(n)} < y_2^{(n)} < \ldots < y_n^{(n)} < 1.$$

We denote by $\nu_{n,w}$ the normalized zero-counting measure of $U_{n,w}$. Then for all $n = 1, 2, \ldots,$

$$D[\nu_{n,w} - \mu] \leq c\sqrt{\varepsilon_n(w)} \tag{5}$$

with some absolute constant $c > 0$ (Theorem 2 in [2]).

***Proof of Theorem* 1.** Let

$$\varphi(x) := \begin{cases} \operatorname{sgn} U_{n-m+d_{n,m}+1,w}(x) & \text{for } x \text{ with } U_{n-m+1,w}(x) \neq 0, \\ 1 & \text{elsewhere.} \end{cases}$$

Then φ satisfies the conditions of Lemma 1.

Because of (3), 0 is a best L^1_w-approximation to $f - r$ from the space $\mathcal{P}_{n-m+d_{n,m}}$. Thus, there exists a sign function $\psi \perp \mathcal{P}_{n-m+d_{n,m}}$ such that

$$\psi(x) = \operatorname{sgn}(f - r)(x) \quad \text{for all} \quad x \in [-1, 1] \setminus Z(f - r)$$

(cf. [4], Proposition 20).

Then either $\psi = \pm\varphi$ a.e. on $[-1, 1]$ or, due to Lemma 1, ψ has a sign change in each interval

$$\left(y_i^{(n-m+d_{n,m}+1)},\ y_{i+1}^{(n-m+d_{n,m}+1)}\right), \quad 0 \leq i \leq n - m + d_{n,m} + 1,$$

where

$$-1 < y_1^{(n-m+d_{n,m}+1)} < \ldots < y_{n-m+d_{n,m}+1}^{(n-m+d_{n,m}+1)} < 1$$

are the zeros of $U_{n-m+d_{n,m}+1,w}(x)$ and

$$y_0^{(n-m+d_{n,m}+1)} = -1, \quad y_{n-m+d_{n,m}+2}^{(n-m+d_{n,m}+1)} = 1.$$

Finally, we get from (5) the desired discrepancy result.

Corollary 1 is an immediate consequence of Theorem 1. Theorem 2 follows from a result in [2], stating that, for generalized Jacobi weights,

$$D[\nu_{n,w} - \mu] \leq c\frac{(\log n)^2}{n}$$

for all $n = 2, 3, \ldots$, where c is again an absolute constant.

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