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SOME MOMENT RESULTS ABOUT THE LIMIT OF A MARTINGALE RELATED TO THE SUPERCRITICAL BRANCHING RANDOM WALK AND PERPETUITIES

ДЕЯКІ РЕЗУЛЬТАТИ ПРО МОМЕНТИ ГРАНИЦІ МАРТИНГАЛА, ПОВ'ЯЗАНОГО З НАДКРИТИЧНИМ ГІЛЛЯСТИМ ВИПАДКОВИМ БЛУКАННЯМ, ТА РОЗВ'ЯЗКІВ ДЕЯКИХ СТОХАСТИЧНИХ РІЗНИЦЕВИХ РІВНЯНЬ

Let $\mathcal{M}^{(n)}, n = 1, 2, \ldots$, be the supercritical branching random walk in which the family sizes may be infinite with positive probability. Assume that a natural martingale related to $\mathcal{M}^{(n)}$ converges almost surely and in mean to a random variable W. For a large subclass of nonnegative and concave functions f, we provide a criterion for the finiteness of $\mathbb{E}Wf(W)$. The main assertions of the present paper generalize some results obtained recently in Kuhlbusch's Ph.D. thesis as well as previously known results for the Galton – Watson processes. In the process of the proof, we study the existence of the f-moments of perpetuities.

Нехай $\mathcal{M}^{(n)}$, $n = 1, 2, \ldots$, — надкритичне випадкове блукання, у якому розмір родини може бути нескінченним з додатною ймовірністю. Припустимо, що стандартний мартингал, пов'язаний з $\mathcal{M}^{(n)}$, збігається майже напевно і в середньому до випадкової величини W. Для великого підкласу невід'ємних та вгнутих функцій f наведено критерій скінченності $\mathbb{E}Wf(W)$. Основні твердження роботи узагальнюють деякі результати, отримані в дисертації Кульбуша, а також результати, відомі для процесів Гальтона – Ватсона. У процесі доведення досліджується існування f-моментів розв'язків деяких стохастичних різницевих рівнянь.

1. Introduction and results. Assume that an initial ancestor of some population is placed at the origin of the real line. She produces offspring who form the first generation of the population. Each individual of the first generation in her turn gives birth to children too. All children of the individuals of the first generation constitute the second generation and so on. A point process $\mathcal{M}$ with points $\{A_i, i = \overline{1, \mathcal{M}(\mathbb{R})}\}$ controls the location of the population over the real line in such a way. For $i = 1, 2, \ldots$, the displacements of the individuals of the i-th generation relative to positions of their mothers (they reside in the $i - 1$-th generation) are given by independent copies of $\mathcal{M}$. The sequence of the point processes $\mathcal{M}^{(n)}$, $n = 1, 2, \ldots$, which define positions of the n-th generation individuals is called *the branching random walk* (the BRW, in short). Many references related to the BRW can be found in [1 – 3].

In the sequel, for $n = 1, 2, \ldots, \mathcal{F}^n$ denotes a σ-field containing all information about the first n generations. The position of the individual u is denoted by A_u; the symbol $|u| = n$ means that the individual u resides in the n-th generation; the symbol $\sum_{|u|=n}$ denotes the summation over all individuals of the n-th generation.

Set $K := \mathcal{M}(\mathbb{R})$ and $q := \mathbb{P}\{K < \infty\} \in [0, 1]$. In this paper, we only consider the *supercritical* BRW. Therefore, if $q = 1$, we additionally assume that $\mathbb{E}K > 1$. Recall that the supercriticality ensures the survival of the population with a positive probability.

Define the function

$$m(y) := \mathbb{E}\sum_{i=1}^{K} e^{yA_i} \in (0, \infty], \quad y > 0.$$

Assume that $m(\gamma) < \infty$ for some $\gamma > 0$ and set

$$W_n^{(\gamma)} := m(\gamma)^{-n} \sum_{|u|=n} e^{\gamma A_u}, \quad n = 1, 2, \ldots.$$

The sequence $(W_n^{(\gamma)}, \mathcal{F}^n)$, $n = 1, 2, \ldots$, is a nonnegative martingale (Kingman [4] and Biggins [5] were the first to study such a martingale). Since γ and $\mathcal{F}^n$ will be the same from line to line, in what follows the martingale is denoted just by W_n. This martingale converges either almost surely to zero or almost surely and in mean to a random variable W which is positive with positive probability (throughout the text we use words "positive" and "increasing" in a strict sense). Put $Y_i := e^{\gamma A_i}/m(\gamma)$. The (probability) distribution of the W satisfies the equality

$$W \stackrel{d}{=} \sum_{i=1}^{K} Y_i W^{(i)},$$

where, given $\mathcal{F}^1$, $W^{(1)}, W^{(2)}, \ldots$ are conditionally independent copies of the W.

The papers [5–7] provide *conditions* for the martingale convergence in mean. However, all these authors required more or less restrictive additional assumptions. Our Proposition 1 can be read from Theorem 2 [3], where the *criterion* of the above mentioned convergence has been obtained (but in other terms).

The equality

$$\mathbb{E} \sum_{i=1}^{K} Y_i t(Y_i) = \mathbb{E} t(Z), \tag{1}$$

which is assumed to hold for bounded Borel functions t, defines the distribution of a random variable Z. Notice that

$$\mathbb{P}\{Z = 0\} = 0.$$

In the sequel, we additionally always assume that

$$\mathbb{P}\{Z = 1\} < 1, \qquad \mathbb{P}\{W_1 = 1\} < 1.$$

As soon as the distribution of Z was defined, we can permit for (1) to hold for any Borel function t. In that case, we assume that if the left-hand side is infinite or does not exist, the same is true for the right-hand side.

Let $T_n, n = 0, 1, \ldots$, be the random walk starting at zero with a step distributed like $V := -\log Z$. Define the function

$$A_Z(y) := \int_0^y \mathbb{P}\{V > x\} dx, \quad y > 0.$$

Relevant properties of this function can be found in [8].

Proposition 1. *The martingale W_n converges in mean if and only if*

$$\lim_{n\to\infty} T_n = +\infty \ \text{a.s.}; \qquad \int_{(1,\infty)} \frac{x \log x}{A_Z(x)} d\mathbb{P}\{W_1 \le x\} < \infty, \tag{2}$$

or equivalently if and only if either

i) $\mathbb{E}V \in (0, \infty)$ *and* $\mathbb{E}W_1 \log^+ W_1 < \infty$, *or*

ii) $\mathbb{E}V = \infty$ *and* $\int_{(1,\infty)} \frac{x \log x}{A_Z(x)} d\mathbb{P}\{W_1 \le x\} < \infty$, *or*

iii) $\mathbb{E}V$ *does not exist and* $\mathbb{E}\left(\frac{\log^+ Z}{A_Z(\log^+ Z)}\right) < \infty$, *and*

$$\int\limits_{(1,\infty)} \frac{x \log x}{A_Z(x)} d\mathbb{P}\{W_1 \leq x\} < \infty.$$

***Remark* 1.** In any case, the classical $x \log^+ x$ condition together with the condition $\lim\limits_{n\to\infty} T_n = +\infty$ a.s. are sufficient for the mean convergence of the martingale. A quite remarkable fact is that when $\mathbb{E}V$ is infinite or does not exist, the $x \log^+ x$ condition is no longer necessary. Thus we come to a bit discouraging conclusion: the weaker moment restriction is imposed on V, the weaker moment condition may be put on W_1.

As soon as the problem of existence of somewhere positive W is settled, it is natural to want to investigate moments of W. Following this principle, in this paper we will study f-moments of W. Consequently, the description of appropriate functions f will be given next.

Throughout the text, we assume that one of the following two assumptions is in force.

Assumption A. *Function* $f > 0$ *is nondecreasing and concave on* $[0, \infty)$, $\lim\limits_{x\to\infty} f(x) = \infty$. *For fixed* $B, d \geq 0$ *a new function* g *is defined by*

$$g(x) := B + \int\limits_d^x (f(y)/y)dy \quad \textit{for} \quad x \geq d; \qquad g(x) = 0 \quad \textit{for} \quad x < d.$$

Assumption B. *Function* f *is nondecreasing and concave on* $[0, \infty)$, $\lim\limits_{x\to\infty} f(x) = \infty$ *and* $f(0) = 0$. *Additionally, there exists* $p > 0$ *such that*

$$f(xy) \leq pf(x)f(y) \quad \textit{for all} \quad x, y > 0. \tag{3}$$

In this paper, $\psi(x)$, $x \geq 0$, is called a submultiplicative function if $\psi(x)$ is finite, positive, and Borel measurable and

$$\psi(0) = 1 \quad \text{and} \quad \psi(x + y) \leq \psi(x)\psi(y).$$

Recall that for a submultiplicative function ψ, there exists a limit

$$\lim_{x\to\infty} \frac{\log \psi(x)}{x} \in [0, \infty).$$

Inequality (3) implies that

$$h(x + y) := f(e^{x+y}) \leq ph(x)h(y) \quad \text{for all} \quad x, y > 0.$$

As pointed out by Sgibnev on [9, p. 85], the latter implies that there exists a nondecreasing submultiplicative function ψ such that

$$p_1\psi(x) \leq h(x) \leq p_2\psi(x) \quad \text{for some positive constants} \quad p_1,\ p_2. \tag{4}$$

Therefore, we can define a constant $r \in [0, 1]$ by

$$r := \lim_{x\to\infty} \frac{\log h(x)}{x}.$$

In what follows, $F \asymp G$ means that

$$0 < \liminf_{x\to\infty} \frac{F(x)}{G(x)} \leq \limsup_{x\to\infty} \frac{F(x)}{G(x)} < \infty.$$

We are now ready to present our first main result.

Theorem 1. *Let f satisfy Assumption* A *or* B.

a) *If $\mathcal{M}(-\infty, -\gamma^{-1}\log m(\gamma)) = 0$ a.s., assume that Assumption* A *holds. If the integral in* (2) *converges, then*

$$\mathbb{E}W_1 g(W_1) < \infty \Rightarrow \mathbb{E}Wf(W) < \infty;$$

if $\mathbb{E}V \in (0, \infty)$, then

$$\mathbb{E}W_1 g(W_1) < \infty \Leftarrow \mathbb{E}Wf(W) < \infty.$$

In particular, if $\mathbb{E}V \in (0, \infty)$ and $\mathbb{E}W_1 \log^+ W_1 < \infty$, then both implications hold and we have the equivalence. If $f \asymp g$, then we have the equivalence under the weaker assumption that the integral in (2) *converges.*

b) *If $\mathcal{M}(-\infty, -\gamma^{-1}\log m(\gamma)) \geq 1$ with positive probability, assume that Assumption* B *holds. If $\mathbb{E}V \in (0, +\infty]$, then*

$\text{b}_1)$ *if $r > 0$, then*

$$\mathbb{E}W_1 f(W_1) < \infty, \qquad \mathbb{E}Z^r < 1 \Rightarrow \mathbb{E}Wf(W) < \infty;$$

$\text{b}_2)$ *if $r = 0$, then*

$$\mathbb{E}W_1 f(W_1) < \infty \Rightarrow \mathbb{E}Wf(W) < \infty.$$

If $\mathbb{E}V \in (0, +\infty)$ in both cases $r > 0$ and $r = 0$, the converse implications hold, and we in fact have the equivalence.

***Remark* 2.** In case $f(x) = x^a$, $a \in (0, 1]$ Theorem 1 reduces to the well-known equivalence

$$\mathbb{E}W^{a+1} < \infty \Leftrightarrow \mathbb{E}W_1^{a+1} < \infty, \quad \mathbb{E}Z^a < 1$$

(see, for example, Proposition 4 [3]).

There are many results in the spirit of Theorem 1 related to the Galton – Watson process (see[10, 11] for recent developments). In the context of the branching random walk, our Theorem 1 generalizes a statement in Section 4 [12], Corollary 10 [13], Theorems 4.4.1 and 4.5.1 [14]. The best previously known results like our Theorem 1 were recently obtained in Kuhlbusch's Ph.D. thesis [14]. In Section 2, we partially compare our results to Kuhlbusch's ones.

The technique developed in this work is an extension of the approach proposed in [11] for the case of the Galton – Watson processes and in [3]. It should be noted that independently and at the same time a similar technique has also been used in [15] in the context of branching diffusions. Our method of proof consists in comparing (under the appropriate change of measure proposed in [7]) the random variable W with a so called *perpetuity*. Keeping this in mind we find it useful to study the existence of the f-moments of perpetuities.

Let $(Q_1, M_1), (Q_2, M_2), \ldots$ be independent copies of a random vector (Q, M). Let Z_0 be a random variable which is independent of (Q, M).

Set $\Pi_0 := 1$ and $\Pi_n := M_1 M_2 \ldots M_n$, $n = 1, 2, \ldots$, $X := -\log|M|$, $A_M(y) := \int_0^y \mathbb{P}\{X > x\}dx$, $y > 0$. The following proposition is a selection from Theorem 2.1 [8].

Proposition 2. *The following assertions are equivalent:*

$$\lim_{n\to\infty}\Pi_n = 0 \quad a.s., \qquad \int\limits_1^\infty \left(\frac{\log q}{A_M(\log q)}\right) d\mathbb{P}\{|Q|\le q\} < \infty, \tag{5}$$

$$\sum_{n=1}^{\infty}|\Pi_{n-1}Q_n| < \infty \quad a.s.$$

Each of these ensures

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\Pi_{n-1}Q_n + \Pi_n Z_0\right) = Z_\infty \quad a.s.,$$

where

$$Z_\infty := \sum_{n=1}^{\infty}\Pi_{n-1}Q_n.$$

In the literature, there exist several results about the existence of moments (or the tail behaviour) of the random variable Z_∞ called a *perpetuity*. We only mention two of them:

a) $\mathbb{E}|M|^p < 1$ and $\mathbb{E}|Q|^p < \infty \Leftrightarrow \mathbb{E}|Z_\infty| < \infty, p > 0$ (in [16] this has been shown in case $M, Q \ge 0$, in [17] the implication $\Rightarrow$ has been proved in case $p > 1$);

b) if $\mathbb{P}\{|M| \le 1\} = 1$ and $\mathbb{E}e^{\epsilon|Q|} < \infty$ for some $\epsilon > 0$, then $\mathbb{E}e^{\rho|Z_\infty|} < \infty$ for $0 \le \rho < \sup\{\theta : \mathbb{E}e^{\theta|Q|}|M| < 1\}$ (this fact follows from Theorem 2.1 [18]; this work implicitly contains some other results related to moments).

Note that so far the existence of the f-moments of Z_∞ has not been investigated (the only exception being the case $f(x) = x^a$, $a \in (0,1]$).

In the sequel, we assume that

$$\mathbb{P}\{M = 0\} = 0, \qquad \mathbb{P}\{Q = 0\} < 1,$$

and the distribution of Z_∞ is nondegenerate.

The second main result is as follows.

Theorem 2. *Let f satisfy Assumption A or B.*

a) *If*

$$\mathbb{P}\{|M| \le 1\} = 1 \quad \textit{and} \quad \mathbb{P}\{|M| = 1\} < 1, \tag{6}$$

assume Assumption A holds. If the integral in (5) *converges, then*

$$\mathbb{E}g(|Q|) < \infty \Rightarrow \mathbb{E}f(|Z_\infty|) < \infty;$$

if $\mathbb{E}X \in (0,\infty)$, then

$$\mathbb{E}g(|Q|) < \infty \Leftarrow \mathbb{E}f(|Z_\infty|) < \infty.$$

In particular, if $\mathbb{E}X \in (0,\infty)$ and $\mathbb{E}\log^+|Q| < \infty$, then both implications hold and we have the equivalence. If $f \asymp g$ then we have the equivalence under the weaker assumption that the integral in (5) *converges.*

b) *If $\mathbb{P}\{|M| > 1\} > 0$, assume that Assumption* B *holds. If $\mathbb{E}X \in (0,+\infty]$ then*

b$_1$) *if $r > 0$ then*

$$\mathbb{E}f(|Q|) < \infty, \qquad \mathbb{E}f(|M| \vee 1) < \infty, \qquad \mathbb{E}|M|^r < 1 \Rightarrow \mathbb{E}f(|Z_\infty|) < \infty;$$

b$_2$) *if* $r = 0$ *then*

$$\mathbb{E}f(|Q|) < \infty, \qquad \mathbb{E}f(|M| \vee 1) < \infty \Rightarrow \mathbb{E}f(|Z_\infty|) < \infty.$$

If $\mathbb{E}X \in (0, +\infty)$ *in both cases* $r > 0$ *and* $r = 0$ *the converse implications hold, and we in fact have the equivalence.*

The rest of the paper is organized as follows. Section 2 contains some relevant properties of functions f and g. In Section 3, after giving a preliminary result, we study the f-moments of perpetuities and prove Theorem 2. In Section 4, we provide a careful description of a change of measure construction and prove Theorem 1.

2. Properties of functions f and g, and examples. To give a better feeling of the results obtained, we first point out some pairs (f, g) which satisfy Assumption A. These examples are taken from Section 3 [11].

1. For $p \in (0, 1)$,

$$f(x) = x^p, \qquad g(x) = x^p/p.$$

2. For $p \in (0, 1]$,

$$f(x) = e^{-1}x1_{\{x\in[0,e)\}} + \log^p x1_{\{x\geq e\}}, g(x) = (p+1)^{-1}(\log^{p+1} x - 1)1_{\{x\geq e\}};$$

for $p > 1$,

$$f(x) = p(p-1)e^{1-p}x1_{\{x\in[0,e^{p-1})\}} + (\log^p x + (p-1)^2)1_{\{x\geq e^{p-1}\}},$$
$$g(x) = ((p+1)^{-1}(\log^{p+1} x - (p-1)^p) + (p-1)^2(\log x - p + 1))1_{\{x\geq e^{p-1}\}},$$

therefore, for $p > 0$,

$$f(x) \asymp \log^p x, \qquad g(x) \asymp \log^{p+1} x.$$

3. For $\beta > 0$ and $c := (\beta/e)^\beta - \beta$,

$$f(x) = \beta^\beta e^{-\beta-\exp(\beta)}x1_{\{x\in[0,\exp(e^\beta))\}} + (\log^\beta \log x + c)1_{\{x\geq\exp(e^\beta)\}},$$

$$g(x) = \left(\log x \log^\beta \log x - \beta e^\beta - \beta \int\limits_{e^\beta}^{\log x} \log^{\beta-1} u du + c(\log x - e^\beta)\right) 1_{\{x\geq e^\beta\}},$$

therefore,

$$f(x) \asymp \log^\beta \log x, \qquad g(x) \asymp \log x \log^\beta \log x.$$

As it follows from Theorems 1 (a) and 2 (a), it is important to know when $f \asymp g$, if (f, g) satisfy Assumption A. A simple sufficient condition for this to hold was given in Corollary 1.2 [11]: if there exists an $\alpha \in (0, 1)$ such that $x^{-\alpha}f(x)$ does not decrease for large x, then $f \asymp g$.

Now we would like to explain the point of using Assumption B. To prove Theorem 2 (b), it would be highly desirable if functions f possessed two properties: (4) and $f(x) \asymp \int_0^x (f(u)/u)du$. Assumption B appears to be the weakest possible one to ensure that these properties do hold. The next lemma collects some properties of functions satisfying Assumption B.

Lemma 1. *Let f satisfy Assumption* B. *Then*

a) f *and* $g(x) := \int_0^x (f(u)/u)du$ *satisfy Assumption* A *with* $B = d = 0$; *moreover,* $f \asymp g$;

b) $\lim_{x\to\infty} \frac{f(x)}{\log^\epsilon x} = \infty$ *for every* $\epsilon > 0$.

Proof. The first part of (a) is obvious. Let us verify that $f \asymp g$. Since $f(x)/x$ is nonincreasing, we have

$$g(x) = \int_0^x (f(u)/u)du \geq (f(x)/x) \int_0^x du = f(x).$$

Using now (3), we obtain

$$g(x) = \int_0^1 (f(tx)/t)dt \leq pf(x) \int_0^1 (f(t)/t)dt = \mathrm{const} f(x).$$

From these two inequalities, we obtain the needed. The following result can be derived from the proof of Proposition 2 [13]: if $H : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function with concave derivative, $H(0) = 0$, and there exists a positive constant c such that $H(xy) \leq cH(x)H(y)$ for all $x, y \in \mathbb{R}^+$, then

$$\lim_{x\to\infty} \frac{H(x)}{x\log^\epsilon x} = \infty \quad \text{for every} \quad \epsilon > 0. \tag{7}$$

Set $H(x) := \int_0^x f(u)du$. Since

$$2^{-1}xf(x) \leq H(x) \leq xf(x), \tag{8}$$

we have

$$H(xy) \leq xyf(xy) \overset{(3)}{\leq} pxyf(x)f(y) \leq 4pH(x)H(y).$$

Therefore, the so defined H possesses all the properties described above. This gives (7) and, in view of (8), the statement follows.

As was indicated in Introduction, some results related to our Theorem 1 were given in [14]. Kuhlbusch studied the ϕ-moments of W when ϕ is a regularly varying function with index α subject to additional restrictions. If $\alpha \in (1, 2)$, his Theorems 4.5.1 and 4.4.1 are contained in our Theorem 1 (a) and Theorem 1 (b) correspondingly. Indeed, it is well known that given a regularly varying function t with index $\alpha \in (1, 2)$, there exists a concave function z such that $t(x) \asymp xz(x)$. On the other hand, a concave function need not be regularly or slowly varying. Although it is a quite obvious fact, we propose a simple example (due to Professor Oleg Zakusylo) of positive, increasing, and concave function which is not regularly varying. Define

$$q(x) := 2^{-k}x + 2^{k+1} - 3, \quad \text{if} \quad x \in [4^k, \quad 4^{k+1}), k = 0, 1, \ldots.$$

The first three stated properties are easily seen. To check the absence of regular variation, set $x_n := 4^n$, $y_n := 3 \cdot 4^n$, $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} \frac{q(2x_n)}{q(x_n)} = \frac{4}{3}$ and $\lim_{n\to\infty} \frac{q(2y_n)}{q(y_n)} = \frac{8}{5}$.

Another example could be given which, however, requires more computations (omitted here). Take

$$q(x) := x^{\beta}(1 + a_{\beta}\sin(\log x) + b_{\beta}\cos(\log x)), \quad \beta \in (0,1),$$

with appropriate parameters a_{β}, b_{β}.

3. Moments of perpetuities. Let $\{(\xi_k, \eta_k),\ k = 1,2,\ldots\}$ be independent copies of a random vector (ξ, η). Set $R_n := \xi_1 + \ldots + \xi_n$, $n = 1,2,\ldots$, $R_0 := 0$. Lemma 2 given next is needed for the proof of Proposition 3. Note that this lemma generalizes Proposition 7 [19].

Lemma 2. *Assume that* $\lim\limits_{n\to\infty} R_n = \infty$ *and* $\zeta := \sup\limits_{k\geq 0}(-R_k + \eta_{k+1}) < \infty$ *a.s. Then*

$$\mathbb{P}\{\zeta > x\} \geq \mathbb{P}\{\eta > x\} + \int\limits_{-\infty}^{x} \mathbb{P}\{\sup_{k\geq 0}(-R_k) > x - y\} d\mathbb{P}\{\eta \leq y\}, \quad x \in \mathbb{R}.$$

Proof. For every $n = 1,2,\ldots$, put $M_n := \sup\left\{k \geq 0 : -R_k = \max\limits_{0\leq l\leq n}(-R_l)\right\}$. Then

$$\zeta \geq \max_{0\leq k\leq n}(-R_k + \eta_{k+1}) \geq -R_{M_n} + \eta_{M_n+1}.$$

Therefore,

$$\mathbb{P}\{\zeta > x\} \geq \mathbb{P}\{-R_{M_n} + \eta_{M_n+1} > x\} =$$

$$= \sum_{m=0}^{n} \mathbb{P}\{-R_{M_n} + \eta_{M_n+1} > x, M_n = m\} =$$

$$= \sum_{m=0}^{n} \mathbb{P}\{-R_m + \eta_{m+1} > x,\ M_n = m\} =$$

$$= \sum_{m=0}^{n} \int\limits_{\mathbb{R}} \mathbb{P}\{-R_m > x - y, M_n = m\} d\mathbb{P}\{\eta_{m+1} \leq y\} =$$

$$= \int\limits_{\mathbb{R}} \sum_{m=0}^{n} \mathbb{P}\{-R_m > x - y, M_n = m\} d\mathbb{P}\{\eta \leq y\} =$$

$$= \int\limits_{\mathbb{R}} \mathbb{P}\Big\{\max_{0\leq k\leq n}(-R_k) > x - y\Big\} d\mathbb{P}\{\eta \leq y\}.$$

Since R_k drifts to ∞, $\sup\limits_{k\geq 0}(-R_k) < \infty$ a.s. Letting $n \to \infty$ and using Fatou's lemma allow us to conclude that

$$\mathbb{P}\{\zeta > x\} \geq \int\limits_{\mathbb{R}} \mathbb{P}\Big\{\sup_{k\geq 0}(-R_k) > x - y\Big\} d\mathbb{P}\{\eta \leq y\} =$$

$$= \mathbb{P}\{\eta > x\} + \int\limits_{-\infty}^{x} \mathbb{P}\Big\{\sup_{k\geq 0}(-R_k) > x - y\Big\} d\mathbb{P}\{\eta \leq y\}.$$

The proof is complete.

Consider independent and identically distributed random vectors

$$(\grave{M}_k, \grave{Q}_k) := (M_{2k-1}M_{2k}, M_{2k-1}Q_{2k} + Q_{2k-1}), \quad k = 1, 2, \ldots,$$

and set

$$\grave{\Pi}_0 := 1, \qquad \grave{\Pi}_n := \grave{M}_1 \grave{M}_2 \ldots \grave{M}_n, \quad n = 1, 2, \ldots.$$

Proposition 3. *Assume that h does not decrease, has one-sided derivatives which coincide a.e., and for large x and some $c > 0$,*

$$h(2x) \le ch(x). \tag{9}$$

Then $\mathbb{E}h(|Z_\infty|) < \infty$ implies:

$$\mathbb{E}h(|Q|) < \infty \quad \textit{and either} \quad \mathbb{E}h\left(\sup_{n\ge 0}|\Pi_n|\right) < \infty \quad \textit{or} \quad \mathbb{E}h\left(\sup_{n\ge 0}|\grave{\Pi}_n|\right) < \infty;$$

$$\textit{either} \quad \mathbb{E}h\left(\sup_{n\ge 1}|\Pi_{n-1}||Q_n^s|\right) < \infty \quad \textit{or} \quad \mathbb{E}h\left(\sup_{n\ge 1}|\grave{\Pi}_{n-1}||\grave{Q}_n^s|\right) < \infty, \tag{10}$$

where $Q_n^s := Q_n - Q_n'$; $(M_n, Q_n) \overset{d}{=} (M_n, Q_n')$; given M_n, Q_n and Q_n' are conditionally independent; *$\grave{Q}_n^s$ and $\grave{Q}_n'$ have the same meaning but in terms of $\grave{M}_n$ and $\grave{Q}_n$.*

Proof. As in [8, p. 1212, 1213], by using a regular conditional distribution for Q given M, we can construct the sequence $\{Q_j', j = 1, 2, \ldots\}$ such that $(M_j, Q_j'), j = 1, 2, \ldots,$ are independent copies of (M, Q); given M_j, Q_j' and Q_j are conditionally independent.

Consider two cases:

1. Q is not a Borel function of M. Let us define conditionally symmetrized random variables

$$Q_j^s := Q_j - Q_j', \qquad Z_n^s := \sum_{k=1}^{n} \Pi_{k-1} Q_k^s, \quad n = 1, 2, \ldots.$$

Note that $\mathcal{L}(Q_j^s) \ne \delta_0$. Let $\mathcal{B}_n = \sigma(M_1, \ldots, M_n)$, $n = 1, 2, \ldots$. By symmetrization inequalities [20] for $n = 1, 2, \ldots$

$$\mathbb{P}\left\{\max_{1\le k\le n}|\Pi_{k-1}Q_k^s| > x|\mathcal{B}_n\right\} \le$$

$$\le 2\mathbb{P}\left\{|Z_n^s| > x|\mathcal{B}_n\right\} \le 4\mathbb{P}\left\{\left|\sum_{k=1}^{n}\Pi_{k-1}Q_k\right| > x/2|\mathcal{B}_n\right\}.$$

Taking expectations and then letting n go to ∞ gives

$$\mathbb{P}\left\{\sup_{k\ge 1}|\Pi_{k-1}Q_k^s| > x\right\} \le 2\mathbb{P}\left\{|Z_\infty^s| > x\right\} \le 4\mathbb{P}\left\{|Z_\infty| > x/2\right\}, \quad x \in \mathbb{R}. \tag{11}$$

These inequalities hold for all x, as the distribution of Z_∞ is continuous and the sequence $\left\{\max_{1\le k\le n}|\Pi_{k-1}Q_k^s|,\ n = 1, 2, \ldots\right\}$ is monotone.

Assume that $\mathbb{E}h(|Z_\infty|) < \infty$. Condition (11) implies

$$\mathbb{E}h\left((1/2)\sup_{k\ge 1}|\Pi_{k-1}Q_k^s|\right) < \infty.$$

Taking into account (9), we have

$$\infty > \mathbb{E}h\left(\sup_{k\ge 1}|\Pi_{k-1}Q_k^s|\right) = \mathbb{E}q\left(\sup_{k\ge 0}(-S_k + \log|Q_{k+1}^s|)\right), \qquad (12)$$

where $q(x) := h(e^x)$, $S_k := -\log|\Pi_k|$, $k = 0, 1, \ldots$, is a random walk with a step distributed like X. Since $|Z_\infty| < \infty$ a.s., Proposition 2 ensures that $\lim_{k\to\infty} S_k = \infty$ a.s. According to Lemma 2, we have

$$\mathbb{P}\left\{\sup_{k\ge 0}(-S_k + \log|Q_{k+1}^s|) > x\right\} \ge$$

$$\ge \mathbb{P}\{\log|Q^s| > x\} + \int_{-\infty}^{x} \mathbb{P}\left\{\sup_{k\ge 0}(-S_k) > x - y\right\} d\mathbb{P}\{\log|Q^s| \le y\}, \quad x \in \mathbb{R}. \qquad (13)$$

Let q' be any of one-sided derivatives of q. In view of (12),

$$\infty > \int_{\mathbb{R}} q'(x)\mathbb{P}\left\{\sup_{k\ge 0}(-S_k + \log|Q_{k+1}^s|) > x\right\} dx \overset{(13)}{\ge}$$

$$\overset{(13)}{\ge} \int_{\mathbb{R}} q'(x) \int_{-\infty}^{x} \mathbb{P}\left\{\sup_{k\ge 0}(-S_k) > x - y\right\} d\mathbb{P}\{\log|Q^s| \le y\} dx =$$

$$= \int_{\mathbb{R}} d\mathbb{P}\{\log|Q^s| \le y\} \int_{0}^{\infty} q'(x+y)\mathbb{P}\left\{\sup_{k\ge 0}(-S_k) > x\right\} dx =$$

$$= \mathbb{E}q\left(\log U + \sup_{k\ge 0}(-S_k)\right) = \mathbb{E}h\left(U\sup_{k\ge 0}|\Pi_k|\right),$$

where U is a random variable which is independent of $\sup_{k\ge 0}|\Pi_k|$ and distributed like $|Q^s|$. Therefore, $\mathbb{E}h\left(\sup_{k\ge 0}|\Pi_k|\right) < \infty$ and $\mathbb{E}h(|Q_1^s|) < \infty$.

The latter inequality ensures $\mathbb{E}\big(h(|Q_1^s|)|\mathcal{B}_1\big) < \infty$ a.s. Hence,

$$\mathbb{E}h(|Q - Q^*|) < \infty, \qquad (14)$$

where Q^* is independent copy of Q. In the same way as formula (5.7) in [20] has been proved, we can verify that there exists $a \in \mathbb{R}$ such that

$$2\mathbb{P}\big\{|Q - Q^*| > x - a\big\} \ge \mathbb{P}\big\{|Q| > x\big\}. \qquad (15)$$

Monotonicity of h and condition (9) imply that, for fixed $b \in \mathbb{R}$,

$$h(x) \asymp h(x - b). \qquad (16)$$

From this, (14) and (15), we have $\mathbb{E}h(|Q|) < \infty$.

2. $Q = r(M)$ for some Borel function r. We have

$$\sum_{k=1}^{2n} \Pi_{k-1}Q_k = \sum_{k=1}^{n} \grave{\Pi}_{k-1}\grave{Q}_k,$$

where the random variables on the right-hand side were defined before the proposition. If $\grave{Q}_k = s(\grave{M}_k)$ for some Borel function s, then

$$M_1Q_2 + Q_1 = s(M_1M_2)$$

and, by Proposition 1 [21], either $Q + cM = c$ a.s. for some c or $(M, Q) = (1, c_1)$ for some c_1. The first of these is excluded by our assumption before Theorem 2. The second is incompatible with $|Z_\infty| < \infty$ a.s. by Proposition 1. Therefore, $\grave{Q}_k$ is not a Borel function of $\grave{M}_k$. Now, the first part of the proof can be applied (on $(\grave{M}_k, \grave{Q}_k)$ instead of (M_k, Q_k)) which yields $\mathbb{E}h(|\grave{Q}|) < \infty$, $\mathbb{E}h\left(\sup\limits_{n\geq 0}\grave{\Pi}_n\right) < \infty$ and $\mathbb{E}h\left(\sup\limits_{n\geq 1}|\grave{\Pi}_{n-1}||\grave{Q}^s_n|\right) < \infty$. Taking into account (9), (16) and the equality

$$\mathbb{E}h(|\grave{Q}|) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \mathbb{E}h(|mQ_2 + q|)\mathbb{P}\{M_1 \in dm,\ Q_1 \in dq\},$$

we conclude that $\mathbb{E}h(|Q|) < \infty$. Proposition 3 has been proved.

Now we are ready to give proof of Theorem 2(a).

***Proof of Theorem* 2 (a)**. *Necessity.* Assume that

$$\mathbb{E}g(|Q|) < \infty. \tag{17}$$

In view of (6), $\lim\limits_{n\to\infty} \Pi_n = 0$ a.s. By Proposition 2, this together with the assumption that the integral in (5) converges, ensures that $|Z_\infty| < \infty$ a.s.

Condition (17) implies $J := \int_d^\infty \mathbb{P}\{|Q| > v\}g'(v)dv < \infty$. By assumption, $m := \mathbb{E}|M| \in (0, 1)$. Let us check that for arbitrary $n \in \mathbb{N}$ and fixed $c > f(d)$,

$$I_n := (1 - m)\mathbb{E}\left(f\left|\sum_{k=1}^n |\Pi_{k-1}Q_k|\right| \vee c\right) \leq J. \tag{18}$$

We have

$$J = \int\limits_d^\infty \mathbb{P}\{|Q| > v\}(f(v)/v)dv \geq \tag{19}$$

$$\geq (1 - m)\int\limits_d^\infty \mathbb{P}\{|Q| > v\}\mathbb{E}\left(\sum_{k=1}^n |\Pi_{k-1}|f'(|\Pi_{k-1}|v)\right)dv \geq$$

$$\geq (1 - m)\mathbb{E}\sum_{k=1}^n \int\limits_{d/|\Pi_{k-1}|}^\infty \mathbb{P}\{|Q| > v\}|\Pi_{k-1}|f'(|\Pi_{k-1}|v)dv \geq \tag{20}$$

$$\geq (1 - m)\int\limits_c^\infty \sum_{k=1}^n \mathbb{E}\mathbb{P}\{|Q_k| > f^{-1}(x)/|\Pi_{k-1}|\}dx =$$

$$= (1 - m)\sum_{k=1}^n \int\limits_c^\infty \mathbb{P}\{f(|\Pi_{k-1}||Q_k|) > x\}dx =$$

$$= (1-m)\mathbb{E}\sum_{k=1}^{n}(f(|\Pi_{k-1}||Q_k|)\vee c) \geq \tag{21}$$

$$\geq (1-m)\mathbb{E}\left(f\left(\sum_{k=1}^{n}|\Pi_{k-1}||Q_k|\right)\vee c\right) \geq$$

$$\geq (1-m)\mathbb{E}\left(f\left|\sum_{k=1}^{n}\Pi_{k-1}Q_k\right|\vee c\right)$$

which proves (18). Inequality (19) above has been obtained as follows: f' does not increase, the sequence $|\Pi_k(\omega)|, k=0,1,\ldots$, does not increase and $0<\Pi_k(\omega)\leq 1$ a.s. Hence,

$$f(v) \geq \mathbb{E}\int\limits_{|\Pi_n|v}^{v} f'(y)dy = \mathbb{E}\sum_{k=1}^{n}\int\limits_{|\Pi_k|v}^{|\Pi_{k-1}|v} f'(y)dy \geq$$

$$\geq (1-m)v\mathbb{E}\sum_{k=1}^{n}|\Pi_{k-1}|f'(|\Pi_{k-1}|v).$$

Inequality (20) follows by change of variable $x = f(v|\Pi_{k-1}|)$. Inequality (21) follows from the fact that the function $x\to|x|$ is subadditive, and the functions $x\to x\vee c$ and $f(x)$ are subadditive and nondecreasing. (Take for simplicity of explanation $n=2$ and set $x:=|Q_1|, y:=|M_1||Q_2|$. Inequality (21) is implied by the inequalities $(f(|x|)\vee c)+$ $+ (f(|y|)\vee c) \geq (f(|x|)+f(|y|)\vee c) \geq (f(|x+y|)\vee c) \geq (|f(x+y)|\vee c)$.)

Thus, I_n is bounded from the above by the constant J that does not depend on n. By the assumptions of the theorem and Proposition 2, the series $\sum_{k=1}^{n}|\Pi_{k-1}||Q_k|$ is a.s. convergent. Since f is continuous, we have that as $n\to\infty$, the sequence $f\left(\left|\sum_{k=1}^{n}\Pi_{k-1}Q_k\right|\right)$ converges a.s. to $f\left(\left|\sum_{k=1}^{\infty}\Pi_{k-1}Q_k\right|\right)$. An appeal to Fatou's lemma gives

$$\mathbb{E}f(|Z_\infty|) = \mathbb{E}f\left(\left|\sum_{k=1}^{\infty}\Pi_{k-1}Q_k\right|\right) \leq J < \infty.$$

Sufficiency. Assume that $\mathbb{E}f(|Z_\infty|)<\infty$. By Proposition 3, $\mathbb{E}f(|Q|)<\infty$. Thus, if $f\asymp g$, combining this observation with the previous part of the theorem, we deduce that $\mathbb{E}f(|Z_\infty|)<\infty \Leftrightarrow \mathbb{E}f(|Q|)<\infty$ under the sole condition that the integral in (5) converges.

Consider now the general case. If the support of a distribution of Q is bounded, then $|Z_\infty|$ has finite moments of all positive integer orders and the result of the theorem is trivial. Hence, in what follows, we assume that the support of a distribution of $|Q|$ is unbounded from the above. In that case, there exists $s>d$ such that $\mathbb{P}\{|Q|>t\}>0$ for all $t\geq s$.

Assume that the distribution of M is nondegenerate. By Proposition 3, either

$$f\left(\sup_{n\geq 1}|\Pi_{n-1}||Q_n^s|\right)<\infty \quad \text{or} \quad \mathbb{E}f\left(\sup_{n\geq 1}|\grave{\Pi}_{n-1}||\grave{Q}_n^s|\right)<\infty.$$

We only invesigate the second (harder) possibility. The strong law of large numbers implies that there almost surely exists $L>0$ such that $|\grave{\Pi}_k|\geq e^{-4\mu k}$ for $k\geq L$, where $\mu := \mathbb{E}(-\log|M|) \in (0,\infty)$. Therefore, we can choose $k_0 < \infty$ such that $|\grave{\Pi}_k| \ge e^{-4\mu(k\vee k_0)}$ for every $k = 0, 1, \ldots$. Since

$$\sup_{k\ge1}|\grave{\Pi}_{k-1}||\grave{Q}_k^s| \ge \sup_{k\ge k_0+2}|\grave{\Pi}_{k-1}||\grave{Q}_k^s| \ge \sup_{k\ge k_0+2} e^{-4\mu(k-1)}|\grave{Q}_k^s|,$$

$$\sup_{k\ge k_0+2} e^{-4\mu(k-1)}|\grave{Q}_k^s| \overset{d}{=} e^{-4\mu k_0}\sup_{k\ge1} e^{-4\mu k}|\grave{Q}_k^s|,$$

and f does not decrease, $\mathbb{E}f\left(\sup_{n\ge1}|\grave{\Pi}_{n-1}||\grave{Q}_n^s|\right) < \infty$ implies

$$\mathbb{E}f\left(\sup_{k\ge1} e^{-4\mu k}|\grave{Q}_k^s|\right) < \infty. \tag{22}$$

Since $\mu \in (0,\infty)$ and $|Z_\infty| < \infty$ almost surely, Proposition 2 allows us to conclude that $\mathbb{E}\log^+|Q| < \infty$. The function $x \to \log(1+|x|)$ is subadditive and $\log(1+|xy|) \le$ $\le \log(1+|x|)+\log(1+|y|)$. Therefore, $\mathbb{E}\log^+|\grave{Q}| = \mathbb{E}\log^+|Q_1+M_1Q_2| < \infty$. Taking conditional expectations and using the symmetrization inequality allow us to check that $\mathbb{E}\log^+|\grave{Q}^s| < \infty$. The latter implies

$$C := \prod_{k=1}^{\infty}\mathbb{P}\{e^{-4\mu k}|\grave{Q}_k^s| \le s\} > 0.$$

Furthermore, we have for $t \ge s$

$$\mathbb{P}\left\{\sup_{k\ge1} e^{-4\mu k}|\grave{Q}_k^s| > t\right\} = \sum_{k=1}^{\infty}\mathbb{P}\left\{|\grave{Q}^s| > e^{4\mu k}t\right\}\prod_{j=1}^{k-1}\mathbb{P}\left\{|\grave{Q}^s| \le e^{4\mu j}t\right\} \ge$$

$$\ge \sum_{k=1}^{\infty}\mathbb{P}\left\{|\grave{Q}^s| > e^{4\mu k}t\right\}\prod_{j=1}^{k-1}\mathbb{P}\left\{|\grave{Q}^s| \le e^{4\mu j}s\right\} \ge C\sum_{k=1}^{\infty}\mathbb{P}\left\{|\grave{Q}^s| > e^{4\mu k}t\right\}. \tag{23}$$

Assume now that the distribution of M is degenerate. By assumption, $\mathbb{P}\{|M| = 1\} <$ < 1. Consequently, $\mathbb{P}\{|M| = \gamma\} = 1$ for some $\gamma \in (0,1)$. An easy calculation reveals that, in this case, the analogue of (23) holds with $e^{-4\mu} = \gamma$.

Recall that $g'(u) = u^{-1}f(u)$ for $u > s$. Let us show that

$$\int_s^{\infty} g'(u)\mathbb{P}\{e^{-4\mu}|\grave{Q}^s| > u\}du < \infty. \tag{24}$$

Inequality (24) follows from the inequalities

$$\infty \overset{(22)}{>} \int_s^{\infty} f'(t)\mathbb{P}\{\sup_{k\ge1} e^{-4\mu k}|\grave{Q}_k^s| > t\}dt \overset{(23)}{\ge}$$

$$\overset{(23)}{\ge} C\int_s^{\infty} f'(t)\sum_{k=1}^{\infty}\mathbb{P}\{e^{-4\mu k}|\grave{Q}_k^s| > t\}dt \ge$$

$$\ge \frac{C}{e^{4\mu}-1}\int_s^{\infty} f'(t)\int_{e^{4\mu}t}^{\infty}\frac{\mathbb{P}\{|\grave{Q}^s| > z\}}{z}dzdt \ge$$

$$\geq \text{const} + \frac{C}{e^{4\mu}-1}\int\limits_s^\infty \frac{f(t)}{t}\mathbb{P}\{e^{-4\mu}|\grave{Q}^s| > t\}dt.$$

Now (24) implies $\mathbb{E}g(|\grave{Q}^s|) < \infty$. By taking conditional expectations and using (15), we can prove that

$$\mathbb{E}g(|\grave{Q}|) = \mathbb{E}g(|Q_1 + M_1Q_2|) < \infty.$$

The latter implies $\mathbb{E}g(|Q|) < \infty$ (a similar situation has been treated at the end of the proof of Proposition 3). The proof of Theorem 2 (a) is complete.

For later use, it is worth recording the following corollary which can be read from the previous proof.

***Corollary* 1.** *Assume that Assumption* A *and* (6) *hold, and* $\mathbb{E}X \in (0,\infty)$ *and* $\mathbb{E}\log^+|Q| < \infty$. *Then* (10) *implies* $\mathbb{E}g(|Q|) < \infty$.

Proposition 4. *Assume that Assumption B holds and* $\lim_{n\to\infty}\Pi_n = 0$ *a.s. Then*

$$\mathbb{E}f(|Z_\infty|) < \infty \Leftarrow \mathbb{E}f(|Q|) < \infty, \qquad \mathbb{E}f\left(\sup_{n\geq 0}|\Pi_n|\right) < \infty.$$

Proof. Define the random times $N_0 := 0$,

$$N_{i+1} := \inf\{n > N_i : |\Pi_n| < |\Pi_{N_i}|\}, i = 0, 1, \ldots.$$

Clearly, $\mathbb{E}N_i < \infty, i = 1, 2, \ldots$. For $k = 1, 2, \ldots$ set

$$M'_k := |M_{N_{k-1}+1}|\ldots|M_{N_k}|, \qquad \Pi'_0 := 1, \qquad \Pi'_k := M'_1\ldots M'_k,$$

$$Q'_k := |Q_{N_{k-1}+1}| + |M_{N_{k-1}+1}||Q_{N_{k-1}+2}| + \ldots + |M_{N_{k-1}+1}|\ldots|M_{N_k-1}||Q_{N_k}|.$$

Then we have that (M'_k, Q'_k) are independent copies of $\left(|\Pi_{N_1}|, \sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)$ and, moreover,

$$\sum_{k=1}^\infty |\Pi_{k-1}||Q_k| = \sum_{k=1}^\infty \Pi'_{k-1}Q'_k. \tag{25}$$

Let us show that we can use the implication $\Rightarrow$ of Theorem 2(a) on the vector $\left(|\Pi_{N_1}|, \sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)$. Since $|\Pi_{N_1}| \in (0,1)$ a.s. and $\mathbb{P}\{|\Pi_{N_1}| = 1\} = 0$, it remains to verify that the integral in (5) $\left(\text{with } (|M|, |Q|) \text{ replaced by } \left(|\Pi_{N_1}|, \sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)\right)$ converges. By Lemma 1, $f \asymp g$, where $g(x) = \int_0^x (f(u)/u)du$, and f grows faster than any power of logarithm. Thus, if we can show that

$$\mathbb{E}f\left(\sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right) < \infty, \tag{26}$$

then

1) (26) implies that $\mathbb{E}\log^+\left(\sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)<\infty$; this, in turn, allows us to conclude that the integral in (5) converges and, therefore, the needed part of Theorem 2 (a) applies;

2) since (26) is equivalent to $\mathbb{E}g\left(\sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)<\infty$, by Theorem 2 (a), (26) implies $\mathbb{E}f\left(\sum_{k=1}^{\infty}\Pi'_{k-1}Q'_k\right)<\infty$ and, hence, $\mathbb{E}f\left(\sum_{k=1}^{\infty}|\Pi_{k-1}||Q_k|\right)<\infty$ in view of (25).

Let U be a random variable distributed like $|Q|$ and independent of $\sup\limits_{k\ge 0}|\Pi_k|$. We now prove (26):

$$\mathbb{E}f\left(\sum_{k=1}^{N_1}|\Pi_{k-1}||Q_k|\right)\le\mathbb{E}\sum_{k=1}^{N_1}f(|\Pi_{k-1}||Q_k|)=$$

(subadditivity of f and $N_1<\infty$ a.s.)

$$=\sum_{n=1}^{\infty}\mathbb{E}\left(\sum_{k=1}^{n}f(|\Pi_{k-1}||Q_k|)\right)1_{\{N_1=n\}}=\sum_{n=1}^{\infty}\mathbb{E}\left(f(|\Pi_{n-1}||Q_n|)\right)1_{\{N_1>n-1\}}=$$

(the change of order of summation is justified by the fact that all summands are nonnegative)

$$=\sum_{n=1}^{\infty}\int\limits_0^{\infty}\mathbb{E}(f(|\Pi_{n-1}|q)1_{\{N_1>n-1\}})d\mathbb{P}\{|Q_n|\le q\}=$$

(Q_n is independent of both Π_{n-1} and $1_{\{N_1>n-1\}}$)

$$=\int\limits_0^{\infty}\sum_{n=0}^{\infty}\mathbb{E}f(|\Pi_n|q)1_{\{N_1>n\}}d\mathbb{P}\{U\le q\}=\mathbb{E}N_1\int\limits_0^{\infty}\mathbb{E}f(q\sup_{k\ge 0}|\Pi_k|)d\mathbb{P}\{U\le q\}=$$

(this, is in fact, Lemma 2 of [22]:

$$\mathbb{E}f\left(q\sup_{k\ge 0}|\Pi_k|\right)=\mathbb{E}f\left(q\exp(-\inf_{k\ge 0}S_k)\right)=$$

$$=(\mathbb{E}N_1)^{-1}\sum_{n=0}^{\infty}\mathbb{E}f(q\exp(-S_n))1_{\{S_0\le 0,\ldots,S_n\le 0\}}=$$

$$=(\mathbb{E}N_1)^{-1}\sum_{n=0}^{\infty}\mathbb{E}f(q|\Pi_n|)1_{\{N_1>n\}};$$

Keener assumed that $\mathbb{E}S_1$ exists, but this condition is not needed)

$$=\mathbb{E}N_1\mathbb{E}f(U\sup_{k\ge 0}|\Pi_k|)\le p\mathbb{E}N_1\mathbb{E}f(|Q|)\mathbb{E}f(\sup_{k\ge 0}|\Pi_k|)<\infty$$

(we have used (3) and the assumptions of the proposition). The proof of Proposition 4 is finished.

We are now ready to give proof of Theorem 2 (b).

***Proof of Theorem* 2 (b).** *Necessity.* Let $\mathbb{E}f(|Q|) < \infty$, $\mathbb{E}f(|M| \vee 1) < \infty$, and $\mathbb{E}|M|^r < 1$ if $r > 0$. Using Lemma 1 (a), we conclude that $\mathbb{E}g(|M| \vee 1) < \infty$, where $g(x) = \int_0^x (f(u)/u)du$. Let us check that

$$\widetilde{a} := \int\limits_0^\infty h(x)\mathbb{P}\{\log^+|M| > x\}dx < \infty,$$

where $h(x) = f(e^x)$. Note first that

$$g(e^x) = \int\limits_{-\infty}^{x} f(e^u)du = \text{const} + \int\limits_0^x f(e^u)du.$$

Further, we have $\widetilde{a} = \mathbb{E}\int_0^{\log^+|M|} f(e^x)dx = -\text{const} + \mathbb{E}g(e^{\log^+|M|}) = -\text{const} + \mathbb{E}g(|M| \vee 1) < \infty$. Inequality (4) now implies that

$$\int\limits_0^\infty \psi(x)\mathbb{P}\{\log^+|M| > x\}dx < \infty.$$

According to Theorem 2 of [23], the latter condition together with the conditions $\mathbb{E}|M|^r < 1$, if $r > 0$, and $\mathbb{E}\log|M| \in [-\infty, 0)$ allows us to conclude that $\mathbb{E}\psi\left(\sup_{n\geq 0}(-S_n)\right) < \infty$, where $S_n = -\log|\Pi_n|$. From (4) it follows that

$$\infty > \mathbb{E}f\left(\exp(\sup_{n\geq 0}(-S_n)\right) = \mathbb{E}f\left(\sup_{n\geq 0}|\Pi_n|\right).$$

It remains to apply Proposition 4 to conclude that $\mathbb{E}f(|Z_\infty|) < \infty$.

Sufficiency. Let now $\mathbb{E}f(|Z_\infty|) < \infty$. By Proposition 3, $\infty > \mathbb{E}f(|Q|)$ and either

$$\infty > \mathbb{E}f\left(\sup_{n\geq 0}|\Pi_n|\right) = \mathbb{E}h\left(\sup_{n\geq 0}(-S_n)\right) \tag{27}$$

or

$$\infty > \mathbb{E}f\left(\sup_{n\geq 0}|\grave{\Pi}_n|\right) = \mathbb{E}h\left(\sup_{n\geq 0}(-\grave{S}_n)\right), \tag{28}$$

where $\grave{S}_n = -\log|\grave{\Pi}_n|$ is the random walk with a step distributed like $-\log|M_1M_2|$, and $h(x) = f(e^x)$. Let us check that (28) ensures $\mathbb{E}f(|M| \vee 1) < \infty$ and, if $r > 0$, $\mathbb{E}|M|^r < 1$. The proof for (27) is simpler.

In view of (4), we have

$$\mathbb{E}\psi\left(\sup_{n\geq 0}(-\grave{S}_n)\right) < \infty. \tag{29}$$

By the assumption of the theorem, $\mathbb{E}(-\grave{S}_1) = \mathbb{E}\log|M_1M_2| \in (-\infty, 0)$. An appeal to Theorem 1 [23] allows us to conclude that

$$\int\limits_0^\infty \psi(x)\mathbb{P}\{\log^+|M_1M_2| > x\}dx < \infty.$$

According to (4), $\widehat{b} := \int_0^\infty h(x)\mathbb{P}\{\log^+|M_1M_2| > x\}dx < \infty$. But

$$\widehat{b} = \mathbb{E}\int\limits_0^{\log^+|M_1M_2|} f(e^x)dx = -\text{const} + \mathbb{E}g(e^{\log^+|M_1M_2|}) =$$

$$= -\text{const} + \mathbb{E}g(|M_1M_2| \vee 1)$$

and $f \asymp g$. Therefore, $\mathbb{E}f(|M_1M_2| \vee 1) < \infty$ and

$$\mathbb{E}f(|M| \vee 1) \le (\mathbb{P}\{|M| > 1\})^{-1}\mathbb{E}f(|M_1M_2| \vee 1) < \infty.$$

If $r > 0$, then (29) implies $1 > \mathbb{E}|M_1M_2|^r = (\mathbb{E}|M|^r)^2$ (see Remark 1 [23]). The proof of Theorem 2 (b) is finished.

4. Proofs related to the BRW. Let t_r be a rooted family tree associated with a point process $\mathcal{M}$. We say that (t_r, X) is a labelled tree if each individual (vertex) $\theta \in t_r\backslash\{0\}$ is assigned its displacement $X(\theta)$ from its parent. The BRW defines a probability measure μ on the set of labelled trees. Lyons in [7] constructed a new probability measure $\widehat{\mu}^*$ on the set of infinite labelled trees with distinguished rays (a ray is an infinite line of descent starting from the root). It is under this measure $\widehat{\mu}^*$ we can successfully bound the martingale limit W by a perpetuity from the above, and $\sup\limits_{k\ge0} W_k$ by a largest summand of a perpetuity from below.

Under $\widehat{\mu}^*$, the usual family tree is replaced by a size-biased tree that has a ray with a special status. This single ray is often called a trunk or spine.

For $k = 1, 2, \ldots$, let v_k be the individual belonging to the trunk and sitting at the k-th generation, v_0 be an initial ancestor; and let $A_{v_k,i}$, $i = 1, 2, \ldots$, be the displacements of children of v_k from v_k. For $k = 1, 2, \ldots$, let $\widehat{M_k}$ be the random variable which gives the displacement of v_k from her mother divided by $m(\gamma)$ and $\widehat{Q}_k = m^{-1}(\gamma)\sum_i e^{\gamma A_{v_{k-1},i}}$. Note that, by construction, the random vectors $(\widehat{M_k}, \widehat{Q}_k)$, $k = 1, 2, \ldots$, are independent and identically distributed[1].

Let $\mathcal{G}$ be the σ-field generated by the reproduction of v_k, $k = 1, 2, \ldots$, i.e., by the sequence of independent point processes $\widehat{\mathcal{M}_k}$ with points $A_{v_{k-1},i}$, $i = 1, 2, \ldots$. Set $\widehat{\Pi}_0 = 1$ and $\widehat{\Pi}_k := \widehat{M_1}\ldots\widehat{M_k}$, $k = 1, 2, \ldots$. Now we can write the two essential inequalities which, in fact, were found by Lyons:

$$\mathbb{E}_{\mu^*}(W_n|\mathcal{G}) \le \sum_{k=1}^n \widehat{\Pi}_{k-1}\widehat{Q}_k, \tag{30}$$

$$\sup_{k\ge0} W_k \ge \sup_{k\ge0} \widehat{\Pi}_k\widehat{Q}_{k+1} \ \text{ under } \ \widehat{\mu}^*. \tag{31}$$

[1] Note that the proof of Theorem 2 [3] gives erroneous impression that $\widehat{M}_k$ and $\widehat{Q}_k$ are independent as well. Fortunately, this gap does not affect the proof of that result.

***Proof of Theorem* 1 (b).** *Necessity.* Under $\widehat{\mu}^*$, $\widehat{M_1} \overset{d}{=} Z$ and $\widehat{Q}_1$ has the size-biased distribution corresponding to the distribution of W_1. Thus, the assumptions of the theorem can be rewritten in terms of $\widehat{M}$ and $\widehat{Q}$ as follows:

$$\widehat{\mu}^*\{\widehat{M} > 1\} > 0, \qquad \mathbb{E}_{\mu^*} \log \widehat{M} \in [-\infty, 0), \qquad \mathbb{E}_{\mu^*} f(\widehat{Q}) < \infty$$

and if $r > 0$

$$\mathbb{E}_{\mu^*} \widehat{M}^r < 1.$$

Since $W_1 = \sum_{i=1}^{K} Y_i$, we have $Y_i \leq W_1$, $i = 1, 2, \ldots$. Hence,

$$\mathbb{E}_{\mu^*} f(\widehat{M} \vee 1) = \mathbb{E} f(Z \vee 1) = \mathbb{E} \sum_{i=1}^{\infty} Y_i f(Y_i \vee 1) \leq \mathbb{E} W_1 f(W_1 \vee 1) < \infty.$$

We conclude that all the assumptions of necessity of Theorem 2 (b) hold. Therefore, the right-hand side of (30) converges $\widehat{\mu}^*$ a.s. to a random variable $\widehat{Z_\infty}$, say and $\mathbb{E}_{\mu^*} f(\widehat{Z_\infty}) <$ $< \infty$. From the results of Lyons it follows that W_n converges to W $\widehat{\mu}^*$ a.s. (apply Fatou's lemma to (30) to conclude $\mathbb{E}_{\mu^*} \left(\liminf_{n\to\infty} W_n | \mathcal{G}\right) \leq \widehat{Z_\infty}$; hence, $\liminf_{n\to\infty} W_n < \infty$ under $\widehat{\mu}^*$; from the change of measure construction it follows that $1/W_n$ is a positive $\widehat{\mu}^*$ martingale with respect to an appropriate filtration; hence the statement). By Fatou's lemma, we have

$$\mathbb{E}_{\mu^*}(W|\mathcal{G}) \leq \widehat{Z_\infty}.$$

Since f is nondecreasing and concave, then using Jensen's inequality and applying the expectation operator one more results in

$$\mathbb{E}_{\mu^*}(f(W)|\mathcal{G}) \leq f(\mathbb{E}_{\mu^*}(W|\mathcal{G})) \leq f(\widehat{Z_\infty}),$$

$$\mathbb{E}_{\mu^*} f(W) \leq \mathbb{E}_{\mu^*} f(\widehat{Z_\infty}). \tag{32}$$

We have already proved that the right-hand side of (32) is finite. Hence, we have

$$\mathbb{E}_{\mu^*} f(W) < \infty.$$

Using the Laplace – Stieltjes transforms, it can be easily checked that $\mathbb{E} W_n f(W_n) =$ $= \mathbb{E}_{\mu^*} f(W_n)$, $n = 1, 2, \ldots$, implies

$$\mathbb{E} W f(W) = \mathbb{E}_{\mu^*} f(W). \tag{33}$$

This completes the proof of this part of Theorem 1 (b).

To prove Theorem 1 (b) in the reverse direction, we need a lemma. It proposes a $\widehat{\mu}^*$-counterpart of the inequality obtained in Lemma 2 [12].

Lemma 3. *For each $a > 0$ small enough, there exists $B > 1$ such that whenever $t > 1$, the following inequalities hold*:

$$\widehat{\mu}^*\{W > t\} \leq \widehat{\mu}^* \left\{\sup_{n\geq 0} W_n > t\right\} \leq B\widehat{\mu}^*\{W > at\}.$$

In particular, for any nonnegative, nondecreasing and anti-starshaped (*in particular, concave*) *function h,*

$$\mathbb{E}_{\mu^*} h(W) < \infty \Leftrightarrow \mathbb{E}_{\mu^*} h\left(\sup_{n\geq 0} W_n\right) < \infty.$$

Proof. The left-hand side inequality is obvious. Let us prove the rest. It can be checked that under $\widehat{\mu}^*$, the following equality of distributions holds:

$$W \stackrel{d}{=} \frac{1}{m^n(\gamma)} e^{\gamma v_n} W + \frac{1}{m^n(\gamma)} \sum_{|u|=n}{}' e^{\gamma A_u} V_u, \tag{34}$$

where $\sum_{|u|=n}'$ denotes the summation over all individuals of the n-th generation (of the size-biased tree) but v_n; given the information about first n generations in the size-biased tree V_u are independent copies of a random variable V with distribution $\mathbb{P}\{W \in dx\}$ which is also independent of W.

In what follows, we write $\mathbb{E}$ and $\mathbb{P}$ instead of $\mathbb{E}_{\mu^*}$ and $\widehat{\mu}^*$. We can choose $c,\ d > 0$ such that $r := \mathbb{E}(V \wedge c) = \mathbb{E}(W \wedge d) \in (0,1)$. Fix any $a \in (0, r)$. Consider the events $E_n := \left\{ \max\limits_{0 \le i \le n-1} W_i \le t,\ W_n > t \right\}$, $n = 1, 2, \ldots$. Without loss of generality, we can assume that the set $\{u : |u| = n\}$ is enumerated in some way such that v_n is the first individual. Keeping this in mind, denote by $\{\alpha_k, k = 1, 2, \ldots\}$ realizations of $\left\{ \frac{1}{m^n(\gamma) W_n} e^{\gamma A_u},\ |u| = n \right\}$. Note that $\sum_k \alpha_k = 1$. Define the event

$$D := \left\{ \frac{1}{m^n(\gamma) W_n} e^{\gamma v_n} (W \wedge d) + \frac{1}{m^n(\gamma) W_n} \sum_{|u|=n}{}' e^{\gamma A_u} (V_u \wedge c) > a \right\}.$$

Then almost surely

$$\mathbb{P}\{D | \mathcal{F}^n\} = \mathbb{P}\left\{ \eta := \alpha_1 (W \wedge d) + \sum_{k=2}^{\infty} \alpha_k (V_k \wedge c) - a > 0 \right\} \ge \frac{1}{B},$$

where $\frac{1}{B} := \frac{(r-a)^2}{(\mathbb{E}(V \wedge c - a)^2) \vee (\mathbb{E}(W \wedge d - a)^2)} \in (0,1)$. To get the latter inequality, we have used

$$\mathbb{P}\{\eta > 0\} \ge \frac{(\mathbb{E}\eta)^2}{\mathbb{E}\eta^2},$$

which is applicable as $\mathbb{E}\eta = r - a > 0$. $\mathbb{E}\eta^2$ is estimated as follows:

$$\mathbb{E}\eta^2 = \alpha_1^2 \mathbb{E}(W \wedge d - a)^2 + \mathbb{E}(V \wedge c - a)^2 \sum_{k=2}^{\infty} \alpha_k^2 + 2\Big(\mathbb{E}(V \wedge c - a)\Big)^2 \sum_{1 \le i < j} \alpha_i \alpha_j \le$$

$$\le \Big((\mathbb{E}(W \wedge d - a)^2\Big) \vee \Big(\mathbb{E}(V \wedge c - a)^2)\Big) \sum_{k=1}^{\infty} \alpha_k^2 + 2\Big(\mathbb{E}(V \wedge c - a)\Big)^2 \sum_{1 \le i < j} \alpha_i \alpha_j \le$$

$$\le \Big((\mathbb{E}(W \wedge d - a)^2\Big) \vee \Big(\mathbb{E}(V \wedge c - a)^2)\Big) \left(\sum_{k=1}^{\infty} \alpha_k^2 + 2 \sum_{1 \le i < j} \alpha_i \alpha_j \right) =$$

$$= \Big((\mathbb{E}(W \wedge d - a)^2\Big) \vee \Big(\mathbb{E}(V \wedge c - a)^2)\Big).$$

Since $E_n \in \mathcal{F}^n$, we have $\mathbb{P}\{D \bigcap E_n\} = \mathbb{E}\mathbb{P}\{D | \mathcal{F}^n\} 1_{E_n} \ge (1/B)\mathbb{P}\{E_n\}$. If $\mathbb{P}\{E_n\} \ne$ $\ne 0$, the latter implies

$$\mathbb{P}\{D|E_n\} \geq \frac{1}{B}. \tag{35}$$

For $t > 1$, we have

$$\mathbb{P}\{W > at|E_n\} \overset{(34)}{\geq} \mathbb{P}\left\{\frac{1}{m^n(\gamma)W_n}e^{\gamma v_n}W + \frac{1}{m^n(\gamma)W_n}\sum_{|u|=n}' e^{\gamma A_u}V_u > \frac{at}{W_n}\middle|E_n\right\} \geq$$

$$\geq \mathbb{P}\{D|E_n\} \overset{(35)}{\geq} \frac{1}{B}.$$

The inequality

$$\mathbb{P}\{W > at\} \geq \sum_{n=1}^{\infty}\mathbb{P}\{W > at|E_n\}\mathbb{P}\{E_n\} \geq \left(\frac{1}{B}\right)\mathbb{P}\left\{\sup_{n\geq 0}W_n > t\right\}$$

completes the proof of the first part of Lemma 3. To prove the second part, we should only note that the implication $\Rightarrow$ follows from the inequality $\mathbb{E}h(\sup_{n\geq 0}W_n) \leq B\mathbb{E}h(W/a) \leq$ $\leq (B/a)\mathbb{E}h(W)$.

***Proof of Theorem* 1 (b).** *Sufficiency.* Assume now that $\mathbb{E}\log Z \in (-\infty, 0)$ and $\mathbb{E}Wf(W) < \infty$. Then $\mathbb{E}_{\mu^*}\log\widehat{M} \in (-\infty, 0)$ and in view of (33), $\mathbb{E}_{\mu^*}f(W) < \infty$. Therefore, by Lemma 3, $\mathbb{E}_{\mu^*}f\left(\sup_{k\geq 0}W_k\right) < \infty$. In view of (31),

$$\infty > \mathbb{E}_{\mu^*}f\left(\sup_{k\geq 0}W_k\right) \geq \mathbb{E}_{\mu^*}f\left(\sup_{k\geq 0}\widehat{\Pi}_k\widehat{Q}_{k+1}\right).$$

Let us now apply Proposition 3 on the pair $(\widehat{M}, \widehat{Q})$ to get $\mathbb{E}_{\mu^*}f(\widehat{Q}) < \infty$ and either $\mathbb{E}_{\mu^*}f\left(\sup_{n\geq 0}\widehat{\Pi}_n\right) < \infty$ or $\mathbb{E}_{\mu^*}f\left(\sup_{n\geq 0}\grave{\widehat{\Pi}}_n\right) < \infty$. Exactly the same analysis as in the proof of Theorem 1 (b) (implication $\Leftarrow$) shows that $\mathbb{E}_{\mu^*}f(\widehat{M}\vee 1) < \infty$ and, if $r > 0$, $\mathbb{E}_{\mu^*}\widehat{M}^r < 1$. It remains to recall that

$$\mathbb{E}_{\mu^*}f(\widehat{Q}) = \mathbb{E}W_1f(W_1), \qquad \mathbb{E}_{\mu^*}f(\widehat{M}\vee 1) = \mathbb{E}f(Z\vee 1), \qquad \mathbb{E}_{\mu^*}\widehat{M}^r = \mathbb{E}Z^r.$$

However, the condition $\mathbb{E}f(Z\vee 1) < \infty$ can be omitted as it is implied by $\mathbb{E}W_1f(W_1) <$ $< \infty$. The proof is complete.

***Proof of Theorem* 1 (a)** goes the similar but simpler way as that of Theorem 1 (b). The only difference is that while Theorem 1 (b) uses Theorem 2 (b) and Proposition 3, Theorem 1 (a) appeals to Theorem 2 (a) and Corollary 1. We can use these statements as the condition $\mathcal{M}(-\infty, -\gamma^{-1}\log m(\gamma)) = 0$ a.s. implies that the distribution of the random variable Z (and hence of $\widehat{M}$) is concentrated on $[0, 1]$. Recall that, throughout the paper, we assumed that $\mathbb{P}\{Z = 1\} < 1$.

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