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ON ITERATION STABILITY OF THE BIRKHOFF CENTER WITH RESPECT TO POWER 2*

ІТЕРАЦІЙНА СТІЙКІСТЬ ЦЕНТРА БІРКГОФА ВІДНОСНО СТЕПЕНЯ 2

It is proved that the Birkhoff center of a homeomorphism on an arbitrary metric space coincides with the Birkhoff center of its power 2.

Доведено, що центр Біркгофа гомеоморфізму на довільному метричному просторі збігається з центром Біркгофа його степеня 2.

It is known that nonwandering set of a homeomorphism can change when we take a power of the homeomorphism. Such examples are shown in [1, 2]. On the other hand, the sets of recurrent points [3], chain recurrent points [4], and the limit set do not change when we take a power. A natural question arises: What can happen with the Birkhoff center of a dynamical system when we take its power? Here, we prove that the Birkhoff center of dynamical systems on arbitrary metric spaces coincides with the Birkhoff center of their power 2.

Let M be a metric space and let $f, g : M \to M$ be homeomorphisms of M. Denote by $\Omega(f)$ the set of nonwandering points of f. Iterate the construction of the nonwandering set. Let $\Omega_1(f) = \Omega(f)$. Define by induction $\Omega_{n+1}(f) = \Omega\left(f\big|_{\Omega_n(f)}\right)$. Denote by $\Omega_\omega(f)$ the intersection of the obtained sequence of embedded closed invariant sets. This process can be continued using transfinite induction. According to the Zorn lemma, the process will stop at an ordinal number α for which $\Omega_\alpha(f) = \Omega\left(f\big|_{\Omega_\alpha(f)}\right)$. The obtained closed invariant set is called *the Birkhoff center* and is denoted by $BC(f)$.

Lemma 1. $\Omega(g^n) \subseteq \Omega(g)$.

Proof. Let $x \in M \setminus \Omega(g)$ be a wandering point of g. By definition, there exists a neighborhood U of x such that $g^k(U) \cap U = \varnothing$. Then $g^{nk}(U) \cap U = f^k(U) \cap U = \varnothing$, $k \in \mathbb{Z}$, and x is a wandering point of f. Thus, $M \setminus \Omega(g) \subseteq M \setminus \Omega(g^n)$ and $\Omega(g) \supseteq \Omega(g^n)$.

Definition 1. *A point* ξ *is called tied with a point* μ *by* g *if for all neighborhoods* $U(\mu)$ *and* $V(\xi)$ *of points* μ *and* ξ, *correspondingly, there exists* $N \in \mathbb{Z}$ *such that* $U(\mu) \cap g^N(V(\xi)) \neq \varnothing$.

Definition 2. *If one can choose* $N \in \mathbb{Z}^-$ ($N \in \mathbb{Z}^+$) *in the previous definition, then the point* ξ *is called* α*-tied* (ω*-tied*) *with the point* μ. *The point* α- *and* ω*-tied with the point* μ *is called bi-tied with the point* μ.

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Definition 3. *Points* $\xi_0, \ldots, \xi_{n-1}$ *are called cyclically tied with a cycle length* n *if there exists* $k \in 1, \ldots, n-1$ *such that for each* $i = 1, \ldots, n-1$, *the point* ξ_i *is* ω- (α-)*tied with the point* ξ_m, $m = k + i \pmod n$.

Lemma 2. *Let* $\xi \in \Omega(g) \setminus \Omega(g^n)$. *Then the trajectories* $O_{g^n}\left(g^i(\xi)\right)$, $i = 1, \ldots$ $\ldots, n-1$, *related to* g^n *on which the trajectory* $O_g(\xi)$ *is split are wandering cyclically tied ones.*

Proof. Let $\xi \in \Omega(g) \setminus \Omega(g^n)$. Since $\xi \in \Omega(g)$, we have $\forall\, U(\xi)\ \exists\, m$: $g^m(U(\xi)) \cap U(\xi) \neq \varnothing$. Further, since $\xi \notin \Omega(g^n)$, $\forall\, l \in \mathbb{Z}: g^{ln}(U(\xi)) \cap U(\xi) \neq \varnothing$. We obtain that $m \neq 0 \bmod n$ and, hence, ξ and $g^m(\xi)$ belong to different trajectories related to g^n. Choose $U(\xi)$ to be the set $B_{1/p}(\xi)$ of $1/p$-neighborhoods of the point $\xi, p \in \mathbb{N}$. Since n is finite, but the set of values of p is infinite, there exists $k \neq 0$ mod n such that $m(p) = k \bmod n$ for infinite number of values of p in the previous formulae. Show that ξ is tied with $g^k(\xi)$. Let $V_1(\xi)$ and $V_2(g^k(\xi))$ be arbitrary neighborhoods of points ξ and $g^k(\xi)$. Choose p such that $m(p) = tn + k$ to be large enough for inclusions $B_{1/p}(\xi) \subset V_1(\xi)$ and $g^k(B_{1/p}(\xi)) \subset V_2(g^k(\xi))$. Then $g^{tn+k}(B_{1/p}(\xi)) \cap B_{1/p}(\xi) \neq \varnothing$ and, hence, $V_1(\xi) \cap g^{tn}\left(V_2\left(g^k(\xi)\right)\right) \neq \varnothing$.

Cyclicity of tying follows from the fact that g maps tied points into tied ones.

The lemma is proved.

Theorem 1. *A nonwandering set coinciding with the whole space is iteration stable with respect to power* 2.

Proof. Suppose on the contrary that $M = \Omega(f) \neq \Omega(f^2)$.

Let $x \in \Omega(f) \setminus \Omega(f^2)$. Then, according to Lemma 2, x and $y = f(x)$ are tied: $\forall\, n \geq 1\ \ \exists\, a_n \in B_{1/n}(x)\ \ \exists\, A_n$: $f^{A_n}(a_n) \in B_{1/n}(y)$. Assume for definiteness that x and y are ω-tied (including the bi-tied case too). Then $A_n > 0$. Otherwise, x and y are α-tied, and this case reduces to previous one but with f^{-1} instead of f, because by definition the nonwandering sets of f and f^{-1} coincide. Note that we can assume all a_n to be wandering points of f^2 because the set of wandering points is open.

Denote $b_n = f^{-1}\left(f^{A_n}(a_n)\right)$. The sequence $f^{A_n}(a_n)$ tends to y. The sequence (b_n) tends to x by continuity. Denote with $B_\varepsilon(x)$ the open ball of radius ε around the point x. According to Lemma 2, b_n and $f^{A_n}(a_n)$ are tied. Then $\forall\, n \geq 1$ $\forall\, m \geq \geq 1\ \ \exists\, c_{mn} \in B_{1/m}(b_n)\ \ \exists\, K_{mn}$: $f^{K_{mn}}(c_{mn}) \in B_{1/m}\left(f^{A_n}(a_n)\right)$. Similarly, we can assume c_{mn} to be wandering points of f^2. Since b_n and $f^{A_n}(a_n)$ are tied, there exists a subsequence such that either b_n and $f^{A_n}(a_n)$ are ω-tied or b_n and $f^{A_n}(a_n)$ are α-tied. Switching to such subsequence, we can assume that all b_n and $f^{A_n}(a_n)$ are ω-tied (or α-tied).

Assume that they are ω-tied. Then $K_{mn} > 0$. Consider the sequence (c_{nn}). By construction, (c_{nn}) tends to x and $\left(f^{-A_n}(c_{nn})\right)$ tends to y. But $\left(f^{K_{nn}}(c_{nn})\right)$ also tends to y. Since $A_n > 0$ and $K_{mn} > 0$, $K_{nn} \neq -A_n$. We obtain that the point y is self-tied and, hence, is nonwandering.

Assume that they are α-tied. Consider the sequence $f^{-1}(c_{nn})$. By continuity, it tends to y. According to Lemma 2, $f^{-1}(c_{mn})$ and (c_{mn}) are tied. The further proof is reduced to the considered cases. If we can choose a subsequence from $f^{-1}(c_{nn})$ which consists of points ω-tied to their images, then, repeating the reasons for ω-tied b_n and $f^{A_n}(a_n)$, we obtain that x is nonwandering point. Otherwise, b_n and $f^{A_n}(a_n)$ are α-tied and the sequences $f^{-1}(c_{m\,n})$ and $(c_{m\,n})$ which tie them are also α-tied. This case reduces by substituting f with f^{-1} to already considered one when points are ω-tied and the sequences which tie them are also ω-tied.

The theorem is proved.

Lemma 3. $BC(g^n) \subseteq BC(g)$.

Proof. Consider the family of nonwandering sets in the definition of the Birkhoff center $\Omega(f) = \Omega_1(f) \supseteq \Omega_2(f) \supseteq \ldots \supseteq \Omega_\omega(f) \supseteq \Omega_{\omega+1}(f) \supseteq \ldots$ indexed with ordinal numbers. The family is ordered by inclusion. At that relations, $\Omega_\alpha(f) \supseteq \Omega_\beta(f)$ and $\alpha \leq \beta$ are equivalent. Consider the same family for f^n.

Prove that $\Omega_\lambda(f^n) \subseteq \Omega_\lambda(f)$ for each ordinal λ using transfinite induction. When $\lambda = 1$, it is Lemma 1. Let it be true for each $\alpha < \lambda$. Show that it is true for λ. If λ is limit ordinal, then by construction, $\Omega_\lambda(f) = \bigcap_{\beta<\lambda} \Omega_\beta(f) \supseteq \bigcap_{\beta<\lambda} \Omega_\beta(f^n) = \Omega_\lambda(f^n)$. Otherwise, λ has previous ordinal $\hat{\lambda}$. According to inductive proposition, $\Omega_{\hat{\lambda}}(f) \supseteq \Omega_{\hat{\lambda}}(f^n)$.

Denote $f_{\hat{\lambda}} = f\big|_{\Omega_{\hat{\lambda}}(f)} : \Omega_{\hat{\lambda}}(f) \to \Omega_{\hat{\lambda}}(f)$ and $f'_{\hat{\lambda}} = f^n\big|_{\Omega_{\hat{\lambda}}(f^n)} : \Omega_{\hat{\lambda}}(f) \to \Omega_{\hat{\lambda}}(f)$. Note that $\Omega\left(f_{\hat{\lambda}}\right) \supseteq \Omega\left(\left(f_{\hat{\lambda}}\right)^n\right)$ according to Lemma 1 and $f'_{\hat{\lambda}} = \left(f_{\hat{\lambda}}\right)^n\Big|_{\Omega_{\hat{\lambda}}(f^n)}$. Hence, $\Omega_\lambda(f) = \Omega\left(f\big|_{\Omega_{\hat{\lambda}}(f)}\right) = \Omega\left(f_{\hat{\lambda}}\right) \supseteq \Omega\left(\left(f_{\hat{\lambda}}\right)^n\right) \supseteq \Omega\left(\left(f_{\hat{\lambda}}\right)^n\Big|_{\Omega_{\hat{\lambda}}(f^n)}\right) = \Omega\left(f'_{\hat{\lambda}}\right) = \Omega\left(f^n\Big|_{\Omega_{\hat{\lambda}}(f^n)}\right) = \Omega_\lambda(f^n)$. The second inclusion follows from the fact that the nonwandering set of a map cannot be smaller that the nonwandering set of its restriction.

Let λ, λ' be depths of centers $BC(f)$ and $BC(f^n)$, correspondingly. Denote $\beta = \max(\lambda, \lambda')$. Then $BC(f) = \Omega_\beta(f) \supseteq \Omega_\beta(f^n) = BC(f^n)$.

The lemma is proved.

It follows that the iteration stability of Birkhoff center is equivalent to the iteration stability of nonwandering set coinciding with the whole space. As a consequence, we have the following theorem:

Theorem 2. *The Birkhoff center is iteration stable with respect to power* 2.

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