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CONVERGENCE OF SP-ITERATION FOR GENERALIZED NONEXPANSIVE MAPPING IN BANACH SPACES

ЗБІЖНІСТЬ SP-ІТЕРАЦІЙ ДЛЯ УЗАГАЛЬНЕНИХ НЕРОЗШИРЮЮЧИХ ВІДОБРАЖЕНЬ У БАНАХОВИХ ПРОСТОРАХ

Phuengrattana and Suantai [J. Comput. and Appl. Math., **235**, 3006 – 3014 (2011)] introduced an iteration scheme and they named this iteration as SP-iteration. In this paper, we study the convergence behavior of SP-iteration scheme for the class of generalized nonexpansive mappings. One weak convergence theorem and two strong convergence theorems in uniformly convex Banach spaces are obtained. We also furnish a numerical example in support of our main result. In process, our results generalize and improve many existing results in the literature.

У роботі Phuengrattana і Suantai [J. Comput. and Appl. Math., **235**, 3006 – 3014 (2011)] запропоновано ітераційну схему із назвою SP-ітерація. Нашу статтю присвячено вивченню збіжності цієї схеми SP-ітерацій для класу узагальнених нерозширюючих відображень. Доведено одну теорему про слабку збіжність та дві теореми про сильну збіжність у рівномірно опуклих банахових просторах. З метою ілюстрації основного результату наведено числовий приклад. Отримані результати узагальнюють та удосконалюють багато інших відомих результатів.

1. Introduction. Let X be an arbitrary nonempty set and $T\colon X \to X$. A point $x \in X$ is said to be fixed point of mapping T if $Tx = x$. Fixed point theorems play a very important role in many fields so that discussions and studies on its concept provide wide applications in various areas not only in mathematics but also in other allied subjects. For example, in mathematics, fixed point theorems are vital for the existence of a solution to boundary-value problems and integral equations. In economics, fixed point results are incredibly useful when it comes to prove the existence of a solution for various types of Nash equilibria. Moreover, there are some applications in chemistry, biology, computer science and engineering. The classical contraction mapping principle of Banach is one of the most powerful theorems in fixed point theory. A number of articles in the fixed point theory have been dedicated to the improvement and generalization of this pioneer theorem. It is also well-known that different iteration processes for contraction and nonexpansive mappings have been successfully used to develop efficient and powerful numerical methods for solving various nonlinear equations and variational problems, often of great importance for applications in various areas of pure and applied sciences. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an up-to date literature on this theme, one can refer to Berinde [1].

Let K be a nonempty subset of Banach space X. A mapping $T\colon K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all} \quad x,\, y \in K.$$

It is known that in general, sequence of Picard's iterates defined as (for any $x_1 \in K$)

$$x_{n+1} = T^n x_1, \qquad n \in \mathbb{N},$$

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does not converge for a nonexpansive mapping, e.g., Picard's iterates of nonexpansive mapping $T\colon [-1,1] \to [-1,1]$ defined by $Tx = -x$ does not converge for any nonzero $x \in [-1,1]$ even T has a fixed point.

In an attempt to construct a convergent sequence of iterates in respect of a nonexpansive mapping, Mann [2] defined an iteration method as follows:

$$x_1 \in K,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \qquad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$.

In 1974, Ishikawa [3] introduced a new two step iteration procedure as follows:

$$x_1 \in K,$$

$$y_n = (1-\alpha_n)x_n + \alpha_n T x_n,$$

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \qquad n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$.

Mann and Ishikawa iteration procedures are two basic and most utilized iteration schemes. For a comparison of two iterative schemes in the one-dimensional case, one may refer Rhoades [4] wherein it is shown that under suitable conditions (see part (a) of Theorem 3) rate of convergence of Ishikawa iteration is better than that of Mann iteration. Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been investigated by various authors (c.f. [5–9]).

In 2007, Xu and Noor [10] introduced a three step iteration scheme which is a genuine extension of Mann and Ishikawa schemes and described as follows: (for $x_1 \in K$)

$$y_n = (1-\gamma_n)x_n + \gamma_n T x_n,$$

$$z_n = (1-\beta_n)x_n + \beta_n T y_n,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \subset (0,1)$.

Thianwan [11] introduced the following two step iteration scheme:

$$y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$.

Recently, Phuengrattana and Suantai [12] defined SP-iteration as follows: (for $x_1 \in K$)

$$y_n = (1-\gamma_n)x_n + \gamma_n T x_n,$$

$$z_n = (1-\beta_n)y_n + \beta_n T y_n, \tag{1.2}$$

$$x_{n+1} = (1-\alpha_n)z_n + \alpha_n T z_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\gamma_n \subset (0,1)$.

Phuengrattana and Suantai [12] proved some convergence theorems for SP-iteration. They also proved that the rate of convergence of iterative schemes due to Mann [2], Ishikawa [5], Xu and Noor [10] and SP-iteration [12] is equivalent for nonexpansive mapping but SP-iteration converges better than others for the class of continuous and nondecreasing functions.

On the other hand, Suzuki [13] introduced a new class of mappings which is larger than the class of nonexpansive mappings and named the defining class as condition (C) which also referred as generalized nonexpansive mapping and proved some existence and convergence theorems for Mann iteration.

In this paper, we prove weak as well as strong convergence theorems for SP-iteration (1.2) for generalized nonexpansive mapping. In process, our results generalize several corresponding results contained in [10, 12, 14].

2. Basic definitions and relevant results. In this section, we collect some basic definitions and needed results. We start with the following definition due to Opial [15].

Definition 2.1. *A Banach space X is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in X with $x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) implies that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

Examples of Banach spaces satisfying Opial's condition are Hilbert spaces and all l^p, $1 < p < \infty$, spaces. On the other hand, $L^p[a, b]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

Now, we recall the definition of mapping which satisfies condition (C).

Definition 2.2 [13]**.** *A mapping T defined on a subset K of a Banach space X is said to satisfy condition* (C) *if* (*for all* $x, y \in K$)

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

From the definition, it is easy to see that every nonexpansive mapping satisfies condition (C). If a mapping T satisfies condition (C) and has a fixed point, then T is a quasinonexpansive mapping. But converse need not be true in general. The following examples justify these facts.

***Example* 2.1** [13]**.** Define a self mapping T on $[0, 3] \subset \mathbb{R}$ by

$$Tx = \begin{cases} 0, \text{ when } x \neq 3, \\ 1, \text{ when } x = 3. \end{cases}$$

Then T satisfies condition (C) but T is not a nonexpansive mapping.

***Example* 2.2** [13]**.** Define a self mapping T on $[0, 3] \subset \mathbb{R}$ by

$$Tx = \begin{cases} 0, \text{ when } x \neq 3, \\ 2, \text{ when } x = 3. \end{cases}$$

Then $F(T) \neq \varnothing$ and T is a quasinonexpansive mapping but does not satisfy condition (C).

Suzuki [13] also proved the following existence theorem for generalized nonexpansive mappings, i.e., condition (C).

Theorem 2.1 [13]**.** *Let T be a mapping defined on a convex subset K of a Banach space X which enjoys condition* (C)*. Also, assume that either of the following holds:*

(i) *K is compact*

or

(ii) *K is weakly compact and X has the Opial property.*

Then T has a fixed point in K.

The following theorem is also very important which characterizes the fixed point set of generalized nonexpansive mapping.

Theorem 2.2 [13]**.** *Let T be a mapping defined on a closed subset K of a Banach space X. Assume that T satisfies condition* (C)*. Then $F(T)$ is closed. Moreover, if X is strictly convex and K is convex, then F(T) is also convex.*

The following results are useful and will be used repeatedly.

Lemma 2.1 [13]**.** *Let K be a subset of a Banach space X and $T\colon K \to K$ be a mapping which satisfies condition* (C)*, then for all $x, y \in K$ following holds:*

$$\|x - Ty\| \leq 3\|x - Tx\| + \|x - y\|.$$

The following theorems due Xu [16] and Sun et al. [17] are crucial to prove our results.

Theorem 2.3 [16]**.** *Let X be a Banach space. Then X is uniformly convex if and only if, for any p, $1 < p < \infty$, and $r > 0$, there exists a continuous strictly increasing convex function $g_r\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $g_r(0) = 0$ and*

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - t(1-t)g_r(\|x - y\|)$$

for all $x, y \in B_r[0]$ and $t \in [0, 1]$.

Theorem 2.4 [17]**.** *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing map with $g(0) = 0$. If a sequence $\{x_n\}$ in $[0, \infty)$ satisfies $\lim_{n\to\infty} g(x_n) = 0$, then $\lim_{n\to\infty} x_n = 0$.*

In 1974, Senter and Dotson [18] introduced condition (I) as follows.

Definition 2.3 [18]**.** *A mapping $T\colon K \to K$ is said to satisfy condition* (I) *if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$.*

3. Main results. Firstly, we prove the following auxiliary lemma.

Lemma 3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T\colon K \to K$ be generalized nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1)$. If $\{x_n\}$ is described as in* (1.2)*, then*

(i) *$\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$,*

(ii) *$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. (i) Let $p \in F(T)$. Since,

$$\frac{1}{2}\|p - Tp\| = 0 \leq \|x_n - p\|,$$

which due to condition (C) gives rise $\|Tx_n - Tp\| \leq \|x_n - p\|$.

Similarly, we have $\|Ty_n - Tp\| \leq \|y_n - p\|$ and $\|Tz_n - Tp\| \leq \|z_n - p\|$. By (1.2), we get

$$\|y_n - p\| = \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \leq$$

$$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - Tp\| \leq$$

$$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| = \|x_n - p\|. \tag{3.1}$$

Also, we obtain

$$\|z_n - p\| = \|(1-\beta_n)y_n + \beta_n Ty_n - p\| \leq$$

$$\leq (1-\beta_n)\|y_n - p\| + \beta_n\|Ty_n - Tp\| \leq$$

$$\leq (1-\beta_n)\|y_n - p\| + \beta_n\|y_n - p\| \leq \|y_n - p\|. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

Now, consider

$$\|x_{n+1} - p\| = \|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \leq$$

$$\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - Tp\| \leq$$

$$\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \leq \|z_n - p\|. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

which shows that $\{\|x_n - p\|\}$ is a decreasing sequence of nonnegative reals. Thus, sequence $\{\|x_n - - p\|\}$ is bounded below and decreasing and, hence, it is convergent.

(ii) From part (i), $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. Let us write

$$\lim_{n\to\infty}\|x_n - p\| = c. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\liminf_{n\to\infty}\|z_n - p\| \geq c,$$

but using (3.3) and (3.4), we get

$$\limsup_{n\to\infty}\|z_n - p\| \leq c$$

and, hence,

$$\lim_{n\to\infty}\|z_n - p\| = c. \tag{3.6}$$

Also, by (3.2) and (3.6), we have

$$\liminf_{n\to\infty}\|y_n - p\| \geq c,$$

while (3.1) implies

$$\limsup_{n\to\infty}\|y_n - p\| \leq c.$$

Hence,

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.7}$$

Now, in view of Theorem 2.3, there exists a continuous strictly increasing convex function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$\|y_n - p\|^2 = \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\|^2 \leq$$

$$\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|Tx_n - Tp\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \leq$$

$$\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \leq$$

$$\leq \|x_n - p\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|)$$

yielding there by

$$\gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|y_n - p\|^2,$$

or

$$g(\|x_n - Tx_n\|) \leq \frac{1}{\gamma_n(1-\gamma_n)}\left[\|x_n - p\|^2 - \|y_n - p\|^2\right].$$

As $\{\gamma_n\} \in [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, then

$$g(\|x_n - Tx_n\|) \leq \frac{1}{\varepsilon^2}\left[\|x_n - p\|^2 - \|y_n - p\|^2\right].$$

In view of (3.5) and (3.7), $\lim_{n\to\infty} g(\|x_n - Tx_n\|) = 0$ and, hence, owing to Theorem 2.4, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Lemma 3.1 is proved.

Now, we prove the following weak convergence theorem for SP-iteration scheme.

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition and $T\colon K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.2), *then the sequence $\{x_n\}$ weakly converges to a fixed point of T.*

Proof. From Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(T)$ so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. As X is uniformly convex, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ for some $q \in K$. Now, we show that $q \in F(T)$. Suppose $q \neq Tq$. Then owing to Lemma 2.1, we have

$$\lim_{k\to\infty}\|x_{n_k} - Tq\| \leq \limsup_{k\to\infty}\{3\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tq\|\} \leq$$

$$\leq \limsup_{k\to\infty}\{3\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - q\|\}.$$

On making $k \to \infty$, we get $x_{n_k} \rightharpoonup Tq$ which is a contradiction to uniqueness of limit of convergent sequence and hence $Tq = q$. Now, we prove that $\{x_n\}$ has unique weak subsequential limit in $F(T)$. To show this, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q_1$ and $x_{n_j} \rightharpoonup q_2$. If $q_1 \neq q_2$, then owing to Opial's condition

$$\lim_{n\to\infty}\|x_n - q_1\| = \lim_{j\to\infty}\|x_{n_j} - q_1\| < \lim_{j\to\infty}\|x_{n_j} - q_2\| = \lim_{n\to\infty}\|x_n - q_2\| =$$

$$= \lim_{i\to\infty} \|x_{n_i} - q_2\| < \lim_{i\to\infty} \|x_{n_i} - q_1\| = \lim_{n\to\infty} \|x_n - q_1\|,$$

which is a contradiction and hence $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 3.1 is proved.

By setting $\beta_n = 0$, SP-iteration reduces to two step iteration scheme, i.e., (1.1). Now, from above theorem we can draw the following corollary.

***Corollary* 3.1.** *Let K be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition and $T: K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.1), *then the sequence $\{x_n\}$ weakly converges to a fixed point of T.*

Now, we prove following strong convergence theorem.

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T: K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.2), *then the sequence $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(z, F(T)) = \inf\{\|z-p\| : p \in F(T)\}$.*

Proof. If $\{x_n\}$ converges to a fixed point p of T, then

$$\liminf_{n\to\infty} \|x_n - p\| = 0$$

and, hence,

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, let $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. In view of (3.4) for all $p \in F(T)$, we have

$$\|x_{n+1} - p\| \le \|x_n - p\|,$$

which implies

$$\inf_{p\in F(T)} \|x_{n+1} - p\| \le \inf_{p\in F(T)} \|x_n - p\|.$$

Therefore, we get

$$d(x_{n+1}, F(T)) \le d(x_n, F(T)).$$

Hence, $\lim_{n\to\infty} d(x_n, F(T))$ exists and thus by assumption we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, for any $\varepsilon > 0$, there exists a positive integer k such that (for all $n \ge k$)

$$d(x_k, F(T)) < \frac{\varepsilon}{4}$$

or

$$\inf\{\|x_k - p\| : p \in F(T)\} < \frac{\varepsilon}{4},$$

so that there exists $p \in F(T)$ such that

$$\|x_k - p\| < \frac{\varepsilon}{2}.$$

Now, for all $m,\ n \ge k$, we have

$$\|x_m - x_n\| \le \|x_m - p\| + \|p - x_n\| \le 2\|x_k - p\| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence in K and converges to some x in K. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$ which implies that $d(x, F(T)) = 0$. In view of Theorem 2.2, $F(T)$ is closed and thus $x \in F(T)$.

Again, by setting $\beta_n = 0$, SP-iteration reduces to two step iteration scheme, i.e., (1.1) and we can draw the following corollary.

***Corollary* 3.2.** *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T\colon K \to K$ be a generalized nonexpansive mapping with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.1), *then the sequence $\{x_n\}$ converges to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Now, we prove the following strong convergence theorem using condition (I).

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly Banach space X and $T\colon K \to K$ be a generalized nonexpansive mapping which satisfies condition* (I) *with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.2), *then $\{x_n\}$ converges to a fixed point of T.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and let us take to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then as argued in Theorem 3.2, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Owing to condition (I) there exists a nondecreasing function f such that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

so that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since, f is a nondecreasing function and $f(0) = 0$, therefore, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now, in view of Theorem 3.2, we are through.

Theorem 3.3 is proved.

***Corollary* 3.3.** *Let K be a nonempty closed convex subset of a uniformly Banach space X and $T\colon K \to K$ be a generalized nonexpansive mapping which satisfies condition* (I) *with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ and $\{\gamma_n\}$ be a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. If $\{x_n\}$ is described as in* (1.1), *then $\{x_n\}$ converges to a fixed point of T.*

***Remark* 3.1.** Theorems 3.1 – 3.3 and Corollaries 3.1 – 3.3 generalize and extend the relevant results from [11, 12, 14].

4. Numerical example. To illustrate the genuineness of our results and the convergence behavior of SP-iteration scheme, we furnish following example of Suzuki's generalized nonexpansive mapping which is not a nonexpansive mapping.

***Example* 4.1.** Define a self mapping T on $[0,1]$ by

$$T(x) = \begin{cases} 1 - x, \quad \text{if} \quad x \in \left[0, \dfrac{1}{33}\right), \\[2mm] \dfrac{x+32}{33}, \quad \text{if} \quad x \in \left[\dfrac{1}{33}, 1\right]. \end{cases}$$

Here T is a Suzuki's generalized nonexpansive mapping, but T is not a nonexpansive.

Verification. Take $x = \dfrac{3}{100}$ and $y = \dfrac{1}{33}$, then

$$\|x - y\| = \left\| \frac{3}{100} - \frac{1}{33} \right\| = \frac{1}{3300}$$

and

$$\|Tx - Ty\| = \left\|1 - \frac{3}{100} - \frac{1057}{1089}\right\| = \frac{67}{108900} > \frac{1}{3300} = \|x - y\|.$$

Hence, T is not a nonexpansive mapping.

Claim: T is a Suzuki's generalized nonexpansive mapping.

Case I. Let $x \in \left[0, \frac{1}{33}\right)$. Then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|2x - 1\|$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{1 - 2x}{2} \leq y - x \implies y \geq \frac{1}{2}$ and, hence, $y \in \left[\frac{1}{2}, 1\right]$. Now, we have

$$\|Tx - Ty\| = \left\|1 - x - \frac{y + 32}{33}\right\| = \left\|\frac{y + 33x - 1}{33}\right\| < \frac{1}{33}$$

and

$$\|x - y\| = |x - y| > \frac{31}{66}.$$

Hence, $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Case II. Let $x \in \left[\frac{1}{33}, 1\right]$. Then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\left\|\frac{x + 32}{33} - x\right\| = \left\|\frac{16 - 16x}{33}\right\|$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{16 - 16x}{33} \leq |x - y|$, which gives two possibilities:

(A) Let $x < y$, then $\frac{16 - 16x}{33} \leq y - x$, i.e., $\frac{17x + 16}{33} \leq y \implies y \in \left[\frac{545}{1089}, 1\right] \subset \left[\frac{1}{33}, 1\right]$. So,

$$\|Tx - Ty\| = \frac{1}{33}\|x - y\| \leq \|x - y\|.$$

Hence, $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

(B) Let $x > y$, then $\frac{16 - 16x}{33} \leq x - y$, i.e., $y \leq \frac{49x - 16}{33}$, so $y \in [0, 1]$. Also, $\frac{33y + 16}{49} \leq x$ which gives $x \in \left[\frac{16}{49}, 1\right]$. For $x \in \left[\frac{16}{49}, 1\right]$ and $y \in \left[\frac{1}{33}, 1\right]$ case II(A) can be used. Therefore, consider $x \in \left[\frac{16}{49}, 1\right]$ and $y \in \left[0, \frac{1}{33}\right)$. Then

$$\|Tx - Ty\| = \left\|\frac{x + 32}{33} - 1 + y\right\| = \left\|\frac{x + 33y - 1}{33}\right\| < \frac{1}{33}$$

and

$$\|x - y\| = |x - y| > \frac{479}{1617} > \frac{1}{33}.$$

Hence, $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Thus, T is Suzuki's generalized nonexpansive mapping. It is easy to see that $x = 1$ is fixed point of T. If we choose $\gamma_n = \frac{1}{n + 68}$, $\beta_n = \sqrt{\frac{n}{n + 68}}$ and $\alpha_n = \sqrt{\frac{n + 67}{n + 68}}$, then we have the following table which illustrate that convergence behavior of Mann, Ishikawa and SP-iteration.

Thus, Table 1 and Fig. 1 show that SP-iteration (1.2) converges faster than the Mann and Ishikawa iteration even for Suzuki's generalized nonexpansive mapping as claimed in Theorem 3.7 of Phuengrattana and Suantai [12]. Also, rate of convergence of SP-iteration is better than Noor iteration and in our case mapping is discontinuous.

Table 1

Step	Mann	Ishikawa	Noor	SP
1	0.02	0.02	0.02	0.02
2	0.03391304347826	0.1344537965569	0.9668347163656	0.9681242789516
3	0.04729606625259	0.2763857687951	0.9989280684102	0.9990208877563
4	0.06030781438742	0.4206836352069	0.9999666029046	0.9999712693742
5	0.07296360139904	0.5531477763904	0.999998992079	0.9999991896535
6	0.08527790806373	0.6665975873401	0.9999999704407	0.9999999779427
7	0.09726443834299	0.7586939227074	0.9999999991557	0.9999999994189
8	0.1089361708573	0.8302090841836	0.9999999999765	0.9999999999851

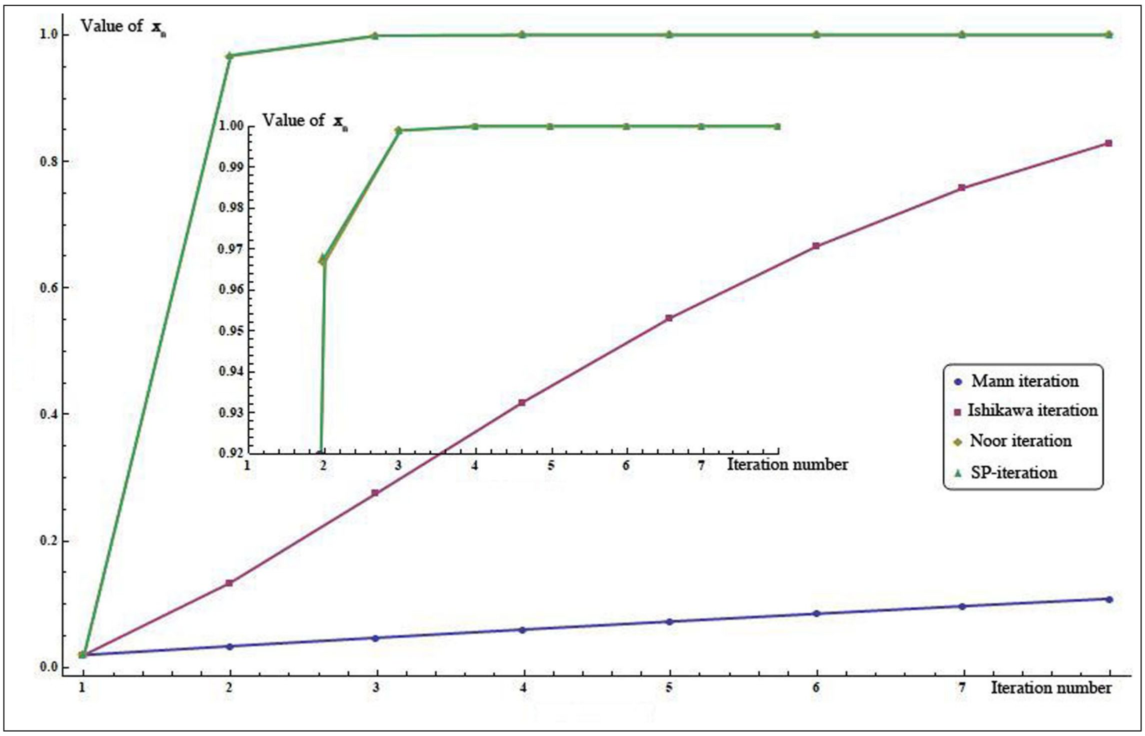

Fig. 1

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