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NATURAL BOUNDARY OF RANDOM DIRICHLET SERIES

НАТУРАЛЬНА ГРАНИЦЯ ВИПАДКОВОГО РЯДУ ДІРІХЛЕ

For the random Dirichlet series

$$\sum_{n=0}^{\infty} X_n(\omega)\, e^{-s\lambda_n} \qquad (s=\sigma+it\in\mathbb{C}, \quad 0=\lambda_0<\lambda_n\uparrow\infty),$$

whose coefficients are uniformly nondegenerate independent random variables, we provide some explicit conditions for the line of convergence to be its natural boundary a.s.

Для випадкового ряду Діріхле

$$\sum_{n=0}^{\infty} X_n(\omega)\, e^{-s\lambda_n} \quad (s=\sigma+it\in\mathbb{C}, \quad 0=\lambda_0<\lambda_n\uparrow\infty),$$

коефіцієнти якого — рівномірно невироджені незалежні випадкові змінні, введено деякі явні умови, за яких лінія збіжності є його натуральною границею майже напевно.

1. Introduction. Consider the random Dirichlet series

$$\sum_{n=0}^{\infty} X_n\, e^{-s\lambda_n}, \tag{1.1}$$

where the coefficients $\{X_n\}$ are complex-valued random variables, $0=\lambda_0<\lambda_1<$ $<\lambda_2<\ldots$ is an increasing sequence of real numbers such that $\lambda_n\to\infty$ and $s=\sigma+it$ is the complex variable. Let

$$\sigma_c(\omega)=\inf\left\{\sigma\in\mathbb{R}:\sum_{n=0}^{\infty} X_n(\omega)\, e^{-\sigma\lambda_n}\text{ converges}\right\}, \tag{1.2}$$

then it is well known that (1.1) converges at $s=\sigma+it$ if $\sigma>\sigma_c(\omega)$ and diverges at s if $\sigma<\sigma_c(\omega)$. Hence $\sigma_c(\omega)$ is called the abscissa of convergence and the line $s=\sigma_c(\omega)+it$, $t\in\mathbb{R}$, is called the convergence line of the random Dirichlet series (1.1). It is known that the sum $f_\omega(s)$ of (1.1) is analytic in the half-plane $\{s=\sigma+it\colon \sigma>\sigma_c(\omega)\}$.

When the coefficients $\{X_n,\, n\ge 0\}$ are independent, the Kolmogorov zero-one law implies that $\sigma_c(\omega)=\sigma_c$ a.s. for some constant $-\infty\le\sigma_c\le\infty$. It has been of importance to answer the following two natural questions: (a) how do we determine σ_c? (b) when is the line of convergence $\sigma=\sigma_c$ the natural boundary of the series (1.1) itself?

In the special case of $\lambda_n=n$ for all $n\ge 0$, (1.1) is reduced to a random Taylor series in $z=e^s$. The natural boundary problem for random Taylor series $F(z)=\sum_{n=0}^{\infty} X_n z^n$, $z\in\mathbb{C}$, has been investigated by several authors. We refer to Kahane [1] and the references therein for more information. A fundamental theorem in this area is due to Ryll-Nardzewski [2] who proved the following conjecture of Blackwell: If the coefficients $\{X_n,\, n\ge 0\}$ are independent, then there exists a deterministic Taylor series $f(z)=\sum_{n=0}^{\infty} a_n\, z^n$ such that

(i) the circle of convergence of $\sum_{n=0}^{\infty}(X_n-a_n)\, z^n$ is a.s. a natural boundary for $F(z)-f(z)$;

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(ii) the radius of convergence $r(F-f)$ of $\sum_{n=0}^{\infty}(X_n - a_n)\, z^n$ is maximal with respect to the choice of deterministic Taylor series;

(iii) if $g(z)$ is another deterministic Taylor series such that $r(F-g) = r(F-f)$ a.s. then the circle of convergence of $F(z) - g(z)$ is a.s. its natural boundary.

Kahane [1, p. 40, 41] gives a simplified alternative proof of Ryll-Nardzewski's result and, moreover, he proves that if the coefficients $\{X_n,\, n \geq 0\}$ are symmetric, then the circle of convergence is a natural boundary for $F(z)$. As a related result, we also mention that Holgate [3] has proved that Ryll-Nardzewski's result still holds for certain random Taylor series with dependent coefficients.

The question (b) for random Dirichlet series (1.1) with independent coefficients has been addressed by Kahane [1] (Section 6 of Chapter 4). In particular, he has extended the above result of Ryll-Nardzewski to random Dirichlet series and has shown that the line of convergence is the natural boundary for (1.1) provided the coefficients $\{X_n,\, n \geq 0\}$ are independent and symmetric random variables.

However, if the coefficients $\{X_n, n \geq 0\}$ are not assumed to be symmetric, Kahane's result does not give any information about the location of the natural boundary of random Dirichlet series (1.1). Even in the special case of random Taylor series, very little has been known. Sun [4] has considered the natural boundary problem for random Taylor series with nonsymmetric independent coefficients, but his condition is rather restrictive; see Remark 3.2.

The objective of this paper is to investigate the questions (a) and (b) above for random Dirichlet series (1.1) with independent and uniformly nondegenerate coefficients $\{X_n,\, n \geq 0\}$ (see Section 2 for definition). We will provide some explicit conditions for (1.1) to have a.s. the convergence line as its natural boundary. When applied to random Taylor series, our result gives a more convenient condition for the convergence circle to be the natural boundary a.s.

We remark that the existence of singular points on the convergence line for random Dirichlet series (1.1) whose coefficients $\{X_n,\, n \geq 0\}$ form a martingale difference sequence has been considered by Ding [5]. However, as far as we know, no results on the natural boundary of such series have been established. Similarly, the problem of finding the location of natural boundaries for the random Taylor series considered by Holgate [3] is also open. For general references on other aspects of deterministic and random Dirichlet series such as convergence, growth and value distributions, we refer to Mandelbrojt [6] and Yu [7].

2. Main result and its proof. Consider the random Dirichlet series

$$\sum_{n=0}^{\infty} X_n(\omega)\, e^{-s\lambda_n}, \quad s = \sigma + it, \;\; 0 = \lambda_0 < \lambda_n \uparrow \infty, \tag{2.1}$$

where $\{X_n,\, n \geq 0\}$ is a sequence of independent, complex-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\sup_{n\geq 0} \sup_{a\in\mathbb{C}} \mathbb{P}\{X_n = a\} < 1. \tag{2.2}$$

Any sequence $\{X_n,\, n \geq 0\}$ of random variables satisfying (2.2) is said to be *uniformly nondegenerate.* Clearly, the Rademacher sequence, Steinhaus sequence and, more generally, any sequence of continuous random variables are uniformly nondegenerate in the above sense. It is verified in Lemma 2.1 that the condition (2.2) holds if and only if there is a sequence $\{R_n\}$ of positive numbers such that

$$\sup_{n\geq 0}\sup_{a\in\mathbb{C}}\mathbb{P}\big\{|X_n-a|\leq R_n\big\}<1. \tag{2.3}$$

This property is connected with the Stein property of $\{X_n\}$; see [8].

Let $\sigma_c^{(1)}$ be the abscissa of convergence of the series (2.1), which is a constant or infinity a.s. by Kolmogorov's zero-one law. Denote by $\sigma_c^{(2)}$ the abscissa of convergence of the Dirichlet series

$$\sum_{n=0}^{\infty} R_n^2\, e^{-2s\lambda_n}. \tag{2.4}$$

The main result of the paper is as follows.

Theorem 2.1. *Let $\{X_n,\, n\geq 0\}$ be a sequence of independent complex-valued random variables satisfying the condition* (2.2). *Then*:

(i) $\sigma_c^{(1)}\geq\sigma_c^{(2)}$ *a.s.*;

(ii) *if $\sigma_c^{(1)}<\infty$, then with probability one for every rational number t the disc*

$$\Big\{s\in\mathbb{C}:\big|s-(\sigma_c^{(1)}+1+it)\big|\leq\sigma_c^{(1)}+1-\sigma_c^{(2)}\Big\}$$

contains at least one singularity of the series (2.1);

(iii) *if $\sigma_c^{(1)}=\sigma_c^{(2)}<\infty$, then with probability one the line $\operatorname{Re}s=\sigma_c^{(1)}$ is the natural boundary of the series* (2.1).

In Section 3, we will give some sufficient conditions on the sequences $\{X_n,\, n\geq 0\}$ and $\{\lambda_n,\, n\geq 0\}$ so that $\sigma_c^{(1)}=\sigma_c^{(2)}$ a.s.

For the proof of Theorem 2.1, we need the following two lemmas.

Lemma 2.1. *The condition* (2.2) *holds if and only if there is a sequence $\{R_n\}$ of positive numbers such that* (2.3) *holds.*

Proof. The condition (2.3) implies clearly the condition (2.2). The reverse follows immediately from the following proposition[1]: If a random variable X has the property that for some constant $\varepsilon\in(0,1)$,

$$\sup_{a\in\mathbb{C}}\mathbb{P}\big\{X=a\big\}\leq 1-\varepsilon, \tag{2.5}$$

then there exists a constant $R>0$ such that

$$\sup_{a\in\mathbb{C}}\mathbb{P}\big\{|X-a|\leq R\big\}\leq 1-\frac{\varepsilon}{2}. \tag{2.6}$$

Assume the above proposition does not hold. Then for every integer $n\geq 1$, there exists $a_n\in\mathbb{C}$ such that

$$\mathbb{P}\Big\{|X-a_n|\leq\frac{1}{n}\Big\}>1-\frac{\varepsilon}{2}. \tag{2.7}$$

We may choose $\{a_n,\, n\geq 1\}$ such that either $\lim\limits_{n\to\infty}|a_n|=\infty$ or $\lim\limits_{n\to\infty}a_n=a_\infty\in\mathbb{C}$. In the first case, (2.7) implies that for every positive number C, $\mathbb{P}\big\{|X|\geq C\big\}\geq 1-\dfrac{\varepsilon}{2}$, which is impossible. In the second case, we note that for an arbitrary $\delta>0$, we can choose $n>\dfrac{2}{\delta}$ such that $|a_\infty-a_n|<\dfrac{\delta}{2}$. Hence by (2.7) we derive

[1] The proof of (2.6) is suggested by the referee. It is simpler than our original proof.

$$\mathbb{P}\{|X-a_\infty|\le\delta\}\ge\mathbb{P}\Big\{|X-a_n|\le\frac{1}{n}\Big\}>1-\frac{\varepsilon}{2}.$$

Letting $\delta\to 0+$, we obtain that $\mathbb{P}\{X=a_\infty\}\ge 1-\dfrac{\varepsilon}{2}$, which contradicts (2.5). The proof of Lemma 2.1 is completed.

Lemma 2.2 below is a consequence of Lemma 2.1 and Corollary 2 of Burkholder [8]. It extends Theorem 1 in Chapter 2 of Kahane [1].

Lemma 2.2. *Let $\{X_n,\, n\ge 0\}$ be a sequence of independent complex-valued random variables satisfying the condition* (2.2). *Suppose $\{b_{mn},\, m,n\ge 0\}$ is a double sequence of complex numbers such that for each $m\ge 0$ the series $\sum_{n=0}^{\infty} b_{mn}X_n$ converges a.s. to a random variable Y_m. If the sequence $\{Y_m,\, m\ge 0\}$ is convergent a.s., then there is a sequence $\{b_n,\, n\ge 0\}$ of complex numbers such that*

$$\lim_{m\to\infty}\sum_{n=0}^{\infty}|b_{mn}-b_n|^2 R_n^2=0 \quad \textit{and} \quad \sum_{n=0}^{\infty}|b_n|^2R_n^2<\infty. \tag{2.8}$$

Proof. Put $Z_n=X_n/R_n$. Then Lemma 2.1 implies that

$$\sup_{n\ge 0}\sup_{a\in\mathbb{C}}\mathbb{P}\left\{|Z_n-a|<1\right\}<1.$$

Now by applying Corollary 2 of Burkholder [8] to the series $\sum_{n=0}^{\infty}(b_{mn}R_n)Z_n$ we complete the proof.

***Proof of Theorem* 2.1.** In order to prove (i), we assume $\sigma_c^{(1)}<\infty$; otherwise there is nothing to prove. For any $\sigma>\sigma_c^{(1)}$ the random Dirichlet series (2.1) converges a.s. for $s=\sigma$. By applying Lemma 2.2 to $b_{mn}=e^{-\sigma\lambda_n}$, we see that the series (2.4) converges at $s=\sigma$. Hence $\sigma_c^{(2)}\le\sigma$ so that (i) holds.

Next we prove (ii). Let $f_\omega(s)$ be the sum function of the random Dirichlet series (2.1) for $\operatorname{Re}s>\sigma_c^{(1)}$. For $t\in\mathbb{R}$, put

$$R(t,\omega)=\left[\limsup_{n\to\infty}\left|\frac{f_\omega^{(n)}(\sigma_c^{(1)}+1+it)}{n!}\right|^{\frac{1}{n}}\right]^{-1}, \tag{2.9}$$

where $f^{(n)}$ denotes the n-th derivative of f. Then $R(t,\omega)$ is the radius of convergence of the Taylor expansion of $f_\omega(s)$ around $s_0=\sigma_c^{(1)}+1+it$. Suppose (ii) is not true, then there is a rational number t_0 such that, with positive probability, the disc

$$\Big\{s\in\mathbb{C}:\ \big|s-\big(\sigma_c^{(1)}+1+it_0\big)\big|\le\sigma_c^{(1)}+1-\sigma_c^{(2)}\Big\}$$

does not contain any singularities of the series (2.1). This implies that

$$\mathbb{P}\Big\{R(t_0,\omega)>\sigma_c^{(1)}+1-\sigma_c^{(2)}\Big\}>0. \tag{2.10}$$

However, by Kolmogorov's zero-one law and (2.10), $R(t_0,\omega)$ is a.s. a constant $R(t_0)$ so that

$$R(t_0,\omega)=R(t_0)>\sigma_c^{(1)}+1-\sigma_c^{(2)} \quad \text{a.s.} \tag{2.11}$$

Now we choose $\sigma_0<\sigma_c^{(2)}$ such that $R(t_0)>\sigma_c^{(1)}+1-\sigma_0$ and let

$$\Omega_0=\Big\{\omega: R(t_0,\omega)>\sigma_c^{(1)}+1-\sigma_0\Big\}.$$

Then (2.11) implies $\mathbb{P}\{\Omega_0\}=1$.

Put $\zeta_0 = \sigma_c^{(1)}+1+it_0$. Since $\sigma_0 < \sigma_c^{(2)}$ we can choose s_1 $\left(\text{e.g. } s_1 = \dfrac{\sigma_0+\sigma_c^{(2)}}{2}+it_0\right)$ such that

$$|s_1-\zeta_0| < \sigma_c^{(1)}+1-\sigma_0, \quad \sigma_0 < \operatorname{Re} s_1 = \sigma_1 < \sigma_c^{(2)}. \tag{2.12}$$

For each $\omega \in \Omega_0$, we consider the Taylor series

$$Y(\omega) = \sum_{n=0}^{\infty} \frac{f_\omega^{(n)}(\zeta_0)}{n!}(s_1-\zeta_0)^n. \tag{2.13}$$

It follows from (2.11) and $R(t_0,\omega) > \sigma_c^{(1)}+1-\sigma_0 > |s_1-\zeta_0|$ that (2.13) is convergent for every $\omega \in \Omega_0$. Now put

$$b_{mn} = \sum_{k=0}^{m} \frac{\phi_n^{(k)}(\zeta_0)}{k!}\,(s_1-\zeta_0)^k,$$

where $\phi_n(s) = e^{-s\lambda_n}$. Then for every $m \geq 0$, the series $\sum_{n=0}^{\infty} b_{mn}X_n$ converges a.s. to the series

$$\sum_{k=0}^{m} \frac{(s_1-\zeta_0)^k}{k!} \sum_{n=0}^{\infty} \phi_n^{(k)}(\zeta_0)X_n(\omega) = \sum_{k=0}^{m} \frac{(s_1-\zeta_0)^k}{k!} f_\omega^{(k)}(\zeta_0) \mathrel{\hat=} Y_m(\omega).$$

Moreover, (2.13) implies that the sequence $\{Y_m(\omega), m \geq 0\}$ converges a.s. to the random variable $Y(\omega)$. Therefore it follows from Lemma 2.2 and $\lim_{m\to\infty} b_{mn} = \phi_n(s_1) = e^{-s_1\lambda_n}$ that

$$\sum_{n=0}^{\infty} R_n^2\, e^{-2\lambda_n \operatorname{Re} s_1} < \infty$$

so that the series (2.4) converges for $\operatorname{Re} s_1 < \sigma_c^{(2)}$ (see (2.12)). This is a contradiction, which proves (ii).

Finally we prove (iii). Since the series (2.1) converges a.s. in the half plane: $\operatorname{Re} s > \sigma_c^{(1)}$, we have $\inf_{t\in\mathbb{Q}} R(t,\omega) \geq 1$ a.s., where $\mathbb{Q}$ is the set of rational numbers. On the other hand, by (ii) and the hypothesis that $\sigma_c^{(1)} = \sigma_c^{(2)}$, we have $\sup_{t\in\mathbb{Q}} R(t,\omega) \leq 1$ a.s. Therefore $R(t,\omega) = 1$ a.s. for all $t \in \mathbb{Q}$, that is, $\sigma_c^{(1)} + it$ is a singularity of the function $f_\omega(s)$ with probability one. Therefore, (iii) follows readily from the facts that $\mathbb{Q}$ is dense in $\mathbb{R}$ and the set of the singularities of an analytic function is closed. This finishes the proof of Theorem 2.1.

3. Three corollaries. In this section, we apply Theorem 2.1 to random Dirichlet series and random Taylor series with different conditions on their coefficients and to derive more explicit results on the line of convergence and natural boundary.

3.1. First we consider the random Dirichlet series (2.1) with coefficients $\{X_n\}$ which are identically distributed, independent and uniformly nondegenerate random variables. We assume further that either

$$\mathbb{E}\{|X_0|^\beta\} < \infty \quad \text{for some} \quad \beta \in (0,1) \tag{3.1}$$

or

$$\mathbb{E}(X_0) = 0 \quad \text{and} \quad \mathbb{E}\{|X_0|^\beta\} < \infty \quad \text{for some} \quad \beta \in [1,2]. \tag{3.2}$$

Theorem 3.1. *Under the above assumptions, the following statements hold*:

(i) $\dfrac{D}{2} \le \sigma_c^{(1)} \le \dfrac{D}{\beta}$, *where* $D = \limsup_{n\to\infty} \dfrac{\log n}{\lambda_n}$;

(ii) *if* $D = 0$, *the line* $\operatorname{Re} s = 0$ *is a.s. the natural boundary of the series* (2.1).

***Remark* 3.1.** Theorem 3.1 applies directly to random Dirichlet series (2.1) whose coefficients are i.i.d. stable random variables.

Proof. (i) Since the random variables $\{X_n,\, n \ge 0\}$ are identically distributed and uniformly nondegenerate, it follows from Lemma 2.1 that (2.3) is valid for $R_n = R$, where $R > 0$ is some positive constant. Hence the series (2.4) reduces to the series $\sum_{n=0}^{\infty} R^2\, e^{-2s\lambda_n}$. Clearly, its abscissa $\sigma_c^{(2)}$ of convergence is at least 0. When $D > 0$, then it is known (see, e.g., [6, 7]) that $\sigma_c^{(2)} = D/2$. Hence by (i) of Theorem 2.1 we always have $D/2 \le \sigma_c^{(1)}$.

To prove $\sigma_c^{(1)} \le D/\beta$ we assume without loss of generality that $D < \infty$. Note that for any $\varepsilon > 0$, the series

$$\sum_{n=0}^{\infty} \mathbb{E}\Big\{\Big|X_n\, e^{-\frac{(D+\varepsilon)\lambda_n}{\beta}}\Big|^{\beta}\Big\} = \sum_{n=0}^{\infty} \mathbb{E}\big\{|X_n|^{\beta}\big\}\, e^{-(D+\epsilon)\lambda_n},$$

is obviously convergent. It follows from (3.1), (3.2) and Corollary 3 in [9, p. 114] that the series (2.1) converges a.s. for $s = (D+\epsilon)/\beta$. Since $\varepsilon > 0$ is arbitrary, this implies $\sigma_c^{(1)} \le \dfrac{D}{\beta}$, as desired.

When $D = 0$, from the above we have $\sigma_c^{(1)} = \sigma_c^{(2)} = 0$. Hence the statement (ii) follows directly from (i) and Theorem 2.1 (iii).

3.2. Now we consider the random Dirichlet series (2.1) where $\{X_n\}$ is a sequence of independent random variables satisfying the following condition: there exists a constant $\alpha > 0$ such that

$$\mathbb{E}\{X_n\} = 0 \quad \text{and} \quad 0 < \alpha\, \mathbb{E}^{1/2}\big\{|X_n|^2\big\} \le \mathbb{E}\{|X_n|\} \quad (\forall n \ge 0). \tag{3.3}$$

This condition was introduced by Marcinkiewicz and Zygmund [10] to establish maximal inequalities for the weighted partial sums of independent random variables $\{X_n,\, n \ge 0\}$ with mean 0 and variance 1 and to study the convergence of the random series $\sum_{n=0}^{\infty} a_n X_n$. A conditional version of (3.3) was formulated by Gundy [11] (who called it Condition (MZ)*) for extending the results of Marcinkiewicz and Zygmund [10] to sequences of martingale differences. See Stout [12] and the references therein for further information.

The following lemma gives a sufficient condition for (3.3) to hold and it follows from a remark of Marcinkiewicz and Zygmund [10, p. 70].

Lemma 3.1. *Let* $\{X_n, n \ge 0\}$ *be a sequence of independent random variables with mean* 0. *If there exist constants* $0 < \eta < 1$ *and* $K > 0$ *such that*

$$\mathbb{E}\big[|X_n|^2 \mathbb{1}_{A_n}\big] \le \eta \quad \forall n \ge 0, \tag{3.4}$$

where $A_n = \{|X_n| \ge K\, \mathbb{E}^{1/2}(|X_n|^2)\}$. *Then* (3.3) *holds with* $\alpha = (1-\eta)/K$.

Let $\sigma_c^{(3)}$ be the abscissa of convergence of the Dirichlet series

$$\sum_{n=0}^{\infty} \mathbb{E}^2\{|X_n|\}\, e^{-2s\lambda_n}. \tag{3.5}$$

Theorem 3.2. *Let* $\{X_n,\, n \geq 0\}$ *be a sequence of independent random variables satisfying the condition* (3.3). *Then the following statements hold*:

(i) $\sigma_c^{(1)} = \sigma_c^{(3)}$ *a.s.*;

(ii) *if* $\sigma_c^{(3)}$ *is finite, then the line* $\operatorname{Re} s = \sigma_c^{(3)}$ *is a.s. the natural boundary of the series* (2.1).

Proof. For any $\sigma > \sigma_c^{(3)}$, the series $\sum_{n=0}^{\infty} \mathbb{E}^2\{|X_n|\}\, e^{-2\sigma\lambda_n}$ is convergent. It follows from (3.3) that the series

$$\sum_{n=0}^{\infty} \mathbb{E}\big\{|X_n|^2\big\}\, e^{-2\sigma\lambda_n}$$

also converges. Hence the random Dirichlet series (2.1) converges a.s. at $s = \sigma$. Consequently, we have $\sigma_c^{(1)} \leq \sigma_c^{(3)}$.

To prove the reverse inequality, by (i) and (iii) of Theorem 2.1 it is sufficient to prove

$$\sup_{n\geq 0}\sup_{a\in\mathbb{C}} \mathbb{P}\left\{|X_n - a| \leq \frac{\alpha}{4}\, \mathbb{E}^{1/2}\big\{|X_n|^2\big\}\right\} < 1. \tag{3.6}$$

That is, $\{X_n,\, n \geq 0\}$ satisfies (2.3) with $R_n = \dfrac{\alpha}{4}\, \mathbb{E}^{1/2}\big\{|X_n|^2\big\}$. In fact, the inequality (3.6) is a consequence of the following claim: If X is a random variable such that

$$\mathbb{E}\{X\} = 0 \quad \text{and} \quad 0 < \alpha\, \mathbb{E}^{1/2}\{|X|^2\} \leq \mathbb{E}\{|X|\}, \tag{3.7}$$

then

$$\sup_{a\in\mathbb{C}} \mathbb{P}\left\{|X - a| \leq \frac{\alpha}{4}\, \mathbb{E}^{1/2}\big\{|X|^2\big\}\right\} \leq 1 - \frac{1}{8}\Big(\frac{\alpha}{2+\alpha}\Big)^2. \tag{3.8}$$

Now let us prove the above claim. Since $\mathbb{E}\{X\} = 0$, Jensen's inequality and the triangle inequality imply that for all $a \in \mathbb{C}$,

$$\mathbb{E}\{|X - a|\} \geq |a| \quad \text{and} \quad \mathbb{E}\{|X - a|\} \geq \mathbb{E}\{|X|\} - |a|. \tag{3.9}$$

Hence by (3.9) and the easily verifiable inequality

$$\inf_{y\geq 0} \max\{y, x - y\} \geq \frac{x}{2} \quad (\forall\, x > 0),$$

we have

$$\mathbb{E}\{|X - a|\} \geq \max\big\{|a|,\, \mathbb{E}\{|X|\} - |a|\big\} \geq \frac{1}{2}\, \mathbb{E}\{|X|\}. \tag{3.10}$$

It follows from (3.10) and the Paley – Zygmund inequality [13] (cf. [1, p. 8]) that for any $\lambda \in (0, 1)$,

$$\mathbb{P}^{1/2}\left\{|X - a| > \frac{\lambda}{2}\, \mathbb{E}\{|X|\}\right\} \geq \mathbb{P}^{1/2}\big\{|X - a| > \lambda\, \mathbb{E}\{|X - a|\}\big\} \geq$$

$$\geq (1 - \lambda)\, \frac{\mathbb{E}\{|X - a|\}}{\mathbb{E}^{1/2}\big\{|X - a|^2\big\}} \geq \frac{1 - \lambda}{\sqrt{2}}\, \frac{\max\{|a|,\, \mathbb{E}\{|X|\} - |a|\}}{\mathbb{E}^{1/2}\big\{|X|^2\big\} + |a|}.$$

By taking $\lambda = 1/2$ and then using the equality

$$\inf_{y\geq 0}\frac{\max\{y,\,x_1-y\}}{x_2+y}\geq \inf_{y\geq 0}\max\left\{\frac{y}{x_2+y},\;\frac{x_1-y}{x_2+y}\right\}=\left.\frac{y}{x_2+y}\right|_{y=\frac{x_1}{2}}=$$

$$=\frac{x_1}{2x_2+x_1}\quad(\forall x_1>0,\,x_2>0),$$

we have that for all $a\in\mathbb{C}$,

$$\mathbb{P}^{1/2}\Big\{|X-a|>\frac{1}{4}\,\mathbb{E}\,\{|X|\}\Big\}\geq\frac{1}{2\sqrt{2}}\,\frac{\mathbb{E}\,\{|X|\}}{2\mathbb{E}^{1/2}\big\{|X|^2\big\}+\mathbb{E}\,\{|X|\}}. \tag{3.11}$$

It is easy to see that (3.8) follows from (3.7) and (3.11). Therefore the proof of Theorem 3.2 is finished.

3.3. Finally, we apply Theorem 3.2 to the random Taylor series

$$\sum_{n=0}^{\infty}X_n(\omega)\,z^n, \tag{3.12}$$

where the sequence $\{X_n\}$ satisfies the conditions imposed in Theorem 3.2. Consider the auxiliary series

$$\sum_{n=0}^{\infty}\mathbb{E}^2\{X_n\}\,z^{2n}. \tag{3.13}$$

Since the series (3.12) and (3.13) can be regarded as special cases of the series (2.1) and (3.5), respectively, we have the following corollary of Theorem 3.2.

Theorem 3.3. *Under the conditions of Theorem* 3.2,

(i) *the radius of convergence of the series* (3.12) *is a.s.*

$$r=\frac{1}{\limsup\limits_{n\to\infty}\big(\mathbb{E}\{|X_n|\}\big)^{1/n}};$$

(ii) *if the radius r is finite and positive, then the circle $|z|=r$ is a.s. the natural boundary of the random Taylor series* (3.12).

Remark **3.2.** Theorem 3.3 gives more concise information about the natural boundary of random Taylor series than that of Sun [4], who has proved that the natural boundary of the random Taylor series $\sum_{n=0}^{\infty}Y_n(\omega)\,z^n$ is the circle

$$|z|=\frac{1}{\limsup\limits_{n\to\infty}\Big(\sqrt{\mathbb{E}\{|Y_n|^2\}}\Big)^{1/n}},$$

provided $\{Y_n\}$ is a sequence of independent random variables with mean 0 and finite second moments satisfying the following condition: there exists a constant $\delta>0$ such that for every event A with $\mathbb{P}\{A\}<\delta$,

$$\inf_{n\geq 0}\int\limits_{A^c}\frac{|Y_n|^2}{\mathbb{E}\{|Y_n|^2\}}\,d\mathbb{P}>\frac{1}{2}, \tag{3.14}$$

where A^c is the complement of A. Condition (3.14) is equivalent to assuming $\mathbb{E}\big(|Y_n|^2\mathbb{1}_A\big)<\mathbb{E}\big(|Y_n|^2\mathbb{1}_{A^c}\big)$ holds *uniformly* for all events A with $\mathbb{P}\{A\}<\delta$ and all integers $n\geq 0$.

We now show that the condition (3.14) implies (3.3). Without loss of generality, we will assume $\mathbb{E}(|Y_n|^2) = 1$ for all $n \geq 0$. Then Chebyshev's inequality implies that for any $\delta \in (0,1)$, there exists a constant $K > 0$ such that

$$\sup_{n\geq 0} \mathbb{P}\big\{|Y_n| \geq K\big\} < \delta. \tag{3.15}$$

Let $A_n = \{|Y_n| \geq K\}$. It follows from (3.14) and (3.15) that $\mathbb{E}(|Y_n|^2 \mathbb{1}_{A_n}) \leq 1/2$ for all integers $n \geq 0$. Hence by Lemma 3.1 we see that (3.3) holds.

The following is an example of independent random variables $\{Y_n, n \geq 0\}$ that verifies the condition (3.3), but not the condition (3.14).

***Example* 3.1.** Let $\{A_n, n \geq 0\}$ be a sequence of events with $\mathbb{P}(A_n) = (1+n)^{-2}$. We define a sequence of independent real-valued random variables $\{Y_n,\ n \geq 0\}$ with mean 0 such that $|Y_n| = 1 + n$ on A_n^c and $|Y_n| = (1+n)^2$ on A_n. Then we have $\mathbb{E}\big(|Y_n|^2 \mathbb{1}_{A_n}\big) = (1+n)^2$ and $\mathbb{E}\big(|Y_n|^2 \mathbb{1}_{A_n^c}\big) = (1+n)^2(1-(1+n)^{-2})$, so (3.14) does not hold. On the other hand, it is easy to see that the condition (3.3) is verified with $\alpha = (\sqrt{2})^{-1}$.

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