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REGULARITY OF NONLINEAR FLOWS ON NONCOMPACT RIEMANNIAN MANIFOLDS: DIFFERENTIAL GEOMETRY VERSUS STOCHASTIC GEOMETRY OR WHAT KIND OF VARIATIONS IS NATURAL?*

РЕГУЛЯРНІСТЬ НЕЛІНІЙНИХ ПОТОКІВ НА НЕКОМПАКТНИХ РІМАНОВИХ МНОГОВИДАХ: ДИФЕРЕНЦІАЛЬНА ГЕОМЕТРІЯ ПРОТИ СТОХАСТИЧНОЇ АБО ЯКІ ВАРІАЦІЇ Є ПРИРОДНИМИ?

We demonstrate that the geometrically correct study of the regularity of nonlinear differential flows on manifolds and related parabolic equations requires the introduction of a new type of variations with respect to the initial data. These variations are defined via a certain generalization of a covariant Riemannian derivative to the case of diffeomorphisms.

We find how the curvature appears in the structure of higher-order variational equations and determine a family of a priori nonlinear estimates of any-order regularity. Using the relation between differential equations on manifolds and semigroups, we investigate C^∞-regular properties of solutions of the Cauchy parabolic problems with coefficients growing on infinity.

The obtained conditions of regularity generalize the classical coercitivity and dissipativity conditions to the case of manifold and relate in unified way the behaviour of diffusion and drift coefficients with the geometric properties of the manifold without the traditional separation of curvature.

Показано, що геометрично коректне дослідження регулярності нелінійних диференціальних потоків на багатовидах та асоційованих параболічних рівнянь вимагає введення нового типу варіацій за початковими умовами. Ці варіації означені за допомогою певного узагальнення коваріантної похідної Рімана на випадок дифеоморфізмів.

Встановлено, яким чином кривина виникає в варіаційних рівняннях високого порядку, і одержано сім'ю апріорних нелінійних оцінок на регулярність довільного порядку. Використовуючи зв'язок між диференціальними рівняннями на багатовидах і напівгрупами, досліджено C^∞-гладкі властивості розв'язків параболічних задач Коші зі зростаючими на нескінченності коефіцієнтами.

Отримані умови регулярності узагальнюють класичні умови коерцитивності та дисипативності на випадок багатовиду і пов'язують поведінку коефіцієнтів дифузії та зсуву з геометричними властивостями багатовиду, без традиційного відокремлення кривини.

1. Introduction. Up-to-date there are formed a lot of qualitively different approaches to the construction and study of differential equations on manifolds with random terms. In transfer from the linear space $\mathbb{R}^d$ to manifold the main attention was to make consistent the geometrical structures of manifold with the purely stochastic effects, influenced by second order differentials, that arise in Itô formula for coordinate changes.

The already known approaches include, in particular:

purely stochastic, based on the definition of diffusion in a consistent with geometry way by implementation of Stratonovich integrals [1 – 3] or more complicate description of diffusion via Itô equations in local coordinates [4, 5]; in the second case arise special Itô bundles of nontensorial fields, related with diffusion coefficients; to make the picture consistent, a special attention should be devoted to the normal charts, generated by exponential mappings,

more geometric, related, for example,

with the raise of diffusion from manifold M to the orthoframe bundle $O(M)$ over it; the direct advantage of such approach is related with the globally existing horizontal vector fields and possibility to write diffusions with Laplacian generator for manifolds with non-zero Euler number; as it has been becoming clear for years, these geometric

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ideas influenced much the construction of advanced analysis on Wiener space and raise of profound analytical questions of stochastics in the Malliavin calculus [6, 7],

or with the consideration of manifold as embedded into $\mathbb{R}^d$ of higher dimension, e.g. [8, 9]; with interpretation of Itô differential and diffusion equations as defined on the bundle of second-order differential operators [10, 11]; putting forward Itô developments of equations via parallel transitions of orthoframes [12]; with more stress on properties of associated transitional probabilities [13], etc.

One can continue this list, by speaking about peculiarities, related with other infinite-dimensional models [9, 14].

The procedure of correct correspondence between geometry and stochastics was successful in all cases. However, further question of consistency with the problematic of **differential geometry**, namely

$$\text{how the geometrically invariant differentials are constructed from invariant objects,} \tag{1.1}$$

remained in shadow. Of course, one may try to consider the traditional derivatives in directions of vector fields or more advanced covariant and stochastic derivatives, e.g. [4 – 7], but as we will soon see, they all miss an important property of geometric invariance with respect to the diffusion process.

Other approach to define the derivatives via stochastic parallel transport ${}^{\gamma}//_{x}^{y}$ of corresponding derivatives or via Cartan orthoframes, e.g. [6, 7, 12, 13], does not provide a transparent definition of geometrically invariant derivatives. Such transport essentially depends on particular path of process y and therefore on the coefficients of equation. But the correct definition of higher-order derivatives should be quite general and does not depend on particular equation. Such definition is possible.

Let us turn to the corresponding constructions. Consider the following situation. Suppose that some process y_t (of diffusion or any other nature) enters coordinate vicinity $U \subset M$ of manifold with coordinate functions $\varphi = (\varphi^i)_{i=1}^{\dim M}$, $\varphi\colon U \to \mathbb{R}^{\mathrm{dimM}}$, so that one can speak about the coordinates of process $y_t^i = (\varphi^i) \circ y_t$ when it stays in this vicinity.

Now let $\mathcal{D}$ be some first order differentiation operation, correctly defined on process y_t. What kind of differentiation it could be is not essential now, the principal moment is that the first order differentiation must obey chain rule

$$\mathcal{D}(f \circ y) = (f' \circ y)\, \mathcal{D}y.$$

In particular, because the local coordinate changes $y^{i'} = \varphi^{i'}(y_t) = (\varphi^{i'} \circ \varphi^{inv})(y^i)_{i=1}^{\mathrm{dimM}}$ represent a special case of locally defined functions, one has rule

$$\mathcal{D}y^{m'} = \frac{\partial y^{m'}}{\partial y^m} \mathcal{D}y^m.$$

Therefore, though process y_t does not determine some coordinate system, like local coordinate mappings φ, φ' do,

the expression $\mathcal{D}y$ becomes a **vector field** with respect to the “coordinate”
changes $(y) \to (y')$ of ”coordinate” variable y.

By classical arguments of differential geometry, related with the standard construction of covariant derivatives, the only way to give a correct definition of the higher-order derivatives $(\widetilde{\mathcal{D}})^i y$ should use additional terms with connection $\Gamma(y_t)$.

The correct recurrent definition of the invariant higher-order variations will be

$$\widetilde{\mathcal{D}} y^m = \mathcal{D} y^m, \qquad \widetilde{\mathcal{D}}[\,(\widetilde{\mathcal{D}})^i y^m\,] = \mathcal{D}[\,(\widetilde{\mathcal{D}})^i y\,] + \Gamma^{\;m}_{p\;q}(y)\,[\,(\widetilde{\mathcal{D}})^i y^p]\mathcal{D} y^q.$$

Like in the classical differential geometry, additional terms with $\Gamma(y)$ in definition of higher-order derivatives $\widetilde{\mathcal{D}}^n$ guarantee the preservance of vector transformation law with respect to the $(y) \to (y')$ coordinate transformations:

$$(\widetilde{\mathcal{D}})^n y^{m'} = \frac{\partial y^{m'}}{\partial y^m}\,(\widetilde{\mathcal{D}})^n y^m \quad \forall\, n \geq 1.$$

In view of problem (1.1), such invariance with respect to the changes of local coordinates $(y) \to (y')$ in vicinity, where comes process y, represents a new and purely geometric requirement of first priority. From another side, the relation between differentials in time of this change $(y) \to (y')$, given by Itô formula, reflects the behaviour on time coordinate and is *secondarily*. It is purely stochastic and related with the nonvoidness of quadratic variation processes.

Finally, let us remark that the above construction and the way to introduce the new type derivatives is independent on particular approach we choose to define the diffusion on manifold, actually

it works for any differential equations (higher-order, etc.) on manifolds,

because, by consideration above, symbol $y \in M$ must have values in manifold, but nothing more. This is especially underlined in the article by **the use of notation** y instead of traditional Greek letters, like ξ, η, ζ, for stochastic processes.

In this article we discuss more concrete case of general diffusion process y_t^x on noncompact manifold. We investigate its regular dependence on initial data x and provide the geometrically correct construction of higher-order variations.

Consider the first order variation $\dfrac{\partial (y_t^x)^m}{\partial x^k}$, that represents a vector field on index m for $(y) \to (y')$ "coordinate" transformations and covector field on index k for $(x) \to (x')$ coordinate changes. From arguments above we can immediately conclude that the definition of *geometrically invariant higher-order variations* must include terms with $\Gamma(x)$ and $\Gamma(y)$ to guarantee the preservance of tensorial character on both "image" $(y) \to (y')$ and "domain" $(x) \to (x')$ coordinate changes of mapping $x \to y_t^x$.

Definition 1.1. *Higher-order variations* $\mathbb{\nabla}_\gamma^x y_t^x$, $\gamma = \{k_1, \ldots, k_n\}$, *of process* y_t^x *are defined by recurrent relations*

$$\begin{gathered} \mathbb{\nabla}_k^x y^m = \frac{\partial (y_t^x)^m}{\partial x^k}, \\ \mathbb{\nabla}_k^x(\mathbb{\nabla}_\gamma^x y^m) = \nabla_k^x(\mathbb{\nabla}_\gamma^x y^m) + \Gamma^{\;m}_{p\;q}(y_t^x)\mathbb{\nabla}_\gamma^x y^p \frac{\partial y^q}{\partial x^k} \end{gathered} \tag{1.2}$$

where $\nabla_k^x(\mathbb{\nabla}_\gamma^x y^m)$ *represents a classical covariant derivative*

$$\nabla_k^x(\mathbb{\nabla}_\gamma^x y^m) = \partial_k^x(\mathbb{\nabla}_\gamma^x y^m) - \sum_{j \in \gamma} \Gamma^{\;h}_{k\;j}(x)\mathbb{\nabla}^x_{\gamma|_{j=h}} y^m$$

and $\mathbb{\nabla}^x_{\gamma|_{j=h}} y^m$ *means substitution of index* j *by* h.

From the point of view of classical Riemannian geometry such definition of the higher-order invariant variation of y with terms $\Gamma(x)$ and $\Gamma(y)$ provides generalization of the classical covariant derivative. Unlike all already existing torsion, polynomial connection and other generalizations of variation, defined primarily at point x, it depends not only on initial point of differentiation x, but also on behaviour of process at point y.

***Remark* 1.1.** Due to the invariance of higher-order variations $\mathbb{V}^x_{i_1,\ldots,i_s} y^m$ with respect to the $(y) \to (y')$ and $(x) \to (x')$ coordinate changes we can introduce the **invariant norm of the higher-order variation**

$$\|(\mathbb{V}^x)^j y_t^x\|^2 = g_{mn}(y_t^x) \prod_{s=1}^{j} g^{i_s k_s}(x) \mathbb{V}^x_{i_1,\ldots,i_s} y^m \mathbb{V}^x_{k_1,\ldots,k_s} y^n. \tag{1.3}$$

After that the regularity problem becomes well-posed geometrically and the regular behaviour of process y_t^x with respect to the initial data can be expressed in terms of some estimates on higher-order variations $\mathbb{V}^x_{i_1,\ldots,i_s} y^m$.

In this article we also demonstrate how the geometry of manifold and its curvature is reflected in the structure of equations on new type variations. We also find conditions on existence and uniqueness of variational processes, which give a natural generalization of coercitivity and dissipativity conditions to the manifold case and can be used even for noncompact and infinite-dimensional manifolds. In particular, they relate the behaviour of geometry and diffusion in a unified way, without traditional separation of curvature.

We also discuss the consequences of regular dependence of diffusion on initial data for the smooth properties of semigroups. The use of nonlinear symmetries of variational equations and a set of associated nonlinear estimates permits us to study the regular properties of semigroup in the case of globally non-Lipschitz behaviour of nonlinear coefficients on infinity.

The development of advanced constructions of Malliavin calculus to the new type stochastic derivatives $\widetilde{\mathcal{D}}$, that generalize the classical Malliavin and Bismut derivatives, and applications to the raise of smoothness properties of diffusion semigroups is a subject of [15].

2. Invariant representation of semigroup derivatives in terms of new variations. On noncompact connected oriented smooth Riemannian manifold M without boundary consider diffusion y_t^x, written in Stratonovich form

$$y_t^x = x + \int_0^t A_0(y_s^x) ds + \sum_{\alpha=1}^{d} \int_0^t A_\alpha(y_s^x) \delta W_s^\alpha. \tag{2.1}$$

Here A_0, A_α, $\alpha = 1, \ldots, d$, are smooth vector fields, globally defined on M, initial data $x \in M$ and W_s^α denotes the $\mathbb{R}^d$-valued Wiener process.

Equation (2.1) is understood in sense that for any smooth function with compact support $f \in C_0^2(M)$ the following equation:

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x) ds + \sum_{\alpha=1}^{d} \int_0^t (A_\alpha f)(y_s^x) \delta W_s^\alpha \tag{2.2}$$

holds as usual equation in $\mathbb{R}^1$. In particular, one can take functions $f^i(x) = x^i$ to be local coordinates and find generator of y_t^x

$$Lf = A_0 f + \frac{1}{2}\sum_\alpha A_\alpha(A_\alpha f). \tag{2.3}$$

Below we are going to study how the properties of nonlinear diffusion A_α and drift A_0 coefficients should be related with the geometric properties of manifold to lead to the regular dependence of process y_t^x on initial data and smooth properties of corresponding diffusion semigroup

$$(P_t f)(x) = \mathbf{E} f(y_t^x) \tag{2.4}$$

in some scales of continuously differentiable functions on manifold. To obtain the regular properties of semigroup one should consider its higher-order derivatives.

Taking formally the first-order derivative of (2.4) we have

$$\nabla_k P_t f(x) = \frac{\partial}{\partial x^k}\mathbf{E} f(y_t^x) = \mathbf{E}\frac{\partial f(y_t^x)}{\partial y^m}\frac{\partial (y_t^x)^m}{\partial x^k}. \tag{2.5}$$

This representation is invariant with respect to the local coordinates transformations $(x) \to \to (x')$, because in (2.5) the first-order variation $\dfrac{\partial (y_t^x)^m}{\partial x^k}$ of diffusion with respect to the initial data is

covector field on index k with respect to coordinate transformations of domain $(x) \to \to (x')$;

vector field on index m with respect to the choice of local coordinate vicinity for diffusion $(y) \to (y')$.

To find the higher-order representation of semigroup derivatives let us write the second order covariant derivative of semigroup

$$\nabla_k \nabla_j P_t f(x) = \left\{\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j} - \Gamma^{\,h}_{k\,j}(x)\frac{\partial}{\partial x^h}\right\} P_t f(x) =$$

$$= \mathbf{E}\left\{\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j} - \Gamma^{\,h}_{k\,j}(x)\frac{\partial}{\partial x^h}\right\} f(y_t^x) =$$

$$= \mathbf{E}\left\{\frac{\partial f(y)}{\partial y^m}\frac{\partial^2 y^m}{\partial x^k \partial x^j} + \frac{\partial^2 f(y)}{\partial y^m \partial y^n}\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial y^j} - \Gamma^{\,h}_{k\,j}(x)\frac{\partial f(y)}{\partial y^m}\frac{\partial y^m}{\partial x^h}\right\}, \tag{2.6}$$

where the covariant derivative of a tensor field is defined in a standard way

$$\nabla_k^x u^{i_1,\ldots,i_p}_{j_1,\ldots,j_q} = \frac{\partial}{\partial x^k} u^{i_1,\ldots,i_p}_{j_1,\ldots,j_q} + \sum_{s=1}^{p}\Gamma^{i_s}_{k\,\ell}(x) u^{i_1,\ldots,i_p|_{i_s=\ell}}_{j_1,\ldots,j_q} - \sum_{s=1}^{q}\Gamma^{\ell}_{k\,j_s}(x) u^{i_1,\ldots,i_p}_{j_1,\ldots,j_q|_{j_s=\ell}} \tag{2.7}$$

$u^{i_1,\ldots,i_p|_{i_s=\ell}}_{j_1,\ldots,j_q}$ means substitution of index i_s by ℓ, the summation on repeating indexes is implemented, and $\Gamma(x)$ are connection coefficients.

Now let us form the covariant derivatives of f in the right-hand side of (2.6). Using that

$$\nabla^y_\ell f(y) = \frac{\partial}{\partial y^\ell} f(y)$$

and

$$\nabla^y_m \nabla^y_n f(y) = \frac{\partial}{\partial y^m}\frac{\partial}{\partial y^n} f(y) - \Gamma^{\,\ell}_{m\,n}(y)\frac{\partial}{\partial y^\ell} f(y)$$

we can continue (2.6)

$$\nabla_k \nabla_j P_t f(x) = \mathbf{E}\Big\{ (\nabla^y_m \nabla^y_n f(y) + \Gamma^{\ \ell}_{m\ n}(y) \nabla^y_\ell f(y)\,) \frac{\partial y^m}{\partial x^k}\frac{\partial y^m}{\partial x^j} + \tag{2.8}$$

$$+\nabla^y_m f(y)\left(\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h}\right)\Big\} =$$

$$= \mathbf{E}\Big\{ \nabla^y_m \nabla^y_n f(y)\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^j} +$$

$$+\nabla^y_m f(y)\left(\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ n}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j}\right)\Big\}. \tag{2.9}$$

Here we redenoted index ℓ in term with $\Gamma(y)$.

The first term $\nabla^y \nabla^y f \dfrac{\partial y}{\partial x}\dfrac{\partial y}{\partial x}$ is obviously invariant under transformations $(x) \to (x')$ and $(y) \to (y')$, but what one should do with the expression in brackets

$$\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ n}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} \ ? \tag{2.10}$$

There are two ways to collect the terms in brackets.

1$^{\mathrm{st}}$ way. One may form the covariant derivative on x variable from first and second terms

$$\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ n}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} = \nabla^x_k\left(\frac{\partial y^m}{\partial x^j}\right) + \Gamma^{\ m}_{\ell\ h}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j}. \tag{2.11}$$

Such representation is obviously invariant with respect to transformations $(x) \to (x')$.

The third term with connection $\Gamma(y)$ has transformation of coordinates law, that includes the second-order derivatives of coordinate change, similar to Itô formula. Therefore the traditional interpretation of (2.11) was that in the stochastic case one should add terms with $\Gamma(y)$ to the classical covariant derivative to compensate the influence of Itô formula. A concept of stochastic differential geometry as a mixture of classical differential geometry and Itô formula arose [5, 10, 11, 16].

2$^{\mathrm{nd}}$ way. It is not clear, whether representation (2.11) is invariant with respect to the transformations $(y) \to (y')$ in the image. Let us work with (2.10) in other way, by collecting first and third terms together

$$\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ n}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} =$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial y^m}{\partial x^j}\right) - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ h}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} =$$

$$= \frac{\partial y^\ell}{\partial x^k}\frac{\partial}{\partial y^\ell}\left(\frac{\partial y^m}{\partial x^j}\right) - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h} + \Gamma^{\ m}_{\ell\ h}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} =$$

$$= \frac{\partial y^\ell}{\partial x^k}\nabla^y_\ell\left(\frac{\partial y^m}{\partial x^j}\right) - \Gamma^{\ h}_{k\ j}(x)\frac{\partial y^m}{\partial x^h}, \tag{2.12}$$

where we used that $\dfrac{\partial}{\partial x^k} = \dfrac{\partial y^\ell}{\partial x^k}\dfrac{\partial}{\partial y^\ell}$. This representation, in comparison to (2.11), is obviously invariant with respect to the coordinate changes $(y) \to (y')$.

Therefore all terms in (2.10) define a second variation of process y_t^x and represent

vector field on index m with respect to the "Itô" changes of coordinates $(y) \to (y')$;

twice covariant field on indexes k, j with respect to the "differential geometric" changes of coordinates $(x) \to (x')$.

If one knows how to define the second-order covariant derivative, then, by common procedures of differential geometry, its higher-order analogies could be easily written (Definition 1.1). The invariance of (1.2) with respect to $(x) \to (x')$ transformations is obvious, for transformations in image $(y) \to (y')$ one should argue like in (2.12), e.g. [17].

***Remarks.* 2.1.** One should note that arguments above also work for the choice $A_\alpha \equiv \equiv 0$, i.e., in the ordinary differential equations case, when stochastic terms do not appear and there are no complications, related with Itô formula. Therefore the introduction of higher-order variation is a pure question of differential geometry.

2.2. The last term with $\Gamma(y)$ in (1.2) depends on solution y_t^x and ensures that the higher-order variation, similar to the first-order variation, remains a vector field with respect to transformations $(y) \to (y')$. It compensates the inevitably arising derivatives on variable x of jacobians $\dfrac{\partial (y'(x,t))^n}{\partial (y(x,t))^m}$ of coordinate changes $(y) \to (y')$.

2.3. The consideration of classical derivatives along vector fields, covariant derivatives of the first-order variation

$$\nabla^x_{k_n} \ldots \nabla^x_{k_2} \frac{\partial (y_t^x)^m}{\partial x^{k_1}},$$

or similar objects, like in [4, 5], destroys the invariance with respect to transformations $(y) \leftrightarrow (y')$ and leads to the geometrically noninvariant objects. Such approach also hides the curvature in a set of geometrically noninvariant variational equations.

Using variations $\mathbb{V}^x_\gamma y_t^x$, we can now write invariant representations of semigroup's derivatives:

Theorem 2.1. *The covariant derivatives of semigroup action and initial function are related via new type variations by*

$$\nabla^x_\gamma P_t f(x) = \sum_{\delta_1 \cup \ldots \cup \delta_s = \gamma} \mathbf{E}\, (\nabla^y_{\{j_1, \ldots, j_s\}} f)(y_t^x)\, \mathbb{V}^x_{\delta_1} y^{j_1} \ldots \mathbb{V}^x_{\delta_s} y^{j_s}. \tag{2.13}$$

Here $\nabla^x_\gamma = \nabla^x_{k_1} \ldots \nabla^x_{k_n}$ *for* $\gamma = \{k_1, \ldots, k_n\}$.

Proof. This representation is easily verified recurrently. Indeed, suppose it is true for all $|\gamma| \le n$. Similar to (2.8), one should consider next order derivative

$$\nabla^x_k \nabla^x_\gamma P_t f(x) = \partial^x_k \nabla^x_\gamma P_t f(x) - \sum_{j \in \gamma} \Gamma_k{}^h{}_j(x) \nabla^x_{\gamma|_{j=h}} P_t f(x).$$

Then one should substitute expressions (2.13), add and subtract $\Gamma(y)$ to form the higher-order covariant derivatives of f, redenote summation indices and come to (2.13) for $\nabla^x_k \nabla^x_\gamma P_t f$.

3. Tensors on domain (x) and image (y) coordinates and recurrent form of the higher-order variational equations. Being equipped with the new definition of variation with respect to the initial data and corresponding representations of semigroup's derivatives, we can turn to the regularity problem.

First we find the recurrent equations on higher-order variations. Differentiating (2.1) on initial data x we have

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = \left(\frac{\partial}{\partial x^k} A^m_\alpha(y)\right)\delta W^\alpha + \left(\frac{\partial}{\partial x^k} A^m_0(y)\right) dt. \tag{3.1}$$

To proceed further it is necessary to give an invariant sense to the partial derivatives $\dfrac{\partial}{\partial x} A(y^x_t)$ in the above equation. To do this we need a certain generalization of Definition 1.1 for tensors on (x) and (y^x_t) coordinates, given by Definition 3.2.

Definition 3.1. *Object $u^{(i/\alpha)}_{(j/\beta)}$ forms a mixed tensor with respect to the coordinate changes $(x) \to (x')$ and $(\phi) \to (\phi')$ iff its coordinates*

$$u^{(i/\alpha)}_{(j/\beta)} = u^{i_1 \ldots i_p/\alpha_1 \ldots \alpha_r}_{j_1 \ldots j_q/\beta_1 \ldots \beta_s}$$

form $T^{p,q}_x M$ tensor on multiindexes $(i) = (i_1, \ldots, i_p)$, $(j) = (j_1, \ldots, j_q)$ with respect to the local coordinates (x^k) and form $T^{r,s}M$ tensor on multiindexes (α), (β) with respect to the local coordinates (ϕ^m).

In other words, after the simultaneous change of local coordinate systems $(x^k) \to \to (x^{k'})$ and $(\phi^m) \to (\phi^{m'})$ one has transformation law

$$u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial \phi^{(\alpha)}}{\partial \phi^{(\alpha')}} \frac{\partial \phi^{(\beta')}}{\partial \phi^{(\beta)}} u^{(i'/\alpha')}_{(j'/\beta')} \tag{3.2}$$

with jacobians $\dfrac{\partial x^{(i)}}{\partial x^{(i')}} = \dfrac{\partial x^{i_1}}{\partial x^{i'_1}} \cdots \dfrac{\partial x^{i_p}}{\partial x^{i'_p}}$, $\dfrac{\partial \phi^{(\alpha)}}{\partial \phi^{(\alpha')}} = \dfrac{\partial \phi^{\alpha_1}}{\partial \phi^{\alpha'_1}} \cdots \dfrac{\partial \phi^{\alpha_s}}{\partial \phi^{\alpha'_s}}$.

***Examples.* 3.1.** A simple example of mixed tensor provide variations $\mathbb{V}_{j_1} \ldots \ldots \mathbb{V}_{j_k} y^m(x,t)$. They form vector fields on index m in a coordinate chart $y^m(x,t)$ and covector on $j_1, \ldots, j_k$ in coordinate vicinity (x).

3.2. Another example of mixed tensor is given by product of tensors $u^{(\alpha)}_{(\beta)}(y^x_t) v^{(i)}_{(j)}(x)$ in vicinities (x) and (y). The change of coordinates at x does not influence $u^{(\alpha)}_{(\beta)}(y^x_t)$ part, jacobians of coordinate changes arise only near $v^{(i)}_{(j)}(x)$. However the different choice of coordinate vicinities for y evoke the tensorial transformation law for $u^{(\alpha)}_{(\beta)}$ multiple.

Now let us suppose that (ϕ^m) coordinates of the mixed tensor depend in effective way on the coordinates (x^k). An analogue of Definition 1.1 for mixed tensors is given by the following definition.

Definition 3.2. *$\mathbb{V}$-derivative of a mixed tensor is defined by*

$$\mathbb{V}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial}{\partial x^k} u^{(i/\alpha)}_{(j/\beta)} + \sum_{s \in (i)} \Gamma^{\,s}_{k\,h}(x) u^{(i/\alpha)|_{s=h}}_{(j/\beta)} - \sum_{s \in (j)} \Gamma^{\,h}_{k\,s}(x) u^{(i/\alpha)}_{(j/\beta)|_{s=h}} + \tag{3.3}$$

$$+ \sum_{\rho \in (\alpha)} \Gamma^{\,\rho}_{\gamma\,\delta}(\phi(x)) u^{(i/\alpha)|_{\rho=\delta}}_{(j/\beta)} \frac{\partial \phi^\delta}{\partial x^k} - \sum_{\rho \in (\beta)} \Gamma^{\,\gamma}_{\rho\,\delta}(\phi(x)) u^{(i/\alpha)}_{(j/\beta)|_{\rho=\gamma}} \frac{\partial \phi^\delta}{\partial x^k}. \tag{3.4}$$

Line (3.3) corresponds to the covariant derivative on (x^k) coordinates, additional line (3.4) makes the resulting expression to be tensor with respect to the coordinates in image (ϕ^m). One may also note that the connection symbols above depend on different parameters and the additional jacobians $\dfrac{\partial \phi}{\partial x}$ are required in line (3.4).

***Remarks.* 3.1.** The tensorial character of $\mathbb{V}$-derivative is easily checked, like before: $\mathbb{V}$-derivative defines a tensor of higher valence, i.e., the mixed tensor law holds

$$\mathbb{V}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^{(i)}}{\partial x^{(i')}}\frac{\partial x^{(j')}}{\partial x^{(j)}}\frac{\partial \phi^{(\alpha)}}{\partial \phi^{(\alpha')}}\frac{\partial \phi^{(\beta')}}{\partial \phi^{(\beta)}}\mathbb{V}^x_{k'} u^{(i'/\alpha')}_{(j'/\beta')}.$$

The proof of this property is easy by application of the transformation of connection law [17].

3.2. An important property of $\mathbb{V}$-derivative is the superposition rule: let $u^{(\alpha)}_{(\beta)}$ be a tensor on manifold M, then

$$\mathbb{V}^x_k u^{(\alpha)}_{(\beta)}(\phi(x)) = \left(\nabla_\ell u^{(\alpha)}_{(\beta)}\right)(\phi(x))\frac{\partial \phi^\ell}{\partial x^k}. \tag{3.5}$$

To check (3.5) we use Definition (3.3) (3.4) to obtain

$$\mathbb{V}^x_k u^{(\alpha)}_{(\beta)}(\phi(x)) = \partial^x_k u^{(0/\alpha)}_{(0/\beta)}(\phi(x)) +$$

$$+\sum_{\rho\in(\alpha)} \Gamma^{\ \rho}_{\gamma\ \delta}(\phi) u^{(0/\alpha)|_{s=\gamma}}_{(0/\beta)}(\phi)\frac{\partial \phi^\delta}{\partial x^k} - \sum_{\rho\in(\beta)} \Gamma^{\ \gamma}_{\rho\ \delta}(\phi) u^{(0/\alpha)}_{(0/\beta)|_{\rho=\gamma}}\frac{\partial \phi^\delta}{\partial x^k}.$$

By chain rule for ∂^x_k and definition of covariant derivative (2.7) one gets the statement.

As we will soon see, property (3.5) simplifies the geometrically correct calculation of the higher-order variational equations.

After the introduction of mixed tensor and its $\mathbb{V}$-derivative, we can further **transform equation on the first variation (3.1).** By adding and subtracting the terms with $\Gamma(y)$ to single out the $\mathbb{V}$-derivative of vector fields $A_0(y),\ A_\alpha(y)$ on image coordinates (y), we have

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) =$$

$$= \left(\mathbb{V}^x_k A^m_\alpha(y) - \Gamma^{\ m}_{p\ q}(y)A^p_\alpha\frac{\partial y^q}{\partial x^k}\right)\delta W^\alpha + \left(\mathbb{V}^x_k A^m_0(y) - \Gamma^{\ m}_{p\ q}(y)A^p_0\frac{\partial y^q}{\partial x^k}\right)dt.$$

Noting that the terms near connection contain the differential of process y (2.1) and using the symmetry of connection $\Gamma^{\ m}_{p\ q} = \Gamma^{\ m}_{q\ p}$, we find another representation for the equation on first variation

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = -\Gamma^{\ m}_{p\ q}(y)\frac{\partial y^p}{\partial x^k}\delta y^q + \mathbb{V}^x_k\left(A^m_\alpha(y)\right)\delta W^\alpha + \mathbb{V}^x_k(A^m_0(y))\,dt. \tag{3.6}$$

We obtain an additional argument in favor of new type variations and $\mathbb{V}$-derivatives: up to the parallel transition term with $\Gamma(y)$ the increments of first-order variation are determined by $\mathbb{V}$-derivatives of coefficients. We take this observation as the recurrence base for the search of higher-order variational equations.

Theorem 3.1. *Suppose that the equation on* $\mathbb{V}$*-variation* $\mathbb{V}^x_\gamma y^m$, $|\gamma| \geq 1$, *is written in form*

$$\delta(\mathbb{V}^x_\gamma y^m) = -\Gamma^{\ m}_{p\ q}(\mathbb{V}^x_\gamma y^p)\delta y^q + M^{\ m}_{\gamma\ i}\delta W^i + N^m_\gamma dt. \tag{3.7}$$

Then the next order variation $\mathbb{V}_k^x \mathbb{V}_\gamma^x y^m = \mathbb{V}_{\gamma\cup\{k\}}^x y^m$ *fulfills*

$$\delta(\mathbb{V}_{\gamma\cup\{k\}}^x y^m) = -\Gamma_{p\ q}^{\ m}(\mathbb{V}_{\gamma\cup\{k\}}^x y^p)\delta y^q + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}\delta y^q + \\ + (\mathbb{V}_k^x M_{\gamma\ i}^{\ m})\delta W^i + (\mathbb{V}_k^x N_\gamma^m)dt. \tag{3.8}$$

The coefficients of variational equations are recurrently related by

$$M_{\gamma\cup\{k\}\ i}^{\ m} = \mathbb{V}_k^x M_{\gamma\ i}^{\ m} + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_i^q, \\ N_{\gamma\cup\{k\}}^m = \mathbb{V}_k^x N_\gamma^m + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_0^q. \tag{3.9}$$

Here R forms curvature (1,3)-tensor with components

$$R_{1\ 34}^{\ 2} = \frac{\partial\Gamma_{1\ 3}^{\ 2}}{\partial x^4} - \frac{\partial\Gamma_{1\ 4}^{\ 2}}{\partial x^3} + \Gamma_{1\ 3}^{\ j}\Gamma_{j\ 4}^{\ 2} - \Gamma_{1\ 4}^{\ j}\Gamma_{j\ 3}^{\ 2}, \tag{3.10}$$

where for simplicity we only point the positions of corresponding indexes.

***Remark* 3.3.** The additional term with $\Gamma(y)$ in the Definition 3.2 of $\mathbb{V}$-derivative compactificates these noninvariant terms to the compact expressions with curvature. So it becomes possible to find the influence of curvature and nonlinearities of diffusion equation on the any order regularity properties.

The approaches to define the variation to be covariant Riemannian, derivative in the direction of vector field or stochastic derivative did not account the invariance on process y_t^x [4, 5, 7] and inevitably led to the growing number of noninvariant terms in the variational equations. Therefore, it was principally hard to trace the influence of curvature in regular properties.

Proof. For simplification we omit, where possible the dependence of connection Γ on variable y, however the dependence on x is always displayed precisely.

Let us substitute the definition of $\mathbb{V}$-derivative under Stratonovich integral

$$\int \delta(\mathbb{V}_k^x \mathbb{V}_\gamma^x y^m) = \\ = \int \delta\left\{ \partial_k^x \mathbb{V}_\gamma^x y^m + \Gamma_{p\ q}^{\ m}(y)\frac{\partial y^p}{\partial x^k}\mathbb{V}_\gamma^x y^q - \sum_{s\in\gamma}\Gamma_{k\ s}^{\ h}(x)\mathbb{V}_{\gamma|_{s=h}}^x y^m \right\}. \tag{3.11}$$

For the first term in (3.11) we apply the inductive assumption (3.7) and, after differentiation of integral and application of property of Stratonovich integral

$$\int X\,\delta\left(\int Y\,\delta Z\right) = \int XY\,\delta Z \tag{3.12}$$

obtain

$$\int \delta\{\partial_k^x \mathbb{V}_\gamma^x y^m\} = \int \delta\left(\partial_k^x \int\left\{-\Gamma_{p\ q}^{\ m}(\mathbb{V}_\gamma^x y^p)\delta y^q + M_{\gamma\ i}^{\ m}\delta W^i + N_\gamma^m dt\right\}\right) = \\ = -\int \frac{\partial\Gamma_{p\ q}^{\ m}}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^k}(\mathbb{V}_\gamma^x y^p)\delta y^q - \int \Gamma_{p\ q}^{\ m}(\mathbb{V}_\gamma^x y^p)\delta\left(\frac{\partial y^q}{\partial x^k}\right) - \tag{3.13}$$

$$-\int \Gamma_{p\,q}^{\,m}(\partial_k^x \mathbb{V}_\gamma^x y^p)\delta y^q + \int \{\partial_k^x M_{\gamma\ i}^{\ m}\delta W^i + \partial_k^x N_\gamma^m dt\}. \tag{3.14}$$

The second term in brackets in (3.11) is first rewritten by Stratonovich – Itô formula for function $f(x_1, x_2, x_3) = x_1 x_2 x_3$:

$$f \circ X_t = f \circ X_0 + \int_0^t D_j f \circ X\, \delta X^j$$

after that the inductive assumption (3.7) and properties of Stratonovich integrals are applied

$$\int \delta \left\{ \Gamma_{p\,q}^{\,m}(y) \frac{\partial y^p}{\partial x^k} \mathbb{V}_\gamma^x y^q \right\} =$$

$$= \int \Gamma_{p\,q}^{\,m} \frac{\partial y^p}{\partial x^k} \delta(\mathbb{V}_\gamma^x y^q) + \int \Gamma_{p\,q}^{\,m}(\mathbb{V}_\gamma^x y^q)\delta\left(\frac{\partial y^p}{\partial x^k}\right) + \int \frac{\partial y^p}{\partial x^k}(\mathbb{V}_\gamma^x y^q)\delta\Gamma_{p\,q}^{\,m}(y) =$$

$$= \int \Gamma_{p\,q}^{\,m} \frac{\partial y^p}{\partial x^k} \Big\{ -\Gamma_{\ell\,s}^{\,q}(\mathbb{V}_\gamma^x y^\ell)\delta y^s + M_{\gamma\ i}^{\ m}\delta W^i + N_\gamma^q dt \Big\} + \tag{3.15}$$

$$+ \int \Gamma_{p\,q}^{\,m}(\mathbb{V}_\gamma^x y^q)\delta\left(\frac{\partial y^p}{\partial x^k}\right) + \int \frac{\partial y^p}{\partial x^k}(\mathbb{V}_\gamma^x y^q)\frac{\partial \Gamma_{p\,q}^{\,m}}{\partial y^\ell}\delta y^\ell. \tag{3.16}$$

For the last term in (3.11) we use again the inductive assumption (3.7)

$$-\int \delta \left\{ \sum_{s\in\gamma} \Gamma_{k\,s}^{\,h}(x)\mathbb{V}_{\gamma|_{s=h}}^x y^m \right\} =$$

$$= -\sum_{s\in\gamma} \int \Gamma_{k\,s}^{\,h}(x) \Big\{ -\Gamma_{p\,q}^{\,m}(y)(\mathbb{V}_{\gamma|_{s=h}}^x y^p)\delta y^q + M_{\gamma|_{s=h}\ i}^{\ m}\delta W^i + N_{\gamma|_{s=h}}^m dt \Big\}. \tag{3.17}$$

Further we transform the first expression in (3.14) to the $\mathbb{V}$-derivative

$$-\int \Gamma_{p\,q}^{\,m}(\partial_k^x \mathbb{V}_\gamma^x y^p)\delta y^q = -\int \Gamma_{p\,q}^{\,m}(\partial_k^x \mathbb{V}_\gamma^x y^p)\delta y^q =$$

$$= -\int \Gamma_{p\,q}^{\,m}(\mathbb{V}_k \mathbb{V}_\gamma^x y^p)\delta y^q + \int \Gamma_{p\,q}^{\,m}\Gamma_{\ell\,n}^{\,p}\frac{\partial y^\ell}{\partial x^k}(\mathbb{V}_\gamma^x y^n)\delta y^q -$$

$$-\sum_{s\in\gamma} \int \Gamma_{p\,q}^{\,m}(y)\Gamma_{k\,s}^{\,h}(x)(\mathbb{V}_{\gamma|_{s=h}}^x y^p)\delta y^q. \tag{3.18}$$

Notice now that:

a) the second expression in (3.13) contracts with the first expression in (3.16);

b) the third expression in (3.18) contracts with the first expression in (3.17);

c) the second and third terms in (3.14), (3.15) and (3.17) give the $\mathbb{V}$-derivatives of M and N coefficients.

We write the remaining terms, by redenoting indexes and gathering terms with derivatives $\partial\Gamma$ and second powers $\Gamma(y)\Gamma(y)$ of connection

$$\int \delta \left\{ \partial_k^x \mathbb{V}_\gamma^x y^m + \Gamma_{p\,q}^{\,m}(y)\frac{\partial y^p}{\partial x^k}\mathbb{V}_\gamma^x y^q - \sum_{s\in\gamma}\Gamma_{k\,s}^{\,h}(x)\mathbb{V}_{\gamma|_{s=h}}^x y^m \right\} =$$

$$= -\int \Gamma_{p\ q}^{\ m}(\mathbb{V}_k^x \mathbb{V}_\gamma^x y^p)\delta y^q + \int \left\{\mathbb{V}_k^x M_{\gamma\ i}^{\ m} \delta W^i + \mathbb{V}_k^x N_\gamma^m dt\right\} +$$

$$+ \int \frac{\partial y^\ell}{\partial x^k}(\mathbb{V}_\gamma^x y^p)\delta y^q \times$$

$$\times \left\{ \frac{\partial \Gamma_{\ell\ p}^{\ m}(y)}{\partial y^q} - \frac{\partial \Gamma_{p\ q}^{\ m}(y)}{\partial y^\ell} + \Gamma_{s\ q}^{\ m}(y)\Gamma_{\ell\ p}^{\ s}(y) - \Gamma_{\ell\ s}^{\ m}(y)\Gamma_{p\ q}^{\ s}(y) \right\}. \qquad (3.19)$$

The terms in (3.19) appear correspondingly
first one from the second term in (3.16);
second one from the first term in (3.13);
third from the first term in (3.18);
last term from the first term in (3.15).
But the expression in brackets $\{\ldots\}$ gives the curvature (3.10), so we conclude

$$\int \delta(\mathbb{V}_k^x \mathbb{V}_\gamma^x y^m) = -\int \left\{ \Gamma_{p\ q}^{\ m}(\mathbb{V}_k^x \mathbb{V}_\gamma^x y^p)\delta y^q + R_{p\ \ell q}^{\ m}(\mathbb{V}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}\delta y^q + \right.$$

$$\left. + \mathbb{V}_k^x M_{\gamma\ i}^{\ m} \delta W^i + \mathbb{V}_k^x N_\gamma^m dt \right\}.$$

The theorem is proved.

4. Symmetries of variational equations and nonlinear estimate on variations. Now we are going to use the symmetry of variational equations to find a set of nonlinear estimates on variations.

First remark that by (3.6) the recurrence base for the definition of higher-order variational systems (3.7) is given by

$$M_{k\ i}^{\ m} = \mathbb{V}_k^x A_i^m(y_t^x), \qquad N_k^m = \mathbb{V}_k^x A_0^m(y_t^x).$$

Using recurrent properties (3.9) and (3.5) we can determine the nonlinear symmetries of variational equations. Because

$$(\mathbb{V}^x)^n H(y_t^x) = \sum_{j_1+\ldots+j_s=n,\ s=1,\ldots,n} (\nabla^y)^s H(y_t^x) \cdot (\mathbb{V}^x)^{j_1} y \ldots (\mathbb{V}^x)^{j_s} y$$

we see that

the n^{th} order variation in the left-hand side of (3.7) is proportional to the n^{th} power of first variation in the right-hand side,

or

$$\sqrt[n]{(\mathbb{V}^x)^n y_t^x} \sim \mathbb{V}^x y_t^x. \qquad (4.1)$$

Introduce nonlinear expression that reflects this symmetry

$$r_n(y,t) = \sum_{j=1}^n \mathbf{E} p_j(\rho^2(y_t^x, z)) \left\| (\mathbb{V}^x)^j y_t^x \right\|^{q/j} \qquad (4.2)$$

and gives some nonlinear norm on the smoothness of process y_t^x with respect to the initial data. Here $z \in M$ is some fixed point, $\rho(x,y)$ is geodesic distance between points x, y, norm of variation is defined in (1.3).

The following theorem provides necessary conditions for quasicontractive estimate on r_n. Introduce notation

$$\widetilde{A_0} = A_0 + \frac{1}{2}\sum_{\alpha=1}^{d} \nabla_{A_\alpha} A_\alpha. \tag{4.3}$$

Theorem 4.1. *Suppose that the following conditions hold*:

coercitivity: *there exists $z \in M$ such that for any $C \in \mathbb{R}_+$ there exists $K_C \in \mathbb{R}^1$ such that for any $x \in M$*

$$\left\langle \widetilde{A_0}(x), \nabla^x \rho^2(x,z) \right\rangle + C\sum_{\alpha=1}^{d} \|A_\alpha(x)\|^2 \le K_C(1+\rho^2(x,z)); \tag{4.4}$$

dissipativity: *for any $C, C' \in \mathbb{R}_+$ there exists $K_C \in \mathbb{R}^1$ such that for any $x, y \in M$*

$$\left\langle \nabla\widetilde{A_0}[h], h \right\rangle + C\sum_{\alpha=1}^{d} |\nabla A_\alpha[h]|^2 - C'\sum_{\alpha=1}^{d} \langle R(A_\alpha, h)A_\alpha, h\rangle \le K_C\|h\|^2; \tag{4.5}$$

notation $\nabla A[h] = h^\ell \nabla_\ell A$ means covariant directional derivative, $\langle\cdot,\cdot\rangle$ and $|\cdot|$ corresponding Riemannian scalar product and norm, and

$$\big[R(A,B)C\big]^m = R_{i\ jk}^{\ m} A^j B^k C^i$$

denotes curvature operator;

nonlinear behaviour of coefficients and curvature: *for any n there are constants $\mathbf{k}_\bullet$ such that for all $j = 1, \ldots, n$ and $x \in M$*

$$\begin{gathered}
\left\|(\nabla)^j \widetilde{A_0}(x)\right\| \le (1+\rho(x,z))^{\mathbf{k}_0}, \\
\left\|(\nabla)^j A_\alpha(x)\right\| \le (1+\rho(x,z))^{\mathbf{k}_\alpha}, \\
\left\|(\nabla)^j R(x)\right\| \le (1+\rho(x,z))^{\mathbf{k}_R}.
\end{gathered} \tag{4.6}$$

Then there is some $\mathbf{k} = \mathbf{k}(\mathbf{k}_0, \mathbf{k}_\alpha, \mathbf{k}_R)$ such that if monotone polynomials $p_j \ge 1$ in (4.2) *are hierarchied by*

$$\forall\, j_1 + j_s = i \le n\colon \quad \big[p_i(\cdot)\big]^i \big(1+|\cdot|^2\big)^{\mathbf{k}_q} \le \big[p_{j_1}(\cdot)\big]^{j_1} \ldots \big[p_{j_s}(\cdot)\big]^{j_s} \tag{4.7}$$

then the nonlinear estimate on variations holds

$$\exists K_{\mathbf{k}} \quad \forall t \ge 0 \quad r_n(y,t) \le e^{K_{\mathbf{k}} t} r_n(y,0). \tag{4.8}$$

***Remarks.* 4.1.** For $M = \mathbb{R}^d$ with the global Euclidean coordinate system $(x^i)_{i=1}^d$ both connection and curvature vanish. In this case $\rho(x,y) = \|x-y\|$ and one can choose point $z = 0$ to reduce conditions (4.4), (4.5) to the classical conditions of

coercitivity:

$$\left\langle \widetilde{A_0}(x), x \right\rangle + C\sum_{\alpha=1}^{d} \left\|A_\alpha(x)\right\|^2 \le K_C(1+\|x\|^2);$$

dissipativity:

$$\Big\langle \nabla\widetilde{A_0}[h], h\Big\rangle + C\sum_{\alpha=1}^{d} \big\|\nabla A_\alpha[h]\big\|^2 \le K_C\|h\|^2.$$

They naturally arise in the proof of nonexplosion and uniqueness estimates on process y_t^x

$$\mathbf{E}\|y_t^x\|^2 \le e^{Kt}\big(1+\|x\|^2\big), \qquad \mathbf{E}\|y_t^x - y_t^z\|^2 \le e^{K'}\|x-z\|^2. \tag{4.9}$$

Indeed, by coercitivity condition and Itô formula

$$h_t \equiv \mathbf{E}\|y_t^x\|^2 = \|x\|^2 + \int_0^t \mathbf{E}\left\{\Big\langle \widetilde{A_0}(y_s^x), y_s^x\Big\rangle + \frac{1}{2}\sum_{\alpha=1}^{d}\big\|A_\alpha(y_s^x)\big\|^2\right\}ds \le$$

$$\le \|x\|^2 + C\int_0^t (1+h_s)ds. \tag{4.10}$$

Then Gronwall – Bellman inequality leads to the first estimate in (4.9). In a similar way dissipativity condition leads to the second estimate in (4.9), ensuring local uniqueness of solutions.

In this sense the coercitivity and dissipativity conditions are natural for nonlinear diffusion equations. Their generalization for stochastic differential equations in infinite dimensional **linear** spaces was found by Krylov, Pardoux, Rosovskii [18, 19].

4.2. In fact, from coercitivity and dissipativity conditions follows that field $\widetilde{A_0}$ and, therefore, field A_0 should be more than a square of fields A_α times curvature. For example, for C^2 bounded diffusion coefficients $\sup_{x\in M}\big(\|A_\alpha(x)\|,\ \|\nabla A_\alpha(x)\|,$ $\|\nabla\nabla A_\alpha(x)\|\big) < \infty$ and bounded geometry $\sup_{x\in M}\big\|R(x)\big\| < \infty$ only the monotonicity of field $(-A_0)$ is necessary for the validity of coercitivity and dissipativity conditions (C^2 boundedness arises due to the structure of $\widetilde{A_0}$).

In the domain of manifold, where the curvature form $m(h,h) = \sum_{\alpha=1}^{d}\big\langle R(A_\alpha,$ $h)A_\alpha, h\big\rangle$ is positive, the additonal terms with curvature improve the restrictions on diffusion and drift. However, the manifolds with strictly positive curvature forms are compact [20], so in this case process y_t^x actually lives on compact manifold and there are no nonlinear complications.

4.3. Let us turn the attention of reader that the coercitivity and dissipativity assumptions naturally arise in the proof of nonexplosion and uniqueness estimates on the process y_t^x. To check this fact one should proceed similar to (4.9), (4.10), with application of Itô formula to expressions $\mathbf{E}\rho^2(y_t^x, o)$ and $\mathbf{E}\rho^2(y_t^x, y_t^z)$ and further use of estimates (4.31), (4.32). This is a subject of [21].

In Theorem 4.1 we actually state, that the *coercitivity and dissipativity assumptions, combined with* (4.6), are **sufficient for any order regularity of process** y_t^x with respect to the initial data. Moreover, as it will be clear from the proof, dissipativity assumption (4.5) represents the *coercitivity condition for variational processes* ∇y_t^x.

4.4. An example of manifold may be given by $\mathbb{R}^d$ with conformally perturbed Euclidean metric tensor $g_{ij}(x) = e^{2\psi(x)}\delta_{ij}$ (such perturbations preserve angles between vectors). In this case [22] connection coefficients of metric g are nonvoid and equal to

$$\Gamma_{i\,j}^{\,h}(x) = \delta_i^h\partial_j^x\psi(x) + \delta_j^h\partial_i^x\psi(x) - \delta_{ij}\delta^{hk}\partial_k^x\psi(x).$$

The covariant derivative of vector field has representation

$$\nabla_i V^h(x) = \partial_i V^h(x) + \Gamma_{i\,j}^{\,h} V^j = \partial_i V^h + \delta_i^h \langle V, \partial\psi\rangle + \partial_i\psi \cdot V^h - \partial^h\psi \cdot V_i$$

and curvature tensor equals to

$$R_{i\,kj}^{\,h} = \delta_k^h \psi_{ij} \ldots \delta_j^h \psi_{ik} + \psi_k^h \delta_{ij} - \psi_j^h \delta_{ik} + (\delta_k^k \delta_{ij} - \delta_j^h \delta_{ik}) \|\partial\psi\|^2,$$

where we used notations $\psi_{ij} = \partial_i\partial_j\psi - \partial_i\psi \cdot \partial_j\psi$, $\psi_k^h = \delta^{hj}\psi_{jk}$ with Kronecker deltas δ_{ij}, δ^{hk}, δ_j^h.

In this case the dissipativity condition (4.5) adopts form, which could be hard to guess by direct calculations in the global $\mathbb{R}^d$ coordinate system of manifold $(\mathbb{R}^d, e^{2\psi}\delta_{ij})$.

Therefore, the use of geometric invariance arguments of Definition 1.1 actually permits to single out the pointwise conditions on the behaviour of coefficients and curvature, that guarantee the global regularity estimates on diffusion process even for more complicate manifolds.

Proof. First let us note that Itô formula implies that for geodesic distance

$$\rho^2(y_t^x, z) = \rho^2(x, y) + \sum_{j=1}^{d} \int_0^t (A_\alpha^1 \rho^2)(y_s^x, z) dW_s^\alpha + \int_0^t (L^1\rho^2)(y_s^x, z) ds \tag{4.11}$$

with notation L^1 for generator L of diffusion (2.3), acting on first coordinate of metric function.

Therefore, writing the differential of one terms in nonlinear expression (4.2) we have by Itô formula (temporarily $2q = m/i$, $p = p_i$):

$$h(t) = \mathbf{E} p(\rho^2(y_t^x, z)) \big\|(\mathbb{V}^x)^i y_t^x\big\|^{2q} =$$

$$= h(0) + \mathbf{E} \int_0^t \Big\{ p(\rho^2(y_s^x, z))\, d\big\|(\mathbb{V}^x)^i y_s^x\big\|^{2q} + \big\|(\mathbb{V}^x)^i y_s^x\big\|^{2q} dp\big(\rho^2(y_s^x, z)\big) +$$

$$+ \frac{1}{2} d\Big[p(\rho^2(y_s^x, z)), \big\|(\mathbb{V}^x)^i y_s^x\big\|^{2q}\Big]\Big\} =$$

$$= h(0) + \int_0^t \mathbf{E}\Big\{ p(\rho^2(y_s^x, z))(2q \|(\mathbb{V}^x)^i y_s^x\|^{2(q-1)} d\big\|(\mathbb{V}^x)^i y_s^x\big\|^2 + \tag{4.12}$$

$$+ q(2q-2)\big\|(\mathbb{V}^x)^i y_s^x\big\|^{2(q-2)} d\Big[\big\|(\mathbb{V}^x)^i y_s^x\big\|^2, \big\|(\mathbb{V}^x)^i y_s^x\big\|^2\Big] +$$

$$+ \big\|(\mathbb{V}^x)^i y_s^x\big\|^{2q} \Big(p'\big(\rho^2(y_s^x, z)\big) d\rho^2(y_s^x, z) +$$

$$+ \frac{1}{2} p''\big(\rho^2(y_s^x, z)\big) d\big[\rho^2(y_s^x, z), \rho^2(y_s^x, z)\big]\Big) + \tag{4.13}$$

$$+ \frac{1}{2} p'\big(\rho^2(y_s^x, z)\big) \big\|(\mathbb{V}^x)^i y_s^x\big\|^{2(q-1)} d\Big[\rho^2(y_s^x, z), \big\|(\mathbb{V}^x)^i y_s^x\big\|^2\Big]\Big\}. \tag{4.14}$$

Next step is to find from recurrent relations of Theorem 3.1 the expressions for differentials of norms $\big\|(\mathbb{V}^x)^j y_t^x\big\|^2$ (1.3).

Recall, that by (3.7), (3.8) the general form of variational equations looks like

$$\delta(X^m_\gamma) = -\Gamma^{\ m}_{p\ q} X^p_\gamma \delta y^q + M_{\gamma\ \alpha}^{\ m} \delta W^\alpha + N^m_\gamma dt \tag{4.15}$$

with coefficients $M_{\gamma\ \alpha}^{\ m}$, N^m_γ, recurrently determined in (3.9).

To simplify further notations, let us introduce an additional process, that formally corresponds to the index $\gamma = \varnothing$

$$\delta X^m_\varnothing = -\Gamma^{\ m}_{p\ q} X^p_\varnothing \delta y^q + A^m_\alpha \delta W^\alpha + A^m_0 dt.$$

Then the relations of coefficients M, N for the process X^m_γ could be written in the following form:

1) recurrent base:

$$M_{\varnothing\ \alpha}^{\ m} = A^m_\alpha(y^x_t), \qquad N^m_\varnothing = A^m_0(y^x_t); \tag{4.16}$$

2) recurrent step by (3.6) for $\gamma = \varnothing$ and (3.8) for $\gamma \neq \varnothing$

$$M_{\gamma\cup\{k\}\ \alpha}^{\ m} = \begin{cases} \mathbb{\nabla}^x_k M_{\varnothing\ \alpha}^{\ m} & \text{for} \quad \gamma = \varnothing, \\ \mathbb{\nabla}^x_k M_{\gamma\ \alpha}^{\ m} + R_{p\ \ell q}^{\ m} X^p_\gamma (\mathbb{\nabla}^x_k y^\ell) A^q_\alpha & \text{for} \quad \gamma \neq \varnothing, \end{cases} \tag{4.17}$$

$$N^m_{\gamma\cup\{k\}} = \begin{cases} \mathbb{\nabla}^x_k N^m_\varnothing & \text{for} \quad \gamma = \varnothing, \\ \mathbb{\nabla}^x_k N^m_\gamma + R_{p\ \ell q}^{\ m} X^p_\gamma (\mathbb{\nabla}^x_k y^\ell) A^q_0, & \text{for} \quad \gamma \neq \varnothing. \end{cases} \tag{4.18}$$

Lemma 4.1. *The differential of norm of process* X^m_γ (4.15) *has form*

$$d\|X\|^2 = g^{\gamma\varepsilon}(x)\Big\{ g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m}) dW^\alpha +$$

$$+ g_{mn}(X^m_\gamma N^n_\varepsilon + X^n_\varepsilon N^m_\gamma + M_{\gamma\ \alpha}^{\ m} M_{\varepsilon\ \alpha}^{\ n}) dt + \frac{1}{2} g_{mn}(X^m_\gamma P^n_\varepsilon + X^n_\varepsilon P^m_\gamma) dt \Big\}. \tag{4.19}$$

Expressions P^m_γ *are recurrently related by*

$$P^m_k = \mathbb{\nabla}^x_k (\nabla_{A_\alpha} A^m_\alpha) + R_{p\ \ell q}^{\ m} A^p_\alpha A^q_\alpha (\mathbb{\nabla}^x_k y^\ell), \tag{4.20}$$

$$P^m_{\gamma\cup\{k\}} = \mathbb{\nabla}^x_k P^m_\gamma + 2R_{p\ \ell q}^{\ m} M_{\gamma\ \alpha}^{\ p} (\mathbb{\nabla}^x_k y^\ell) A^q_\alpha +$$

$$+ (\nabla_s R_{p\ \ell q}^{\ m}) X^p_\gamma (\mathbb{\nabla}^x_k y^\ell) A^q_\alpha A^s_\alpha + R_{p\ \ell q}^{\ m} X^p_\gamma (\mathbb{\nabla}^x_k A^\ell_\alpha) A^q_\alpha +$$

$$+ R_{p\ \ell q}^{\ m} X^p_\gamma (\mathbb{\nabla}^x_k y^\ell)(\nabla_{A_\alpha} A_\alpha). \tag{4.21}$$

Proof. The detail and bookkeeping proof of this result will appear in [23]. Let us briefly present the main idea.

First of all one verifies formula (4.19) and obtains expression for P^m_γ

$$P^m_\gamma dt = d[M_{\gamma\ \alpha}^{\ m}, W^\alpha] + \Gamma^{\ m}_{p\ q} M_{\gamma\ \alpha}^{\ p} A^q_\alpha dt. \tag{4.22}$$

Then it is necessary to find the recurrent relations for P^m_γ. From (4.17) follows

$$P^m_{\gamma\cup\{k\}} dt = d[M_{\gamma\cup\{k\}\alpha}^{\ m}, W^\alpha] + \Gamma^{\ m}_{p\ q} M_{\gamma\cup\{k\}\alpha}^{\ p} A^q_\alpha dt =$$

$$= d\big[\mathbb{\nabla}^x_k M_{\gamma\ \alpha}^{\ m}, W^\alpha\big] + \Gamma^{\ m}_{p\ q} (\mathbb{\nabla}^x_k M_{\gamma\ \alpha}^{\ p}) A^q_\alpha dt + \tag{4.23}$$

$$+d\Big[R_{p\ \ell q}^{\ m} X_\gamma^p (\mathbb{V}_k^x y^\ell) A_\alpha^q, W^\alpha\Big] + \Gamma_{p\ q}^{\ m} (R_{i\ \ell j}^{\ p} X_\gamma^i (\mathbb{V}_k^x y^\ell) A_\alpha^j) A_\alpha^q dt. \tag{4.24}$$

The last line (4.24) appears only for $\gamma \neq \varnothing$.

From arguments, analogous to the proof of Theorem 3.1, it follows representation

$$(4.23) = \Big\{\mathbb{V}_k^x P_\gamma^m + R_{p\ \ell q}^{\ m} M_{\gamma\ \alpha}^{\ p} (\mathbb{V}_k^x y^\ell) A_\alpha^q\Big\} dt. \tag{4.25}$$

To find the reccurence base it is sufficient to apply definitions (4.16), (4.22) and obtain

$$P_\varnothing^m dt = d[A_\alpha^m(y), W^\alpha] + \Gamma_{\ell\ h}^{\ m} A_\alpha^\ell A_\alpha^h dt =$$

$$= \left[\frac{\partial A_\alpha^m}{\partial y^\ell} + \Gamma_{\ell\ h}^{\ m} A_\alpha^h\right] A_\alpha^\ell dt = (\nabla_\ell A_\alpha^m) \cdot A_\alpha^\ell dt = (\nabla_{A_\alpha} A_\alpha^m) dt. \tag{4.26}$$

Therefore from (4.25) and (4.26) it follows (4.20).

In a similar way, direct calculation of line (4.24) gives the remaining terms in (4.21).

The lemma is proved.

Now we use the result of Lemma 4.1 to single out the dissipativity condition (4.5) in the principal part in the right-hand side of (4.19).

Let $i = 1$ and $X_k^m = \mathbb{V}_l^x y^m$, then by (4.20) and (3.5)

$$P_k^m = \mathbb{V}_k^x (\nabla_{A_\alpha} A_\alpha^m(y)) + R_{p\ \ell q}^{\ m} A_\alpha^p A_\alpha^q \mathbb{V}_k^x y^\ell =$$

$$= \nabla_\ell^y \nabla_{A_\alpha} A_\alpha^m \cdot \mathbb{V}_k^x y^\ell - R(A_\alpha, \mathbb{V}_k^x y) A_\alpha.$$

Therefore, because in (4.21) $P_{\gamma\cup\{k\}}^m = \mathbb{V}_k^x P_\gamma^m + \ldots$, the higher-order coefficient permits representation

$$P_\gamma^m = \nabla_\ell \nabla_{A_\alpha} A_\alpha^m \cdot \mathbb{V}_\gamma^x y^\ell - R(A_\alpha, \mathbb{V}_\gamma^x y) A_\alpha +$$

$$+ \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2} K_{\beta_1,\ldots,\beta_s}(\mathbb{V}_{\beta_1}^x y, \ldots, \mathbb{V}_{\beta_s}^x y)$$

with coefficients $K_{\beta_1,\ldots,\beta_s}$, depending on A_0, A_α, R and their covariant derivatives. Moreover, the dependence of $K_{\beta_1,\ldots,\beta_s}(\mathbb{V}_{\beta_1}^x y, \ldots, \mathbb{V}_{\beta_s}^x y)$ on lower order variations $\mathbb{V}_\beta^x y$ also manifests symmetries (4.1).

In a similar way, due to (3.6)

$$M_{k\ \alpha}^{\ m} = \mathbb{V}_k^x A_\alpha^m(y) = \nabla_\ell^y A_\alpha^m(y) \cdot \mathbb{V}_k^x y^\ell,$$

$$N_k^{\ m} = \mathbb{V}_k^x A_0^m(y) = \nabla_\ell^y A_0^m(y) \cdot \mathbb{V}_k^x y^\ell$$

and relations (3.9), we have analogous asymptotic

$$\begin{aligned} M_{\gamma\ \alpha}^{\ m} &= \nabla_\ell^y A_\alpha^m \big[\mathbb{V}_\gamma^x y^\ell\big] + \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2} K'_{\beta_1,\ldots,\beta_s}(\mathbb{V}_{\beta_1}^x y, \ldots, \mathbb{V}_{\beta_s}^x y), \\ N_\gamma^{\ m} &= \nabla_\ell^y A_\alpha^0 \big[\mathbb{V}_\gamma^x y^\ell\big] + \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2} K''_{\beta_1,\ldots,\beta_s}(\mathbb{V}_{\beta_1}^x y, \ldots, \mathbb{V}_{\beta_s}^x y) \end{aligned} \tag{4.27}$$

with multilinear coefficients K', K'', depending on A_0, A_α, R and their covariant derivatives.

Therefore from (4.19) the principal part of differential is

$$d\big\|(\mathbb{V}^x)^i y_t^x\big\|^2 = 2\Big\langle(\mathbb{V})^i y, \nabla_\ell^y A_\alpha\big[(\mathbb{V})^i y^\ell\big]\Big\rangle dW^\alpha +$$

$$+\left\{2\left\langle(\mathbb{V})^i y, \nabla_\ell^y \widetilde{A_0}\big[(\mathbb{V})^i y^\ell\big]\right\rangle + \sum_{\alpha=1}^d \left\|\nabla A_\alpha\big[(\mathbb{V})^i y\big]\right\|^2 -\right.$$

$$\left. - \sum_{\alpha=1}^d \Big\langle R(A_\alpha, (\mathbb{V})^i y)A_\alpha, (\mathbb{V})^i y\Big\rangle\right\} dt +$$

$$+ \sum_{j_1+\ldots+j_s=i,\ s\geq 2} \Big\langle(\mathbb{V}^x)^i y,\ \Big\{K^1_{j_1,\ldots,j_s,\alpha}\big((\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y\big)\,dW^\alpha +$$

$$+ K^2_{j_1,\ldots,j_s}\big((\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y)dt\Big\}\Big\rangle, \tag{4.28}$$

i.e., the dissipativity condition arises in the principal part. Like before the coefficients K^1, K^2 depend on covariant derivatives of A_0, A_α, R and display symmetry (4.1).

Now we can turn to the estimation of (4.12) – (4.14).

1. Using asymptotic (4.28) we see that terms in (4.12) lead to the dissipativity condition (4.5) in principal part with some constants and additional terms with lower order variations

$$h(0) + \int_0^t \mathbf{E}\Big\{p(\rho^2(y_s^x,z))(2q\|(\mathbb{V}^x)^i y_s^x\|^{2(q-1)} d\big\|(\mathbb{V}^x)^i y_s^x\big\|^2 +$$

$$+ q(2q-2)\big\|(\mathbb{V}^x)^i y_s^x\big\|^{2(q-2)} d\Big[\big\|(\mathbb{V}^x)^i y_s^x\big\|^2, \big\|(\mathbb{V}^x)^i y_s^x\big\|^2\Big] \leq$$

$$\leq K\mathbf{E}\int_0^t p(\rho^2(y_t^x,z))\big\|(\mathbb{V})^i y_t^x\big\|^{2(q-1)} \{\text{dissipativity}\}_{C,C'}((\mathbb{V})^i y_t^x, (\mathbb{V})^i y_t^x)dt +$$

$$+ \sum_{j_1+\ldots+j_s=i,\ s\geq 2} \mathbf{E}\int_0^t p\big(\rho^2(y_t^x,z)\big)\big\|(\mathbb{V})^i y_t^x\big\|^{2(q-1)} \times$$

$$\times \big\langle(\mathbb{V})^i y, K_{j_1,\ldots,j_s}((\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y)\big\rangle\, dt \tag{4.29}$$

with coefficients K like before.

2. Term in (4.13) is transformed by monotonicity and polynomiality of p $\big(\exists\, C$: $p''(u)u \leq Cp'(u)\big)$

$$\int_0^1 \mathbf{E}\big\|(\mathbb{V})^i y\big\|^{2q}\Big\{p'(\rho^2(y,z))d\rho^2(y,z) + \frac{1}{2}p''(\rho^2(y,z))d\big[\rho^2(y,z),\rho^2(y,z)\big]\Big\} =$$

$$= \int_0^1 \mathbf{E}\big\|(\mathbb{V})^i y\big\|^{2q}\Big\{p'(\rho^2(y,z))L^1\rho^2(y,z) +$$

$$+\frac{1}{2}p''(\rho^2(y,z))\rho^2(y,z)\frac{1}{\rho^2(y,z)}\sum_{\alpha=1}^{d}\left(A^1_\alpha\rho^2(y,z)\right)^2\Big\}dt \le$$

$$\le \int_0^t \mathbf{E}\big\|(\mathbb{W})^i y\big\|^{2q} p'(\rho^2(y,z))\Big\{L^1\rho^2(y,z)+\frac{C}{\rho^2(y,z)}\sum_{\alpha=1}^{d}\left(A^1_\alpha\rho^2(y,z)\right)^2\Big\}dt. \quad (4.30)$$

Then we apply results of [21] about upper estimates on second-order operators, acting on metric function.

Theorem 4.2 [21]. *Suppose that the generalized coercitivity and dissipativity conditions* (4.4), (4.5) *hold. Then there is constant K such that*

$$L^1\rho^2(x,y)\le K(1+\rho^2(x,y)). \quad (4.31)$$

Moreover, for any C there exists K_C such that

$$L^1\rho^2(x,y)+C\sum_{\alpha=1}^{d}\frac{(A^1_\alpha\rho^2(x,y))^2}{\rho^2(x,y)}\le K_C(1+\rho^2(x,y)). \quad (4.32)$$

3. Using representation (4.28) and (4.11) we find the principal asymptotic of (4.14). By (4.27)

$$\Big|p'(\rho^2)\,\big\|(\mathbb{W})^i y\big\|^{2(q-1)}d\Big[\rho^2,\big\|(\mathbb{W})^i y\big\|^2\Big]\Big| =$$

$$=\Big|p'(\rho^2)\big\|(\mathbb{W})^i y\big\|^{2(q-1)}\sum_{\alpha=1}^{d}A^1_\alpha\rho^2\cdot 2\Big\langle(\mathbb{W})^i y,\nabla A_\alpha\left[(\mathbb{W})^i y\right]+$$

$$+\sum_{j_1+\ldots+j_s,\ s\ge 2}K'_{j_1,\ldots,j_s}\left((\mathbb{W})^{j_1}y,\ldots,(\mathbb{W})^{j_s}y\right)\Big\rangle| \le$$

$$\le p'(\rho^2)\big\|(\mathbb{W})^i y\big\|^{2q}\sum_{\alpha=1}^{d}\frac{(A^1_\alpha\rho^2))^2}{\rho^2}+$$

$$+\,p'(\rho^2)\rho^2\big\|(\mathbb{W})^i y\big\|^{2(q-1)}\times$$

$$\times\Big\|\nabla A_\alpha\left[(\mathbb{W})^i y\right]+\sum_{j_1+\ldots+j_s,\ s\ge 2}K'_{j_1,\ldots,j_s}\left((\mathbb{W})^{j_1}y,\ldots,(\mathbb{W})^{j_s}y\right)\Big\|^2. \quad (4.33)$$

The first term is added to (4.30), after that (4.32) is applied. The second term is combined with terms in (4.12), (4.29), leading to the dissipativity condition with modified constants.

4. Applying coercitivity and dissipativity (4.4), (4.5) we finally come to estimate

$$h(t)=\mathbf{E}p(\rho^2(y^x_t,z))\|(\mathbb{W})^i y^x_t\|^{2q}\le h(0)+C\int_0^t h(t)dt+$$

$$+\sum_{j_1+\ldots+j_s,\ s\ge 2}\int_0^t \mathbf{E}p(\rho^2(y,z))\big\|(\mathbb{W})^i y\big\|^{2(q-1)}\times$$

$$\times K'_{i;j_1,\ldots,j_s}\big((\mathbb{V})^i y; (\mathbb{V})^{j_1} y, \ldots, (\mathbb{V})^{j_s} y\big)dt,$$

where coefficient K' depends quadratically on lower order variations for the case of (4.33).

It remains to apply estimates (4.6) and symmetry (4.1). By inequality $|x^{q-1}y| \leq \leq \dfrac{|x|^q}{q} + (q-1)\dfrac{|y|^q}{q}$ we have for $a = 1, 2$

$$\mathbf{E}p(\rho^2)\big\|(\mathbb{V})^i y\big\|^{2(q-1)} K_{i;j_1,\ldots,j_s}\big((\mathbb{V})^i y; (\mathbb{V})^{j_1} y, \ldots, (\mathbb{V})^{j_s} y\big) \leq$$

$$\leq \mathbf{E}p(\rho^2)(1+\rho^2)^{\mathbf{k}}\big\|(\mathbb{V})^i y\big\|^{2q-a}\big\|(\mathbb{V})^{j_1} y\big\|^a \ldots \big\|(\mathbb{V})^{j_s} y\big\|^a \leq$$

$$\leq C\mathbf{E}p\big\|(\mathbb{V})^i y\big\|^{2q} + C'\mathbf{E}p(\rho^2)(1+\rho^2)^{2q\mathbf{k}}\big\|(\mathbb{V})^{j_1} y\big\|^{2q} \ldots \big\|(\mathbb{V})^{j_s} y\big\|^{2q}.$$

The first term is already of necessary form, to transform the last term we recall that $2q = m/i$ (4.2), so

$$\|x_{j_1}\|^{m/i} \ldots \|x_{j_s}\|^{m/i} = \Big(\|x_{j_1}\|^{m/j_1}\Big)^{j_1/i} \ldots \Big(\|x_{j_s}\|^{m/j_s}\Big)^{j_s/i}.$$

Then the nonlinear hierarchies of polynomials (4.7) give

$$p_i(\rho^2)(1+\rho^2)^{\mathbf{k}m/i}\big\|(\mathbb{V})^{j_1} y\big\|^{m/i} \ldots \big\|(\mathbb{V})^{j_s} y\big\|^{m/i} \leq$$

$$\leq (p_{j_1}(\rho^2)\big\|(\mathbb{V})^{j_1} y\big\|^{m/j_1})^{j_1/i} \ldots (p_{j_s}(\rho^2)\big\|(\mathbb{V})^{j_s} y\big\|^{m/j_s})^{j_s/i} \leq$$

$$\leq \frac{j_1}{i} p_{j_1}(\rho^2)\big\|(\mathbb{V})^{j_1} y\big\|^{m/j_1} + \ldots + \frac{j_s}{i} p_{j_s}(\rho^2)\big\|(\mathbb{V})^{j_s} y\big\|^{m/j_s},$$

i.e., the differential of each term in (4.2) is estimated by terms of (4.2)

$$h_i(t) = \mathbf{E}p_i(\rho^2)\big\|(\mathbb{V})^i y\big\|^{q/i} \leq h_i(0) + \text{const}\int_0^t r_n(y,s)ds.$$

The theorem is proved.

5. C^∞ regular dependence of diffusion process y_t^x on initial data. Applications to the regularity properties of semigroups. Turning to the questions of existence, uniqueness and differentiability of variational equations with respect to the initial data, one can show these properties under conditions (4.4) – (4.6).

Theorem 5.1. *Under conditions* (4.4) – (4.6) *process y_t^x is C^∞ differentiable with respect to the initial data. Its variations $(\mathbb{V}^x)y_t^x$ represent strong solutions to variational systems* (3.7), (3.8).

Proof. First of all, by Theorem 3.1 and asymptotics (4.27), variational equation on process $(\mathbb{V}^x)^i y_t^x$ represents nonautonomous and inhomogeneous equation on variable $(\mathbb{V}^x)^i y_t^x$, if all lower order variations $(\mathbb{V}^x)^j y_t^x$, $j < i$, are already constructed. The behaviour of nonautonomous part is controlled by coercitivity and dissipativity condition. In a similar way the nonlinear symmetries (4.1) and polynomial behaviour of coefficients (4.6) give a set of optimal estimates on inhomogeneous part, like (4.8). Therefore, like in [15, 17, 21, 24, 25], variational processes $(\mathbb{V}^x)^i y_t^x$, $i \geq 1$, are constructed as strong solutions to systems (3.7), (3.8).

To prove C^∞ differentiability of process y_t^x on initial data, it only remains to show that the solutions of variational equations (3.7), (3.8) represent higher-order $\mathbb{V}$-derivatives of process y_t^x, i.e., its differentiability. Like in [15, 17, 21, 24, 25] this can be obtained by application of nonlinear symmetries (4.1) in a recurrent on the order of differentiation way.

Because we consider the finite-dimensional situation, we can also apply the stopping times techniques, e.g. [26]. They guarantee that derivatives of process with locally C^∞ coefficients are represented as solutions of corresponding variational equations before exit times, i.e., give the required statement.

The theorem is proved.

Now we can turn to the applications of nonlinear estimate (4.8) to the regular properties of semigroup. We construct the spaces of continuously differentiable functions, that are preserved under the action of diffusion semigroup $(P_t f)(x) = \mathbf{E} f(y_t^x)$. Because of globally non-Lipschitz coefficients, such semigroups fail the strong continuity in time property even in space of continuous bounded functions $C_b(M)$. Therefore the application of operator techniques of semigroups theory and corresponding constructions of functional spaces, does not seem to be adapted to this case.

To solve this problem we use pure stochastic representations (2.13). Because in formula (2.13) the derivatives of semigroup are related with derivatives of function via kernels, represented by higher-order $\mathbb{\nabla}$-derivatives of process y_t^x, the nonlinear estimates (4.8) help to solve the situation. Moreover, the structure of topologies in these spaces is influenced by nonlinearity parameters of initial diffusion equation and geometry of manifold, see also [27].

Definition 5.1. *Let $q_0, q_1, \ldots, q_n \geq 1$ be a family of monotone functions of polynomial behaviour, that fulfill*

$$\forall i \geq 1 \quad q_i(b)(1+b)^{\mathbf{k}} \leq q_{i+1}(b) \quad \forall b \geq 0. \tag{5.1}$$

Function $f \in C_{\vec{q}}^n(M)$ iff it is n-times continuously covariantly differentiable and the norm is finite

$$\|f\|_{C_{\vec{q}}^n(M)} = \max_{i=0,\ldots,n} \sup_{x\in M} \frac{\|(\nabla^x)^i f(x)\|}{q_i(\rho^2(x,z))}. \tag{5.2}$$

Due to the triangle inequalities for metric and properties of functions q_i the choice of some point z above does not influence on the topology of space $C_{\vec{q}}^n(M)$, but only the choice of norm.

Next theorem is an application of nonlinear estimate (4.8) to the smooth properties of diffusion semigroups.

Theorem 5.2. *Suppose conditions* (4.4) – (4.6) *hold. Then there is parameter* $\mathbf{k}$ *such that for weights $(q_0, \ldots, q_n)$* (5.1) *the space $C_{\vec{q}}^n(M)$ is preserved under the action of semigroup*

$$\forall t \geq 0 \quad P_t \colon C_{\vec{q}}^n(M) \to C_{\vec{q}}^n(M)$$

and the quasicontractive estimate holds: *there are constants K, M such that*

$$\|P_t f\|_{C_{\vec{q}}^n(M)} \leq K e^{Mt} \|f\|_{C_{\vec{q}}^n(M)} \quad \forall f \in C_{\vec{q}}^n(M). \tag{5.3}$$

Proof. First recall that by Theorem 2.1 the covariant derivatives of semigroup are related with the covariant derivatives of function via the kernels, given by variational processes $(\mathbb{\nabla}^x)^i y_t^x$

$$(\nabla^x)^i P_t f(x) = \sum_{j_1+\ldots+j_s=i} \mathbf{E} \left\langle (\nabla^y)^s f(y_t^x),\ (\mathbb{\nabla}^x)^{j_1} y_t^x \otimes \ldots \otimes (\mathbb{\nabla}^x)^{j_s} y_t^x \right\rangle. \qquad (5.4)$$

To estimate these kernels we use nonlinear estimate (4.8). Let us first note that by Definition 1.1 (1.2) the initial data for variations are:

$$\mathbb{\nabla}_k^x (y_t^x)^m \Big|_{t=0} = \frac{\partial x^m}{\partial x^k} = \delta_k^m,$$

$$\mathbb{\nabla}_{kj}^x (y_t^x)^m \Big|_{t=0} = \mathbb{\nabla}_k^x (\mathbb{\nabla}_j^x y^m) \Big|_{t=0} = \partial_k (\delta_j^m) - \Gamma_{k\ j}^{\ h}(x) \delta_h^m + \Gamma_{p\ q}^{\ m}(y_0^x) \delta_j^p \delta_k^q = 0$$

and

$$(\mathbb{\nabla}^x)^i y_t^x \Big|_{t=0} = 0 \quad \forall i \geq 1.$$

Choose a system of weights

$$\forall j = 1, \ldots, n \quad \widetilde{p}_j(x) = P(x)(1+x)^{\mathbf{k}q(1/j-1/n)}$$

that fulfills hierarchy (4.7). For this choice $\widetilde{p}_n = P$ and nonlinear estimate (4.8) has form

$$\mathbf{E} P\big(\rho^2(y_t^x, z)\big) \big\| (\mathbb{\nabla})^n y_t^x \big\|^{q/n} \leq e^{Mt} \rho_n(y, 0) =$$
$$= e^{Mt} P(\rho^2(x,z)) \big(1 + \rho^2(x,z)\big)^{\mathbf{k}q(n-1)/n}. \qquad (5.5)$$

Now we estimate derivatives (5.4) in topologies $C_{\vec{q}}^n(M)$:

$$\frac{\|(\nabla^x)^i P_t f(x)\|_{T_x^{(0,i)}}}{q_i(\rho^2(x,z))} \leq$$

$$\leq \sum_{j_1+\ldots+j_s,\ s\geq 1} \frac{\|\mathbf{E} < (\nabla^y)^s f(y_t^x), (\mathbb{\nabla}^x)^{j_1} y_t^x \otimes \ldots \otimes (\mathbb{\nabla}^x)^{j_s} y_t^x >_{T_y^{(0,i)}} \|_{T_x^{(0,i)}}}{q_i(\rho^2(x,z))} \leq$$

$$\leq \sum_{j_1+\ldots+j_s,\ s\geq 1} \left(\sup_{y_t^x \in M} \frac{\|(\nabla^y)^s f(y_t^x)\|_{T_y^{(0,s)}}}{q_s(\rho^2(y_s^x, z))} \right) \times$$
$$\times \frac{\mathbf{E} q_s(\rho^2(y_s^x, z)) \|(\mathbb{\nabla}^x)^{j_1} y_t^x\| \ldots \|(\mathbb{\nabla}^x)^{j_s} y_t^x\|}{q_i(\rho^2(x,z))} \leq$$

$$\leq \sum_{j_1+\ldots+j_s,\ s\geq 1} \|f\|_{C_{\vec{q}}^n} \frac{\prod_{\ell=1}^s (\mathbf{E} q_s(\rho^2(y_s^x, z)) \|(\mathbb{\nabla}^x)^{j_\ell} y_t^x\|^{i/j_\ell})^{j_\ell/i}}{q_i(\rho^2(x,z))}.$$

Here we substituted an intermediate weight $q_s(y)$ and at last step applied Hölder inequality.

The last fraction is estimated by (5.5)

$$\frac{1}{q_i(\rho^2(x,z))} \prod_{\ell=1}^s (\mathbf{E} q_s(\rho^2(y_s^x, z)) \|(\mathbb{\nabla}^x)^{j_\ell} y_t^x\|^{i/j_\ell})^{j_\ell/i} \leq$$

$$\leq \frac{1}{q_i(\rho^2(x,z))} \prod_{\ell=1}^s (e^{Mt} q_s(\rho^2(x,z))(1+\rho^2(x,z))^{\mathbf{k}i(j_\ell-1)/j_\ell})^{j_\ell/i} =$$

$$= \text{const}\, e^{Mt} \frac{q_s(\rho^2(x,z))(1+\rho^2(x,z))^{\mathbf{k}(i-s)}}{q_i(\rho^2(x,z))} \leq \text{const}\, e^{Mt},$$

where we applied hierarchy (5.1). Above we used that $q = i$, $n = j_\ell$ in notations of (4.2) and that $j_1 + \ldots + j_s = i$. The last inequality follows from hierarchy (5.1). This gives statement (5.3).

To obtain the continuous differentiability of semigroup on x it is sufficient, similar to [15, 17, 21, 24, 25], to use representations (2.13). The C^∞-differentiability of process y_t^x implies pathwise weak relation between solution of initial equation and first variation process

$$f(y_t^x) - f(y_t^y) = \int\limits_y^x \left\langle \nabla^{y_t^z} f(y_t^z), \frac{\partial y_t^z}{\partial z} \right\rangle dz$$

and similar relations for higher-order variations, e.g. [25] (Theorem 3.4). After that one takes the expectation with respect to the random parameter to obtain

$$P_t f(x) - P_t f(y) = \mathbf{E} f(y_t^x) - \mathbf{E} f(y_t^y) =$$

$$= \mathbf{E} \int\limits_y^x \left\langle \nabla^{y_t^z} f(y_t^z), \frac{\partial y_t^z}{\partial z} \right\rangle dz = \int_y^x \left(\mathbf{E} \left\langle \nabla^{y_t^z} f(y_t^z), \frac{\partial y_t^z}{\partial z} \right\rangle \right) dz$$

and similar relations for the higher-order derivatives.

Therefore the increments of semigroup are represented as integrals $\int_y^x$ of aggregates in the right-hand side of (2.13) and these expressions form the derivatives of semigroup. Final conclusion about continuous differentiability of semigroup follows from estimates on the continuity in mean of variational processes with respect to the initial data. This fact can be proved in a similar to nonlinear estimate (4.8) way, by application of symmetries (4.1), e.g. [15, 17, 21, 24, 25].

The theorem is proved.

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