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MULTILAYER STRUCTURES OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS OF EULER TYPE AND THEIR APPLICATION TO NONLINEAR OSCILLATIONS

БАГАТОШАРОВІ СТРУКТУРИ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ ТИПУ ЕЙЛЕРА ТА ЇХ ЗАСТОСУВАННЯ ДО НЕЛІНІЙНИХ КОЛИВАНЬ

The purpose of this paper is to present new oscillation theorems and nonoscillation theorems for the nonlinear Euler differential equation $t^2x'' + g(x) = 0$. Here we assume that $xg(x) > 0$ if $x \neq 0$, but we do not necessarily require that $g(x)$ be monotone increasing. The obtained results are best possible in a certain sense. To establish our results, we use Sturm's comparison theorem for linear Euler differential equations and phase plane analysis for a nonlinear system of Liénard type.

Наведено нові осциляційні та неосциляційні теореми для нелінійного диференціального рівняння Ейлера $t^2x'' + g(x) = 0$, де припускається, що $xg(x) > 0$ при $x \neq 0$, але вимога про монотонне зростання $g(x)$ не є обов'язковою. Одержані результати є найкращими у певному сенсі. Для їх встановлення використано порівняльну теорему Штурма для лінійних диференціальних рівнянь Ейлера та фазовий площинний аналіз для нелінійної системи типу Льєнарда.

1. Introduction and motivation. Let $f(t)$ be a continuous function defined on $[T, \infty)$ for some $T > 0$. The function $f(t)$ is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $f(t_n) = 0$. Otherwise, $f(t)$ said to be *nonoscillatory*.

A class of linear differential equations of Euler type has a multilayer structure. To explain this fact, we first consider the equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\lambda}{(\log t)^2}\right) y = 0, \tag{1.1}$$

where $' = d/dt$ and λ is a positive parameter. Eq. (1.1) is called the Riemann – Weber version of the Euler differential equation (refer to [1 – 4]). All nontrivial solutions of Eq. (1.1) are oscillatory if and only if $\lambda > 1/4$, because Eq. (1.1) has the general solution

$$y(t) = \begin{cases} \sqrt{t}\{K_1(\log t)^z + K_2(\log t)^{1-z}\} & \text{if } \lambda \neq 1/4, \\ \sqrt{t\log t}\{K_3 + K_4 \log(\log t)\} & \text{if } \lambda = 1/4, \end{cases} \tag{1.2}$$

where K_i, $i = 1, 2, 3, 4$, are arbitrary constants and z is the root of

$$z^2 - z + \lambda = 0. \tag{1.3}$$

Hence, for Eq. (1.1) the critical value of λ is $1/4$. Such a number is generally called the *oscillation constant*.

Letting $s = \log t$ and $u(s) = y(t)/\sqrt{t}$, we can reduce Eq. (1.1) to the basic Euler differential equation

$$\ddot{u} + \frac{\lambda}{s^2} u = 0, \tag{1.4}$$

where $\dot{} = d/ds$. It is well-known that the condition $\lambda > 1/4$ is necessary and sufficient for all nontrivial solutions of Eq. (1.4) to be oscillatory (for example, see [5 –

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7]). In other words, the oscillation constant for Eq. (1.4) is also $1/4$. Let us add the perturbation $\lambda/(s \log s)^2 u$ to the critical case of Eq. (1.4), namely, $\ddot{u} + u/(4s^2) = 0$. Then we get

$$\ddot{u} + \frac{1}{s^2}\left(\frac{1}{4} + \frac{\lambda}{(\log s)^2}\right) u = 0, \tag{1.5}$$

which has the same form of Eq. (1.1). Taking account of this relation between Eqs. (1.1) and (1.4), we may regard Eqs. (1.1) and (1.4) as the first and the second stages of linear differential equations of Euler type, respectively. Then what is the third stage? By putting $t = e^s$ and $y(t) = \sqrt{e^s}\, u(s)$, Eq. (1.5) is transformed into the equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{1}{4(\log t)^2} + \frac{\lambda}{(\log t)^2(\log(\log t))^2}\right) y = 0. \tag{1.6}$$

It is safe to say that Eq. (1.6) is the third stage of Euler's differential equations. Repeating the same transformation, we can derive the nth stage of linear differential equations of Euler type (for details, see [8 – 10]). From the reason above, we see that linear differential equations of Euler type have a multilayer structure.

The authors [9, 10], have compared the solutions of Eq. (1.6) or the nth stage of Euler's differential equations with those of the nonlinear equation

$$x'' + \frac{1}{t^2} g(x) = 0, \tag{1.7}$$

where $g(x)$ satisfies a suitable smoothness condition for the uniqueness of solutions of the initial value problem and the assumption

$$x g(x) > 0 \quad \text{if} \quad x \neq 0, \tag{1.8}$$

and established some oscillation theorems and nonoscillation theorems for Eq. (1.7). For example, we can state the following results which are complementary to each other.

Theorem A. *Assume* (1.8) *and suppose that there exists a* $\lambda > 1/4$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{1}{4(\log x^2)^2} + \frac{\lambda}{(\log x^2)^2(\log(\log x^2))^2}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of Eq.* (1.7) *are oscillatory.*

Theorem B. *Assume* (1.8) *and suppose that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{1}{4(\log x^2)^2} + \frac{1}{4(\log x^2)^2(\log(\log x^2))^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all nontrivial solutions of Eq.* (1.7) *are nonoscillatory.*

***Remark* 1.1.** We can prove that all solutions of Eq. (1.7) exist in the future under the assumption (1.8) (for the proof, see [11]). Hence, it is worth while to discuss whether all nontrivial solutions of Eq. (1.7) are oscillatory or nonoscillatory.

As mentioned above, Euler's differential equations have the multilayer structure which is built up of stages such as Eqs. (1.4), (1.1) and (1.6). A natural question now arises. Is the multilayer structure unique? The answer is a "no". For some $a_1 > 0$, let $t = e^s/a_1$ and $y(t) = \sqrt{e^s}\, u(s)$. Then Eq. (1.4) is transferred to the equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\lambda}{(\log a_1 t)^2}\right) y = 0. \tag{1.9}$$

Rewrite t and y in Eq. (1.9) as s and u, respectively. Then the transformation $t = e^s/a_2$ with $a_2 > 0$ yields the equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{1}{4(\log a_2 t)^2} + \frac{\lambda}{(\log a_2 t)^2(\log(a_1 \log a_2 t))^2}\right) y = 0, \tag{1.10}$$

where $y(t) = \sqrt{e^s}\,u(s)$. Using the same process infinitely many times, we can make another multilayer structure of linear differential equations of Euler type. For further details, see the final section.

It is easy to check that Eq. (1.10) has the general solution

$$y(t) = \begin{cases} \sqrt{t \log a_2 t}\{K_1(\log(a_1 \log a_2 t))^z + K_2(\log(a_1 \log a_2 t))^{1-z}\} & \text{if } \lambda \neq 1/4, \\ \sqrt{t \log a_2 t \log(a_1 \log a_2 t)}\{K_3 + K_4 \log(\log(a_1 \log a_2 t))\} & \text{if } \lambda = 1/4, \end{cases}$$

where K_i, $i = 1, 2, 3, 4$, are arbitrary constants and z is the root of Eq. (1.3). In case $\lambda > 1/4$, Eq. (1.3) has conjugate roots $z = 1/2 \pm i\alpha$, where $\alpha = \sqrt{\lambda - 1/4}$. Hence, by Euler's formula, the real solution can be represented as

$$y(t) = \sqrt{t \log a_2 t \log(a_1 \log a_2 t)}\{k_1 \cos(\alpha \log(\log(a_1 \log a_2 t))) + \\ + k_2 \sin(\alpha \log(\log(a_1 \log a_2 t)))\}$$

for some $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$. If $(k_1, k_2) = (0, 0)$, then $y(t)$ is the trivial solution. On the other hand, if $(k_1, k_2) \neq (0, 0)$, then

$$y(t) = k_3 \sqrt{t \log a_2 t \log(a_1 \log a_2 t)}\, \sin(\alpha \log(\log(a_1 \log a_2 t)) + \beta). \tag{1.11}$$

where $k_3 = \sqrt{k_1^2 + k_2^2}$, $\sin\beta = k_1/k_3$ and $\cos\beta = k_2/k_3$.

Comparing the solutions of Eq. (1.7) with those of Eq. (1.10), we have the following pair of an oscillation theorem and a nonoscillation theorem.

Theorem 1.1. *Assume* (1.8) *and suppose that there exists a* $\lambda > 1/4$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{1}{4(\log a_2 x^2)^2} + \frac{\lambda}{(\log a_2 x^2)^2(\log(a_1 \log a_2 x^2))^2}$$

for $|x|$ *sufficiently large, where* a_1 *and* a_2 *are arbitrary positive numbers. Then all nontrivial solutions of Eq.* (1.7) *are oscillatory.*

Theorem 1.2. *Assume* (1.8) *and suppose that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{1}{4(\log a_2 x^2)^2} + \frac{1}{4(\log a_2 x^2)^2(\log(a_1 \log a_2 x^2))^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large, where* a_1 *and* a_2 *are arbitrary positive numbers. Then all nontrivial solutions of Eq.* (1.7) *are nonoscillatory.*

Remark **1.2.** Let us take $0 < a_i < 1$, $i = 1, 2$. Then we find that Theorem 1.2 is superior to Theorem B.

2. Oscillation theorem. In this section, we will prove Theorem 1.1. To this end, we need the following results (refer to [10, 12 – 15]).

Lemma 2.1. *Let* $f(t)$ *be a positive* C^2*-function defined on* $[T, \infty)$ *for some* $T > 0$. *If the second derivative of* $f(t)$ *is negative for* $t \geq T$, *then* $f(t)$ *is nondecreasing on the interval.*

Lemma 2.2. *Assume* (1.8) *and suppose that Eq.* (1.7) *has an eventually positive solution. Then the positive solution tends to infinity as* $t \to \infty$.

Proof of Theorem **1.1.** By way of contradiction, we suppose that Eq. (1.7) has a nonoscillatory solution $x_0(t)$. Then the solution is positive or negative eventually. We consider only the former, because the latter is carried out in the same way. By assumption, we can find a $\lambda > 1/4$ and an $M > 0$ such that

$$\frac{g(x)}{x} \ \geq \ \frac{1}{4} + \frac{1}{4(\log a_2 x^2)^2} + \frac{\lambda}{(\log a_2 x^2)^2(\log(a_1 \log a_2 x^2))^2} \tag{2.1}$$

for $x > M$. By virtue of Lemma 2.2, there exists a $T > 0$ such that

$$x_0(t) > M \quad \text{for} \quad t \geq T.$$

Changing variable $s = \log t$, we can transform Eq. (1.7) into the equation

$$\ddot{u} - \dot{u} + g(u) = 0, \tag{2.2}$$

where $u(s) = x(t)$. Let $u_0(s)$ be the solution of Eq. (2.2) corresponding to $x_0(t)$. Then we have

$$u_0(\log T) = x_0(T) > M.$$

We next move $u_0(s)$ along the s-axis. Let σ_0 be a number with $0 < \sigma_0 < 2\log M$ and put

$$u_1(s) = u_0(s - \sigma_0 + \log T)$$

for $s \geq \sigma_0$. Needless to say, $u_1(s)$ is also a solution of Eq. (2.2) and it is greater than the number M for $s \geq \sigma_0$.

We will estimate the growth rate of $u_1(s)$ in details. For this purpose, we define

$$\xi(s) = u_1(s)e^{-s/2}.$$

Then, using (2.1), we obtain

$$\ddot{\xi}(s) = \left\{\ddot{u}_1(s) - \dot{u}_1(s) + \frac{1}{4}u_1(s)\right\}e^{-s/2} = \left\{-g(u_1(s)) + \frac{1}{4}u_1(s)\right\}e^{-s/2} \leq$$

$$\leq \left\{-\frac{1}{4(\log a_2 u_1^2(s))^2} - \frac{\lambda}{(\log a_2 u_1^2(s))^2(\log(a_1 \log a_2 u_1^2(s)))^2}\right\}u_1(s)e^{-s/2} < 0$$

for $s \geq \sigma_0$. Hence, it follows from Lemma 2.1 that $\xi(s)$ is nondecreasing for $s \geq \sigma_0$, and therefore, we have

$$\xi(s) \geq \xi(\sigma_0) = u_1(\sigma_0)e^{-\sigma_0/2} = u_0(\log T)e^{-\sigma_0/2} > Me^{-\sigma_0/2} > 1$$

for $s \geq \sigma_0$. From this inequality, we get the lower estimation

$$u_1(s) > e^{s/2} \quad \text{for} \quad s \geq \sigma_0. \tag{2.3}$$

Let us form an upper estimation of $u_1(s)$. By the assumption (1.8), we have

$$\ddot{u}_1(s) - \dot{u}_1(s) = -g(u_1(s)) < 0$$

for $s \geq \sigma_0$. This implies that

$$\dot{u}_1(s) \leq \dot{u}_1(\sigma_0)e^{s-\sigma_0} \quad \text{for} \quad s \geq \sigma_0.$$

Integrate both sides of this inequality to obtain

$$u_1(s) \leq \dot{u}_1(\sigma_0)e^{s-\sigma_0} - \dot{u}_1(\sigma_0) + u_1(\sigma_0).$$

Hence, there exists a $\sigma_1 > \sigma_0$ such that

$$\sqrt{a_2}\,u_1(s) < e^{2s} \quad \text{for} \quad s \geq \sigma_1. \tag{2.4}$$

To get a sharper estimation than (2.4), we define a function $\eta(s)$ by

$$s\eta(s) = u_1(s)e^{-s/2}.$$

Differentiating both sides of the equality twice, we have

$$s\ddot{\eta}(s) + 2\dot{\eta}(s) = \left\{\ddot{u}_1(s) - \dot{u}_1(s) + \frac{1}{4}u_1(s)\right\}e^{-s/2}.$$

Hence, together with (2.1), (2.3) and (2.4), we obtain

$$\frac{d}{ds}(s^2\dot{\eta}(s)) = \left\{\ddot{u}_1(s) - \dot{u}_1(s) + \frac{1}{4}u_1(s)\right\}se^{-s/2} = \left\{-g(u_1(s)) + \frac{1}{4}u_1(s)\right\}se^{-s/2} \le$$

$$\le \left\{-\frac{1}{4(\log a_2u_1^2(s))^2} - \frac{\lambda}{(\log a_2u_1^2(s))^2(\log(a_1\log a_2u_1^2(s)))^2}\right\}u_1(s)se^{-s/2} \le$$

$$\le -\frac{1}{4(\log e^{4s})^2}s < -\frac{1}{64s}$$

for $s \ge \sigma_1$. From this inequality, we get

$$s^2\dot{\eta}(s) \le -\frac{1}{64}\log\frac{s}{\sigma_1} + \sigma_1^2\dot{\eta}(\sigma_1) \quad \text{for} \quad s \ge \sigma_1,$$

and therefore, there exists a $\sigma_2 > \sigma_1$ such that $\dot{\eta}(s) < 0$ for $s \ge \sigma_2$. Hence, we obtain the upper estimation

$$u_1(s) \le \frac{u_1(\sigma_2)}{\sigma_2 e^{\sigma_2/2}}se^{s/2} \quad \text{for} \quad s \ge \sigma_2. \tag{2.5}$$

We now consider the function

$$y(t) = \sqrt{t\log a_2t\log(a_1\log a_2t)}\sin\left(\sqrt{\lambda - 1/4}\log(\log(a_1\log a_2t))\right).$$

Then, as shown in Section 1, the function is an oscillatory solution of Eq. (1.10) because $\lambda > 1/4$ (we may take $k_3 = 1$ and $\beta = 0$ in the representation (1.11). It is clear that $y(t)$ has infinitely many zeros

$$e_m = \frac{1}{a_2}\exp\left(\frac{1}{a_1}\exp\left(\exp\frac{\pi m}{\sqrt{\lambda - 1/4}}\right)\right)$$

for $m \in \mathbb{N}$. Let $s_m = \log e_m$. Then s_m tends to infinity as $m \to \infty$. We can easily check that

$$\frac{e^{s_m/2}}{s_{m+1}} \to \infty \quad \text{as} \quad m \to \infty.$$

Hence, we can choose an $m_0 \in \mathbb{N}$ so that

$$\sigma_2 < s_{m_0} \quad \text{and} \quad \frac{u_1(\sigma_2)}{\sigma_2} < \frac{e^{s_{m_0}/2}}{s_{m_0+1}}. \tag{2.6}$$

For the sake of simplicity, let $\sigma_3 = s_{m_0}$ and $\sigma_4 = s_{m_0+1}$. Note that $\sigma_2 < \sigma_3 < \sigma_4$ and points e^{σ_3} and e^{σ_4} are two successive zeros of $y(t)$.

We translate the positive solution $u_1(s)$ of Eq. (2.2). Let

$$u_2(s) = u_1(s - \sigma_3 + \sigma_2)$$

for $s \ge \sigma_3 - \sigma_2 + \sigma_0$. From (2.5) and (2.6), we see that

$$u_2(s) < \frac{u_1(\sigma_2)}{\sigma_2 e^{\sigma_2/2}}(s - \sigma_3 + \sigma_2)e^{(s-\sigma_3+\sigma_2)/2} = \frac{u_1(\sigma_2)}{\sigma_2}(s - \sigma_3 + \sigma_2)e^{(s-\sigma_3)/2} <$$

$$< \frac{e^{\sigma_3/2}}{\sigma_4}(s - \sigma_3 + \sigma_2)e^{(s-\sigma_3)/2} = \frac{s - \sigma_3 + \sigma_2}{\sigma_4}e^{s/2}$$

for $s \geq \sigma_3$. Hence, we have

$$u_2(s) < e^{s/2} \quad \text{for} \quad \sigma_3 \leq s \leq \sigma_4. \tag{2.7}$$

Let $x(t)$ be the solution of Eq. (1.7) corresponding to $u_2(s)$. Since $x(t)$ is made by a parallel translation of $x_0(t)$, it follows that

$$x(t) > M \quad \text{for} \quad t \geq e^{\sigma_3}. \tag{2.8}$$

From (2.7), it also turns out that

$$x(t) < \sqrt{t} \quad \text{for} \quad e^{\sigma_3} \leq t \leq e^{\sigma_4}.$$

Hence, by (2.1) again, we have

$$\frac{g(x(t))}{x(t)} \geq \frac{1}{4} + \frac{1}{4(\log a_2 x^2(t))^2} + \frac{\lambda}{(\log a_2 x^2(t))^2 (\log(a_1 \log a_2 x^2(t)))^2} >$$

$$> \frac{1}{4} + \frac{1}{4(\log a_2 t)^2} + \frac{\lambda}{(\log a_2 t)^2 (\log(a_1 \log a_2 t))^2} \tag{2.9}$$

for $e^{\sigma_3} \leq t \leq e^{\sigma_4}$. We may regard $x(t)$ as an eventually positive solution of the linear differential equation

$$x'' + \frac{1}{t^2} \frac{g(x(t))}{x(t)} x = 0.$$

Remember that $y(t)$ is an oscillatory solution of Eq. (1.10), whose successive zeros are e^{σ_3} and e^{σ_4}. Hence, by (2.9) and Sturm's comparison theorem, $x(t)$ has at least one zero between e^{σ_3} and e^{σ_4}. This is a contradiction to (2.8). The proof of Theorem 1.1 is now complete.

3. Nonoscillation theorem. As has been mentioned in the proof of Theorem 1.1, the change of variable $s = \log t$ transfers Eq. (1.7) to Eq. (2.2), which is equivalent to the system

$$\dot{u} = v + u,$$
$$\dot{v} = -g(u). \tag{3.1}$$

System (3.1) is of Liénard type. Phase plane analysis is frequently made for the purpose of examining the asymptotic behavior of solutions of system (3.1). We call the projection of a positive semitrajectory of system (3.1) onto the phase plane a *positive orbit*.

Suppose that there exists a nontrivial oscillatory solution $x(t)$ of Eq. (1.7). Let t_0 be the initial time of $x(t)$ and let $\{t_n\}$ be the sequence of zeros of $x(t)$. Take $s_0 = \log t_0$ and $\sigma_n = \log t_n$. Let $(u(s), v(s))$ be the solution of system (3.1) corresponding to $x(t)$. Then, it is clear that $u(\sigma_n) = 0$. Taking account of the vector field of system (3.1), we also see that there exists another sequence $\{\tau_n\}$ with $\sigma_n < \tau_n < \sigma_{n+1}$ such that $v(\tau_n) = 0$. To be precise, the positive orbit of system (3.1) starting at $(u(s_0), v(s_0))$ rotates around the origin $(0, 0)$ clockwise. Since $g(x)$ is smooth enough to guarantee the uniqueness of solutions to the initial value problem and system (3.1) is autonomous, any positive orbit of system (3.1) fails to cross itself and all other positive orbits of system (3.1). To sum up, we have the following result.

Lemma 3.1. *Under the assumption* (1.8), *if Eq.* (1.7) *has a nontrivial oscillatory solution, then all nontrivial positive orbits of system* (3.1) *rotate in a clockwise direction about the origin.*

By means of Lemma 3.1 and phase plane analysis for system (3.1), we give the proof of Theorem 1.2.

***Proof of Theorem* 1.2.** We prove only the case that

$$\frac{g(x)}{x} \le \frac{1}{4} + \frac{1}{4(\log a_2 x^2)^2} + \frac{1}{4(\log a_2 x^2)^2(\log(a_1 \log a_2 x^2))^2} \tag{3.2}$$

for $x > 0$ sufficiently large, because the proof of the other case is essentially the same. The proof is by contradiction. Suppose that there exists a nontrivial oscillatory solution of Eq. (1.7). Then, from Lemma 3.1 we see that all positive orbits go around the origin in clockwise order except the trivial positive orbit, namely, the origin.

We choose a number s_0 so large that (3.2) holds for $x > \sqrt{e^{s_0}/a_2}$ and let

$$P = \left(\sqrt{\frac{e^{s_0}}{a_2}}, \left(-\frac{1}{2} + \frac{1}{2s_0} + \frac{1}{2s_0 \log a_1 s_0} \right) \sqrt{\frac{e^{s_0}}{a_2}} \right).$$

Note that the point P belongs to the region $R \overset{\text{df}}{=} \{ (u, v) : -u/2 \le v < 0 \}$. We consider the solution $(u(s), v(s))$ of system (3.1) satisfying the initial condition

$$(u(s_0), v(s_0)) = P. \tag{3.3}$$

Since the positive orbit of system (3.1) corresponding to $(u(s), v(s))$ also rotates about the origin, it meets the straight line $v = -u/2$ infinitely many times. Let $s_1 > s_0$ be the first intersecting time of the positive orbit with the line, and let $Q = (u(s_1), v(s_1))$. Then, taking the vector field of system (3.1) into consideration, we see that the arc PQ of the positive orbit is in the region R. In other words, $-u(s)/2 \le v(s) < 0$ for $s_0 \le s \le s_1$. Hence, we have

$$\dot{u}(s) \ge \frac{u(s)}{2} \quad \text{for} \quad s_0 \le s \le s_1,$$

and therefore,

$$u(s) \ge u(s_0) e^{(s-s_0)/2} = \sqrt{\frac{e^s}{a_2}} \quad \text{for} \quad s_0 \le s \le s_1. \tag{3.4}$$

We consider the function

$$\xi(s) = \frac{v(s)}{u(s)}.$$

Then, the function is defined on an open interval containing $[s_0, s_1]$. Taking notice of (3.3), we see that

$$\xi(s_0) = -\frac{1}{2} + \frac{1}{2s_0} + \frac{1}{2s_0 \log a_1 s_0}.$$

Since the point Q is on the line $v = -u/2$, we also see that

$$\xi(s_1) = -\frac{1}{2}. \tag{3.5}$$

Differentiating $\xi(s)$ and using (3.2) and (3.4), we have

$$\dot{\xi}(s) = -\xi^2(s) - \xi(s) - \frac{g(u(s))}{u(s)} \ge$$

$$\ge -\left(\xi(s) + \frac{1}{2} \right)^2 - \frac{1}{4(\log a_2 u^2(s))^2} - \frac{1}{4(\log a_2 u^2(s))^2 (\log(a_1 \log a_2 u^2(s)))^2} \ge$$

$$\ge -\left(\xi(s) + \frac{1}{2} \right)^2 - \frac{1}{4s^2} - \frac{1}{4s^2(\log a_1 s)^2} \tag{3.6}$$

for $s_0 \le s \le s_1$. To estimate $\xi(s)$, we define

$$\eta(s) = -\frac{1}{2} + \frac{1}{2s} + \frac{1}{2s\log a_1 s}$$

for $s \ge s_0$. It is clear that $\eta(s_0) = \xi(s_0)$. Also, it turns out that $\eta(s)$ satisfies

$$\dot{\eta}(s) = -\left(\eta(s) + \frac{1}{2}\right)^2 - \frac{1}{4s^2} - \frac{1}{4s^2(\log a_1 s)^2}.$$

Hence, together with (3.6), we obtain

$$\xi(s) \ge \eta(s) > -\frac{1}{2} \quad \text{for} \quad s_0 \le s \le s_1.$$

This is a contradiction to (3.5) at $s = s_1$. Thus, all nontrivial solutions of Eq. (1.7) are nonoscillatory. We have completed the proof of Theorem 1.2.

4. Extension to the n-th stage. Having given the proofs of Theorems 1.1 and 1.2, we may now proceed to natural generalizations. We first make a new multilayer structure of linear differential equations of Euler type. Let $\{a_j\}$ be any sequence with $a_j > 0$, $j \in \mathbb{N}$. For $n \in \mathbb{N}$ fixed, we define

$$\log_0^n w = w, \qquad \log_k^n w = \log(a_{n-k} \log_{k-1}^n w), \qquad k = 1, 2, \dots, n-1.$$

We should notice that $\log_k^n w$ depends on n as well as k. Using the terms, we describe two sequence of functions as follows:

$$L_1^n(w) = 1, \qquad L_{k+1}^n(w) = L_k^n(w)\log_k^n w, \qquad k = 1, 2, \dots, n-1,$$

$$S_1(w) = 0, \qquad S_n(w) = \sum_{k=1}^{n-1} \frac{1}{(L_k^n(w))^2}, \qquad n \ge 2.$$

The sequences are well-defined for $w > 0$ sufficiently large. Note that

$$S_{n+1}(w) \not\equiv S_n(w) + \frac{1}{(L_n^n(w))^2}$$

unless $a_1 = a_2 = \dots = a_n$. To be specific,

$$L_1^1(w) = 1, \qquad L_2^2(w) = \log(a_1 w), \qquad L_3^3(w) = \log(a_2 w)\log\big(a_1 \log(a_2 w)\big),$$

$$L_4^4(w) = \log(a_3 w)\log\big(a_2 \log(a_3 w)\big)\log\big(a_1 \log(a_2 \log(a_3 w))\big);$$

$$S_1(w) = 0, \qquad S_2(w) = 1, \qquad S_3(w) = 1 + \frac{1}{(\log(a_2 w))^2},$$

$$S_4(w) = 1 + \frac{1}{(\log(a_3 w))^2} + \frac{1}{(\log(a_3 w))^2(\log(a_2 \log(a_3 w)))^2},$$

and so on.

Consider the linear equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} S_n(t) + \frac{\lambda}{(L_n^n(t))^2}\right) y = 0, \tag{4.1}$$

which coincides with Eqs. (1.9) and (1.10) when $n = 2$ and $n = 3$, respectively. Then we have the following result.

Lemma 4.1. *Let* $n \geq 2$. *Then Eq.* (4.1) *has the general solution*

$$y(t) = \begin{cases} \sqrt{tL_{n-1}^n(t)}\{K_1(\log_{n-1}^n t)^z + K_2(\log_{n-1}^n t)^{1-z}\} & \text{if } \lambda \neq 1/4, \\ \sqrt{tL_n^n(t)}\{K_3 + K_4 \log(\log_{n-1}^n t)\} & \text{if } \lambda = 1/4, \end{cases}$$

where K_i, $i = 1, 2, 3, 4$, *are arbitrary constants and* z *is the root of Eq.* (1.3).

From Lemma 4.1, we see that all nontrivial solutions of Eq. (4.1) are oscillatory if and only if $\lambda > 1/4$. In case $\lambda > 1/4$, $y(t)$ given in Lemma 4.1 is a complex solution. By using Euler's formula, the real solution of Eq. (4.1) can be written in the form

$$y(t) = \sqrt{tL_n^n(t)}\{k_1 \cos(\alpha \log(\log_{n-1}^n t)) + k_2 \sin(\alpha \log(\log_{n-1}^n t))\}, \tag{4.2}$$

where k_i, $i = 1, 2$, are arbitrary constants and $\alpha = \sqrt{\lambda - 1/4} > 0$.

Theorems 4.1 and 4.2 below are proven in the same manner as Theorems 1.1 and 1.2, respectively. We give a very brief outline of their proofs.

Theorem 4.1. *Let* $\{a_j\}$ *be any sequence with* $a_j > 0$, $j \in \mathbb{N}$. *Under the assumption* (1.8), *if there exists a* $\lambda > 1/4$ *such that*

$$\frac{g(x)}{x} \geq \frac{1}{4} S_n(x^2) + \frac{\lambda}{(L_n^n(x^2))^2}$$

for $|x|$ *sufficiently large, then all nontrivial solutions of Eq.* (1.7) *are oscillatory.*

Theorem 4.2. *Let* $\{a_j\}$ *be any sequence with* $a_j > 0$, $j \in \mathbb{N}$. *Under the assumption* (1.8), *if*

$$\frac{g(x)}{x} \leq \frac{1}{4} S_n(x^2) + \frac{1}{4(L_n^n(x^2))^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large, then all nontrivial solutions of Eq.* (1.7) *are nonoscillatory.*

***Outline of the proof of Theorem* 4.1.** By contradiction, we suppose that Eq. (1.7) has an eventually positive solution $x_0(t)$. Let M be a number so large that

$$\frac{g(x)}{x} \geq \frac{1}{4} S_n(x^2) + \frac{\lambda}{(L_n^n(x^2))^2} \tag{4.3}$$

for $x > M$. From Lemma 2.2, we can choose a $T > 0$ such that

$$x_0(t) > M \quad \text{for} \quad t \geq T.$$

Let $u_0(s)$ be the solution of

$$\ddot{u} - \dot{u} + g(u) = 0$$

corresponding to $x_0(t)$ and put

$$u_1(s) = u_0(s - \sigma_0 + \log T)$$

for $s \geq \sigma_0$, where σ_0 is a number with $0 < \sigma_0 < 2\log M$. As in the proof of Theorem 1.1, we can estimate that

$$u_1(s) \leq \frac{u_1(\sigma_2)}{\sigma_2 e^{\sigma_2/2}} s e^{s/2} \quad \text{for} \quad s \geq \sigma_2,$$

where σ_2 is a number with $\sigma_2 > \sigma_0$.

Since $\lambda > 1/4$, Eq. (4.1) has oscillatory solutions of the form (4.2). We select

$$y(t) = \sqrt{tL_n^n(t)} \sin(\sqrt{\lambda - 1/4} \log(\log_{n-1}^n t))$$

from among them. Let e_m be the zeros of $y(t)$. Then we see that

$$e_m = \exp_{n-1}^n\left(\exp\frac{\pi m}{\sqrt{\lambda - 1/4}}\right)$$

for $m \in \mathbb{N}$, where $\{\exp_{n-1}^n t\}$ is a sequence of function as follows:

$$\exp_0^n t = t, \qquad \exp_k^n t = \frac{1}{a_k}\exp(\exp_{k-1}^n t), \qquad k = 1, 2, \ldots, n-1.$$

Let $s_m = \log e_m$. Then we obtain

$$\frac{e^{s_m/2}}{s_{m+1}} \to \infty \quad \text{as} \quad m \to \infty.$$

Hence, there exists an $m_0 \in \mathbb{N}$ such that

$$\sigma_2 < s_{m_0} \quad \text{and} \quad \frac{u_1(\sigma_2)}{\sigma_2} < \frac{e^{s_{m_0}/2}}{s_{m_0+1}}.$$

Put $\sigma_3 = s_{m_0}$ and $\sigma_4 = s_{m_0+1}$. We define

$$u_2(s) = u_1(s - \sigma_3 + \sigma_2)$$

for $s \geq \sigma_3 - \sigma_2 + \sigma_0$. Then, we get the estimation

$$M < u_2(s) < e^{s/2} \quad \text{for} \quad \sigma_3 \leq s \leq \sigma_4.$$

Let $x(t)$ be the solution of Eq. (1.7) corresponding to $u_2(s)$. Then, we can rewrite the above estimation as

$$M < x(t) < \sqrt{t} \quad \text{for} \quad e^{\sigma_3} \leq t \leq e^{\sigma_4}. \tag{4.4}$$

Hence, by (4.3) we have

$$\frac{g(x(t))}{x(t)} > \frac{1}{4}S_n(t) + \frac{\lambda}{(L_n^n(t))^2}$$

for $e^{\sigma_3} \leq t \leq e^{\sigma_4}$. From this inequality and Sturm's comparison theorem, we see that $x(t)$ has at least one zero between e^{σ_3} and e^{σ_4}, which contradicts (4.4). Thus, Theorem 4.1 is now proved.

***Outline of the proof of Theorem* 4.2.** We give only the proof of the case that

$$\frac{g(x)}{x} \leq \frac{1}{4}S_n(x^2) + \frac{1}{4(L_n^n(x^2))^2} \tag{4.5}$$

for $x > 0$ sufficiently large. The proof is by contradiction. Suppose that there exists a nontrivial oscillatory solution of Eq. (1.7). Let

$$P = \left(\sqrt{\frac{e^{s_0}}{a_{n-1}}}, \left(-\frac{1}{2} + \frac{1}{2}\sum_{k=2}^{n}\frac{1}{L_k^n(e^{s_0}/a_{n-1})}\right)\sqrt{\frac{e^{s_0}}{a_{n-1}}}\right),$$

where s_0 is a number so large that (4.5) holds for $\sqrt{e^{s_0}/a_{n-1}}$. Let $(u(s), v(s))$ be the solution of system (3.1) satisfying the initial condition

$$(u(s_0), v(s_0)) = P.$$

We consider the positive orbit of system (3.1) corresponding to $(u(s), v(s))$. Then, from Lemma 3.1 we see that the positive orbit rotates about the origin, and therefore, it meets the straight line $v = -u/2$ infinitely many times. Let $s_1 > s_0$ be the first intersecting time of the positive orbit with the line. Then we have

$$u(s) \;\ge\; u(s_0)e^{(s-s_0)/2} \;=\; \sqrt{\frac{e^s}{a_{n-1}}} \quad \text{for} \quad s_0 \;\le\; s \;\le\; s_1 . \tag{4.6}$$

As in the proof of Theorem 1.2, we compare the function

$$\xi(s) \;=\; \frac{v(s)}{u(s)}$$

with a solution

$$\eta(s) \;=\; -\frac{1}{2} + \frac{1}{2}\sum_{k=2}^{n} \frac{1}{L_k^n(e^s / a_{n-1})}$$

of the equation

$$\dot\eta(s) \;=\; -\left(\eta(s) + \frac{1}{2}\right)^2 - \frac{1}{4}\sum_{k=2}^{n} \frac{1}{(L_k^n(e^s / a_{n-1}))^2} .$$

It follows from (4.5) and (4.6) that

$$\dot\xi(s) \;\ge\; -\left(\xi(s) + \frac{1}{2}\right)^2 - \frac{1}{4}\sum_{k=2}^{n} \frac{1}{(L_k^n(e^s / a_{n-1}))^2}$$

for $s_0 \le s \le s_1$. Since $\eta(s_0) = \xi(s_0)$, we conclude that

$$\xi(s) \;\ge\; \eta(s) \;>\; -\frac{1}{2} \quad \text{for} \quad s_0 \;\le\; s \;\le\; s_1 .$$

This contradicts the fact that $\xi(s_1) = -1/2$. Thus, all nontrivial solutions of Eq. (1.7) are nonoscillatory, thereby completing the proof of Theorem 4.2.

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