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MARKOV UNIQUENESS AND RADEMACHER THEOREM FOR SMOOTH MEASURES ON INFINITE-DIMENSIONAL SPACE UNDER SUCCESSFUL FILTRATION CONDITION*

МАРКОВСЬКА ЄДИНІСТЬ ТА ТЕОРЕМА РАДЕМАХЕРА ДЛЯ ГЛАДКИХ МІР НА НЕСКІНЧЕННОВИМІРНОМУ ПРОСТОРІ ЗА УМОВИ УСПІШНОЇ ФІЛЬТРАЦІЇ

For a smooth measure on an infinite-dimensional space, the successful filtration condition is introduced and Markov uniqueness and Rademacher theorem for measures satisfying such condition are proved. Some sufficient conditions, such as well-known Höegh-Krohn condition, are also considered. Some examples demonstrating connections between these conditions and applications to convex measures are given.

Для гладкої міри на нескінченновимірному просторі введено умову „успішної фільтрації" та доведено марковську єдиність і теорему Радемахера для мір, що задовольняють цю умову. Розглянуто деякі достатні умови, такі як відома умова Хеег-Крона, наведено приклади, що демонструють зв'язок між цими умовами, та застосування до опуклих мір.

Introduction. The aim of this paper is to clarify some questions connected with the notion of stochastic derivative (Sobolev derivative) on an infinite-limensional space with a smooth measure. There are different closely related approaches to define stochastic derivative (see for example [1], Ch. 7, items A.-E.) and the natural question is whether definitions given by these approaches are equivalent. It appears that such an equivalence, say, for most interesting case of W-derivative and G-derivative (exact definitions will be given below), is nontrivial and requires the initial measure to satisfy some structural conditions. The most known is so-called Höegh-Krohn condition (see Definition 2.1 below), which can be regarded as a demand on the measure „to be close to the product-measure".

In this paper, we show that such condition is (is some situations) too restrictive and can be replaced by another one, which we call „successful filtration" condition. We also give class of the measures satisfying „successful filtration" property, for which Höegh-Krohn condition may fail. We show that, under „successful filtration" condition, equivalence of W- and G-derivatives (so-called „Markov uniqueness" property, see next section for a deteiled discussion) holds true. Under the same condition, we also prove the analogue of Rademacher theorem.

The structure of the paper is following. In Section 1 we give main definitions and prove the description of G-derivative in terms of direction-wise Sobolev derivatives. In Section 2, we prove Markov uniqueness property under the „successful filtration" condition and give some sufficient conditions. In Section 3, the analogue of Rademacher theorem is proved. In Section 4, we give some examples demonstrating connections between different conditions sufficient for „successful filtration" property.

1. Stochastic derivatives on a space with a smooth measure. Let the separable Banach space X and the probability measure μ on the Borel σ-algebra $\mathcal{B}(X)$ be fixed. We suppose that there exists a separable Hilbert space H, densely embedded into X by operator $j \in \mathcal{L}(H, X)$, such that the measure μ is logarithmically differentiable in every direction from jH. This means, by the definition, that there exists the linear map (generalized random element)

$$\rho : H \ni h \mapsto (\rho, h) \in L_1(X, \mu)$$

such that for every function f from the set $C^\infty_{0,\,\mathrm{cyl}}(X)$ of the functions of the type

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$$f(\cdot) = F\big(\langle \cdot, x_1^* \rangle, \ldots, \langle \cdot, x_n^* \rangle\big),$$
$$F \in C_0^\infty(\mathbb{R}^n), \qquad x_1^*, \ldots, x_n^* \in X^*, \qquad n \geq 1 \tag{1.1}$$

(smooth cylindrical functions) the following integration by parts formula holds true:

$$E(\nabla_H f, h)_H = -Ef(\rho, h), \qquad h \in H.$$

Here and below we use probabilistic notation $Ef \equiv \int_X f\, d\mu$, ∇_H is the Gatoux derivative w. r. t. H, which is defined for the functions of the type (1.1) as

$$\nabla_H f(\cdot) = \sum_{k=1}^n F_k'\big(\langle \cdot, x_1^* \rangle, \ldots, \langle \cdot, x_n^* \rangle\big) j^* x_k^*.$$

Throughout the paper we suppose that ρ has all moments, which means that $(\rho, h) \in \bigcap_{p \geq 1} L_p(X, \mu)$, $h \in H$. This implies due to the Banach theorem that for every $p \geq 1$ there exists $c_p < +\infty$ such that

$$\|(\rho, h)\|_{L_p(X, \mu)} \leq c_p \|h\|_H, \qquad h \in H.$$

Consider operators ∇_H and $\nabla_h : f \mapsto (\nabla_H f, h)_H$ as the unbounded densely defined operators

$$\nabla_H : L_p(X, \mu) \supset C_{0,\text{cyl}}^\infty(X) \to L_p(X, \mu, H),$$

$$\nabla_h : L_p(X, \mu) \supset C_{0,\text{cyl}}^\infty(X) \to L_p(X, \mu), \qquad p \geq 1.$$

It follows from the integration by parts formula that for every $p \geq 1$ the adjoint operators to ∇_H, ∇_h have domains, which separate points in $L_p(X, \mu, H)$ and $L_p(X, \mu)$ respectively. Further we will need the explicit form of $(\nabla_H)^*$ on a specific set of functionals. Let us denote by $C_{0,\text{cyl}}^\infty(X, H)$ the set of the elements of the type

$$g = \sum_{k=1}^n f_k h_k, \qquad f_k \in C_{0,\text{cyl}}^\infty(X), \qquad h_k \in H, \qquad k = 1, \ldots, n, \quad n \geq 1, \tag{1.2}$$

due to the integration by parts formula the action of the adjoint operator to ∇_H on the element of the type (1.2) is given by

$$(\nabla_H)^* g = \sum_{k=1}^n \big[-(\rho, h_k) f_k - (\nabla_H f_k, h_k)_H\big].$$

Definition 1.1. *Let* $p \geq 1$ *be fixed, then*

a) *the closure* D_p *of* ∇_H *in* L_p *sense is called Sobolev stochastic derivative* (*W-derivative*);

b) *the adjoint operator* D_p^G *to the operator* $(\nabla_H)^*\big|_{C_{0,\text{cyl}}^\infty(X, H)}$ *is called generalized stochastic derivative* (*G-derivative*);

c) *the closure* $D_{p,h}$ *of* ∇_h *in* L_p *is called stochastic derivative in direction* h*. Domains of the corresponding operators are denoted by* $W_p^1(X, \mu)$, $G_p^1(X, \mu)$, $W_{p,h}^1(X, \mu)$.

The following theorem shows that the notions of generalized derivative and derivative in direction h are closely connected.

Theorem 1.1. *The function* f *belongs to* $G_p^1(X,\mu)$ *if and only if it belongs to* $\bigcap_{h\in H} W_{p,h}^1(X,\mu)$ *and there exists an element* $g_f \in L_p(X,\mu,H)$ *such that* $D_{p,h}f = (g_f, h)_H$ *a. s.,* $h\in H$. *If so,* $g_f = D_p^G f$.

Remarks. **1.1.** One can see from the proof, that for a given p the integrability condition on logarithmic derivative can be weakened to the following one:

$$(\rho, h) \in \left[L_p \cap L_q\right](X,\mu), \qquad h\in H, \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

This condition is close to a necessary one, namely for every $q < 2$ one can give an example of one-dimensional measure μ with logarithmic derivative $\rho \in L_p(\mathbb{R},\mu) \setminus L_q(\mathbb{R},\mu)$ such that $W_p^1(\mathbb{R},\mu) \ne G_p^1(\mathbb{R},\mu)$ (such an example the author first knew in a private conversation with A. Yu. Pilipenko).

1.2. For the case $p = 2$ the statement of the theorem can be obtained using the Dirichlet forms technique, namely it follows from Theorem 5.1 [2] (see also Proposition A.1 from the Appendix there for the case $p > 2$). However, for $p \ne 2$ (especially for $p < 2$) it is difficult to apply such technique because the question about connections between W-, G-derivatives and derivative defined via Dirichlet forms approach is complicated (see [1], Ch. 7). Therefore we suppose that the straightforward proof given below is useful.

The proof of Theorem 1.1 is based on a stratification technique. Let $h\in H$ be fixed, let us choose some representation of X in the form of direct sum $Y + \langle jh\rangle$, where $Y \subset X$ is some subspace. Take isomorphism $X \ni x = y + tjh \mapsto (y,t) \in Y\times\mathbb{R}$, let $\pi: X\to Y$ be natural projection and μ^Y is the image of μ under this projection. It is well known that there exists a family $\{\mu_y, y\in Y\}$ of measures on $\mathbb{R}$ such that for every $A\in\mathcal{B}(X)$ the function $y\mapsto \mu_y(\{t \,|\, (y,t)\in A\})$ is measurable and

$$\mu(A) = \int_X \mu_y(\{t \,|\, (y,t)\in A\})\mu^Y(dy).$$

For every function f on $X \equiv Y\times\mathbb{R}$ denote $f_y(\cdot) = f((y,\cdot))$.

Lemma 1.1. *The measure* μ *has logarithmic derivative* $(\rho, h) \in L_q(X,\mu)$ *if and only if for* μ^Y*-almost all* $y\in Y$ *has logarithmic derivative* (*w. r. t. usual differentiation in* $\mathbb{R}$) ρ_y^h *and*

$$\int_Y \int_{\mathbb{R}} \left|\rho_y^h(t)\right|^q \mu_y(dt)\mu^Y(dy) < +\infty.$$

In this case $(\rho,h)(y,t) = \rho_y^h(t)$ *for* μ*-almost all* $(y,t)\in X$.

For the proof of this lemma see [3].

Denote by $G_{p,h}^1(X,\mu)$ the domain of the operator $D_{p,h}^G$, adjoint in $L_p(X,\mu)$ to operator

$$I_h : C_{0,\mathrm{cyl}}^\infty(X,\mu) \ni f \mapsto (\nabla_H)^*(fh) = -(\rho,h)f - (\nabla_H f, h)_H.$$

Due to Lemma 1.1 for μ^Y-almost all $y\in Y$ families $W_p^1(\mathbb{R},\mu_y)$ and $G_p^1(\mathbb{R},\mu_y)$ can be defined analogously to the families $W_{p,h}^1(X,\mu)$ and $G_{p,h}^1(X,\mu)$, denote corresponding analogues of operators $D_{p,h}$, $D_{p,h}^G$ by D_p^y, $D_p^{G,y}$.

Lemma 1.2. 1. *The function* f *belongs to* $W^1_{p,h}(X,\mu)$ *if and only if for* μ^Y-*almost all* $y \in Y$ $f_y \in W^1_p(\mathbb{R}, \mu_y)$ *and*

$$\int_Y \int_{\mathbb{R}} \left| \left[D^y_p f_y \right](t) \right|^p \mu_y(dt) \mu^Y(dy) < +\infty.$$

In this case

$$D_{p,h} f(y,t) = \left[D^y_p f_y \right](t)$$

for μ-*almost all* $(y,t) \in X$.

2. *The function* f *belongs to* $G^1_{p,h}(X,\mu)$ *if and only if for* μ^Y-*almost all* $y \in Y$ $f_y \in G^1_p(\mathbb{R}, \mu_y)$ *and*

$$\int_Y \int_{\mathbb{R}} \left| \left[D^{G,y}_p f_y \right](t) \right|^p \mu_y(dt) \mu^Y(dy) < +\infty.$$

In this case

$$D^G_{p,h} f(y,t) = \left[D^{G,y}_p f_y \right](t)$$

for μ-*almost all* $(y,t) \in X$.

Although the statement of the lemma is one of the main parts of the proof of Theorem 1.1, we omit its proof because it is quite standard. The proof of statement 2 can be given analogously to the proof of Lemma 1.1 (see [3]). The proof of statement 1 can be obtained using the arguments analogous to given in [4].

***Proof of Theorem* 1.1.** Due to Lemma 1.2 it is enough to prove statement of the theorem when $X = \mathbb{R}$ and μ has logarithmic derivative

$$\rho \in \left[L_p \cap L_q \right](\mathbb{R}, \mu), \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

First let us note that for the Lebesgue measure λ^1 classes $W^1_p(\mathbb{R}, \lambda^1)$ and $G^1_p(\mathbb{R}, \lambda^1)$ (in our previous notations) coincide, one can easily obtain this by taking convolutions of functions from $G^1_p(\mathbb{R}, \lambda^1)$ with smooth kernels. Next, note that μ has the density α such that $\alpha^{1/q} \in W^1_q(\mathbb{R}, \lambda^1)$ and $\left[\alpha^{1/q}\right]' = (1/q)\rho\alpha^{1/q}$, this is a corollary of the previous observation and definition of logarithmic derivative. In particular, α is an absolutely continuous function. Analogously, let $f \in G^1_p(\mathbb{R}, \mu)$, then the function $\bar{f} = f \cdot \alpha^{1/p}$ belongs to $W^1_p(\mathbb{R}, \lambda^1)$ and its (Sobolev) derivative is equal $D^G_p f \cdot \alpha^{1/p} - (1/p) f \rho \alpha^{1/p}$. Thus $\bar{f}$ is absolutely continuous on the open set $\{\alpha > 0\}$. Let us represent this set as a family of nonintersecting open intervals $\{(a_k, b_k),\ k \in \mathcal{J}\}$ and show that $f 1\!\!1_{(a_k,b_k)} \in W^1_p(\mathbb{R}, \mu)$ and

$$D^1_p\left[f 1\!\!1_{(a_k,b_k)} \right] = 1\!\!1_{(a_k,b_k)} D^1_p f.$$

Let us study the case $|a_k|, |b_k| < +\infty$ (the case of unbounded interval can be regarded analogously). Let us fix $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon) > 0$ such that

$$\left[\int_{a_k}^{a_k+\delta} + \int_{b_k-\delta}^{b_k} \right] \left[|f(x)|^p + \left| \tilde{D}_p f(x) \right|^p \right] \mu(dx) < \varepsilon$$

and $\delta < (b_k - a_k)/2$. Choose the function $\varphi_\varepsilon \in C_b^\infty(\mathbb{R})$ such that supp $\varphi_\varepsilon \subset (a_k + \delta, b_k - \delta)$ and

$$\left\| \varphi_\varepsilon - \tilde{D}_p f \mathbb{1}_{(a_k+\delta,\, b_k-\delta)} \right\|_{L_p((a_k, b_k),\, \mu)} + $$
$$+ \left\| \varphi_\varepsilon - \tilde{D}_p f \mathbb{1}_{(a_k+\delta,\, b_k-\delta)} \right\|_{L_p\left((a_k+\delta,\, b_k-\delta),\, \lambda^1\right)} < \varepsilon.$$

Then the function

$$f_\varepsilon(x) = f\left(\frac{a_k + b_k}{2}\right) + \int\limits_{(a_k+b_k)/2}^{x} \varphi_\varepsilon(u) \mathbb{1}_{(a_k,\, b_k)}(u)\, du$$

belongs to $C_b^\infty(\mathbb{R})$ and

$$\int\limits_{a_k}^{b_k} |f_\varepsilon(x) - f(x)|^p + \left| f'_\varepsilon(x) - D_p^G f(x) \right|^p \mu(dx) \to 0, \qquad \varepsilon \to 0+.$$

Therefore it is enough to prove that $\mathbb{1}_{(a_k,\, b_k)} \in W_p^1(\mathbb{R}, \mu)$ and $D_p^1 \mathbb{1}_{(a_k,\, b_k)} = 0$. Let us note that due to the integrability condition on ρ we can take in the previous considerations $f = \alpha$. In this case, as far as $\int_{a_k}^{b_k} \alpha'(x)\, dx = 0$, $f_\varepsilon(b_k) \to 0$, $f_\varepsilon(a_k) \to 0$, $\varepsilon \to 0+$.

Thus the function $\psi_1 = \alpha \cdot \mathbb{1}_{(a_k,\, b_k)}$ belongs to the class $W_p^1(\mathbb{R}, \mu)$. Define

$$\beta_{m,n}(t) = \begin{cases} \left(\dfrac{1}{n}\right)^{1-1/m}, & t \in \left[0, \dfrac{1}{n}\right], \\ t^{1/m}, & t \geq \dfrac{1}{n}. \end{cases}$$

Functions $\beta_{m,n}$ are globally Lipschitz, and therefore $\beta_{m,n}(\psi_1) \in W_p^1(\mathbb{R}, \mu)$. Moreover, one can easily verify that if $n \to \infty$ then

$$\beta_{m,n}(\psi_1) \to \psi_m \equiv \alpha^{1/m} \cdot \mathbb{1}_{(a_k,\, b_k)}$$

and

$$\left[\beta_{m,n}(\psi_1)\right]' \to \psi'_m \equiv \frac{1}{m} \rho \psi_m$$

in $L_p(X, \mu)$. This implies that $\psi_m \in W_p^1(\mathbb{R}, \mu)$. Since $\varphi_m \to \mathbb{1}_{(a_k,\, b_k)}$ and $\varphi'_m \to 0$ in $L_p(\mathbb{R}, \mu)$, $m \to \infty$, $\mathbb{1}_{(a_k,\, b_k)} \in W_p^1(\mathbb{R}, \mu)$ and $D_p^1 \mathbb{1}_{(a_k,\, b_k)} = 0$.

The theorem is proved.

Connection between W- and G-derivative is more complicated.

Definition 1.2. *Measure* μ *has the Markov uniqueness property of the order* $p \in (1; +\infty)$ (*notation* $\mu \in MU_p$) *if* $W_p^1(X, \mu) = G_p^1(X, \mu)$.

Let us explain briefly the origin of the terminology. Let

$$\mathscr{E}_0(f, g) = E(\nabla_H f, \nabla_H g)_H, \qquad f, g \in C_{0,\,\mathrm{cyl}}^\infty(X).$$

Except the fact that the map $\mathscr{E}_0(\cdot, \cdot)$ is not closed, it satisfies all properties of a

Dirichlet form (we don't give here the corresponding definitions, referring the reader if necessary to the book [5]). One can easily see that $\mathcal{E}_0$ is closable and its closure coincides with the form

$$\mathcal{E}(f, g) = E(D_2 f, D_2 g)_H, \qquad f, g \in W_2^1(X, \mu).$$

At the same time, if $G_2^1(X, \mu)$ is strictly larger than $W_2^1(X, \mu)$ (obviously $W_2^1(X, \mu) \subset G_2^1(X, \mu)$), then there exists one more closed extension on $\mathcal{E}_0$, namely

$$\mathcal{E}^+(f, g) = E\left(D_2^G f, D_2^G g\right)_H, \qquad f, g \in G_2^1(X, \mu).$$

Existence of two Dirichlet forms $\mathcal{E}$ and $\mathcal{E}^+$, extending $\mathcal{E}_0$, is equivalent to existence of two different Markov semigroups on $L_2(X, \mu)$, such that their generators, restricted on the set $C_{0,\,\mathrm{cyl}}^\infty(X)$, are both equal to the infinite-dimensional elliptic operator

$$L_0 = \nabla_H^* \nabla_H \equiv \sum_{k=1}^{\infty} \nabla_{h_k}^2 - \sum_{k=1}^{\infty} (\rho, h_k) \nabla_{h_k},$$

here $\{h_k\}$ is an orthonormal basis (ONB) in H.

For the objects, introduced above, the following relations hold true.

Proposition 1.1. *The following properties are equivalent*:

1) $\mu \in MU_2$;

2) $\mathcal{E}$ *is the unique Dirichlet form, coinciding with* $\mathcal{E}_0$ *on* $C_{0,\,\mathrm{cyl}}^\infty(X)$;

3) *there exists only one Markov semigroup on* $L_2(X, \mu)$ *such that the restriction of its generator on* $C_{0,\,\mathrm{cyl}}^\infty(X)$ *is equal to* L_0.

Properties 1 – 3 *hold true if*

4) *operator* L_0 *is essentially self-adjoint on* $C_{0,\,\mathrm{cyl}}^\infty(X)$.

For the proofs and detailed discussions of the related topics we send the reader to [2, 6, 7] (see also the bibliography given there). In [6] the assertion 4 (which is called „strong uniqueness" property) was proved using the parabolic criterium of self-adjointness for Hilbert space X under some regularity conditions imposed on the vector logarithmic derivative $\tilde{\rho}\colon X \to X$. Unfortunately, in some situations these results are hardly applicable, for example in some questions connected with the theory of random operators X is a space of compact operators and is not a Hilbert one. Therefore the natural question is how to obtain conditions, sufficient for Markov uniqueness, in the „inner" terms of the measure, for example in the terms of its logarithmic derivative.

2. Approach to Markov uniqueness based on finite-dimensional filtration. One of the natural ideas about characterization of the functions, belonging to $W_p^1(X, \mu)$ or $G_p^1(X, \mu)$ (which, as we believe, is up to the works [8, 9]), is to take some sequence of finite-dimensional projections in X and give differential properties of f in the terms of the sequence of its correspondent projections (i. e., conditional expectations). Unfortunately, it is hard to obtain on this way a criterium, analogous to given in Lemma 1.2 in the terms of the stratification of the space. This is caused by the fact that the operation of projection (conditional expectation) is connected with derivative in a more complicated way than the operation of stratification. Let us consider this connection in a more details.

Suppose $x_1^*, \ldots, x_n^* \in X^*$ are fixed, $M = \left\langle j^* x_1^*, \ldots, j^* x_n^* \right\rangle \subset H$. Denote by $\mathcal{F}_M$ σ-algebra generated by $x_1^*, \ldots, x_n^*$, $f_M = E_M f \equiv E\left[f \middle| \mathcal{F}_M\right]$, $f \in L_1(X, \mu)$. Denote also for $\varphi, \psi \in L_1(X, \mu)$ such that $\varphi\psi \in L_1(X, \mu)$

$$\mathrm{cov}_M(\varphi, \psi) = E_M[\varphi - E_M\varphi][\psi - E_M\psi]$$

— conditional covariation of φ, ψ under condition $\mathcal{F}_M$.

Lemma 2.1. *Let* $h \in H$, $f \in W^1_{p,h}(X, \mu)$, *then for every* $\tilde{p} \le p$ $f_M \in W^1_{\tilde{p},h}(X, \mu)$ *and*

$$(D_{\tilde{p},h} f_M, h)_H = E_M(D_{\tilde{p}} f, h) + \mathrm{cov}_M(f, (\rho, h)). \tag{2.1}$$

This result is well known (up to notations, in which we change the term $E_M f(\rho, h) - E_M f E_M(\rho, h)$ by $\mathrm{cov}_M(f(\rho, h))$, its proof can be found for instance in [1], Proposition 7.1.9.

Consider the bilinear continuous map

$$R^M_\mu : (h, g) \mapsto \mathrm{cov}_M((\rho, h), (\rho, g)), \qquad h, g \in H.$$

One can easily see that R^μ_M is a weak random operator in H in a sense of Skorokhod (see [10]). Consider operator $\tilde{R}^M_\mu = P_M R^M_\mu P_M$ (P_M is the projector on M in H), since space M is finite-dimensional, $\tilde{R}^M_\mu$ is a bounded random operator in M.

Definition 2.1. *The measure* μ *is said to have the successful filtration property of the order* $p \in (1, +\infty]$ (*notation* $\mu \in SF_p$) *if there exists a sequence* $\{x^*_n\} \subset X^*$, *separating points in* X, *such that for the corresponding sequence* $M_n = \langle j^* x^*_1, \ldots, j^* x^*_n \rangle$

$$\sup_n \left\| \| \tilde{R}^{M_n}_\mu \|_{\mathcal{L}(M_n)} \right\|_{L_{p/2}(X, \mu)} < +\infty,$$

here we denote $\| f \|_{L_r} = \left(E| f |^r\right)^{1/r}$ *for all* $r > 0$, *i. e.,* $\|\cdot\|_{L_r}$ *is not necessarily a norm.*

It was mentioned in [10] that it is difficult to give conditions on a weak random operator, necessary and sufficient for this operator to be bounded. On the other hand, one can give a criterium for the operator to be a random Hilbert – Schmidt operator (see [10], Ch. 1, Sect. 1.2). Similarly, we can give a criterium on a sequence of operators to have uniformly bounded (in $L_{p/2}$ sense) Hilbert – Schmidt norms. Imposing these conditions on $\tilde{R}^{M_n}_\mu$, and then using Jensen's inequality for conditional expectation we obtain the well known condition, which sometimes is named Höegh-Krohn condition.

Definition 2.2. *The measure* μ *satisfies Höegh-Krohn condition of the order* $p \in (1, +\infty]$ (*notation* $\mu \in HK_p$) *if there exists a sequence* $\{x^*_n\} \subset X^*$, *separating points in* X, *such that the sequence* $\{e_n \equiv j^* x^*_n\}$ *is an ONB in* H *and*

$$\sup_{n \ge 1} \left\| \sum_{k=1}^n \left[(\rho, e_k) - E_{M_n}(\rho, e_k)\right]^2 \right\|_{L_{p/2 \vee 1}(X, \mu)} < +\infty.$$

Obviously, condition $\mu \in HK_p$ is more strong than $\mu \in SF_p$ (since condition on an operator to be Hilbert – Schmidt is more strong than condition to be bounded), but it is more easy to check. Let us give another sufficient condition for $\mu \in SF_p$.

Definition 2.3. *The measure* μ *belongs to the class* B_p, $p \in (1, +\infty]$, *if* $(\rho, h) \in G_p^1(X, \mu)$, $h \in H$ *and random operator* B_μ *in* H, *given by*

$$B_\mu h = -D_p^G(\rho, h), \qquad h \in H,$$

is bounded and

$$\Big\| \| B_\mu \|_{\mathcal{L}(H)} \Big\|_{p/2 \vee 1} < +\infty.$$

Let us note that operator B_μ is known as an inner characteristic of the measure, in terms of which some analytical properties can be described in a natural and compact way. Say, if $B_\mu \geq cI_H$, $c > 0$, then the well known Bakry – Emery criterium gives log-Sobolev inequality for stochastic derivative [6], under the same condition the analogue of Clark – Ocone representation theorem hold true [11], the same operator is involved into the formula for the Jacobian for nonlinear transformations of measure μ [12]. Further we give a sufficient condition for $\mu \in SF_p$ in the terms of this object.

For a fixed sequence $\{x_n^*\}$, such that $\{e_n \equiv j^* x_n^*\}$ is ONB in H, denote $X_n = jM_n$, note that the measure $\mu_n = \mu\big|_{\mathcal{F}_{M_n}}$ can be regarded as the projection of the measure μ on X_n w. r. t. canonical decomposition $X = X_n + \langle x_1^*, \ldots, x_n^* \rangle^\perp$. Every measure μ_n, $n \geq 1$, has logarithmic derivative ρ_n w. r. t. directions from M_n and (see [3])

$$(\rho_n, h)_H = E_{M_n}(\rho, h), \qquad h \in M_n, \qquad n \geq 1. \tag{2.2}$$

This allows us to define classes $W_p^1(X_n, \mu_n)$, $G_p^1(X_n, \mu_n)$, $n \geq 1$.

Definition 2.4. *The measure* μ *belongs to the class* B_p^P, $p \in (1, +\infty]$, *if there exists a sequence* $\{x_n^*\} \subset X^*$, *separating points in* X, *such that the sequence* $\{e_n \equiv j^* x_n^*\}$ *is an ONB in* H, *for every* $n \geq 1$ $\mu_n \in B_p$ *and*

$$\sup_n \Big\| \| B_{\mu_n} \|_{\mathcal{L}(M_n)} \Big\|_{p/2 \vee 1} < +\infty.$$

Lemma 2.1. *If* $\mu \in B_p^P$, *then* $\mu \in SF_p$.

Proof. Standard approximation arguments together with statement of Theorem 1.1 imply that if $\mu \in B_p^P$ then $\mu \in B_p$. Due to (2.1) and (2.2)

$$\operatorname{cov}_{M_n}((\rho, h), (\rho, g)) = E_{M_n}(B_\mu h, g)_H - (B_{\mu_n} h, g)_{M_n}, \qquad h, g \in M_n, \tag{2.3}$$

which together with Jensen's inequality gives that

$$\sup_n \Big\| \| \tilde{R}_\mu^{M_n} \|_{\mathcal{L}(M_n)} \Big\|_{L_{p/2}(X, \mu)} \leq$$

$$\leq \Big\| \| B_\mu \|_{\mathcal{L}(H)} \Big\|_{p/2 \vee 1} + \sup_n \Big\| \| B_{\mu_n} \|_{\mathcal{L}(M_n)} \Big\|_{p/2 \vee 1} < +\infty.$$

The lemma is proved.

Theorem 2.1. *Let* $\mu \in SF_p$, *then* $\mu \in MU_p$.

***Remark* 2.1.** It follows from the previous discussion and the statement of the theorem that if either $\mu \in HK_p$ or $\mu \in B_p^P$, then $\mu \in MU_p$.

Proof of the theorem. We follow the scheme given in the proof of Theorem 7.1.11 [1] (except one point, which will be discussed later). Let sequence $\{x_n^*\}$ from Definition 2.1 be fixed, suppose for a while that we already know that

$W_p^1(X_n, \mu_n) = G_p^1(X_n, \mu_n)$ for every $n \geq 1$. Let $f \in G_p^1(X, \mu)$ be fixed, taking if necessary $\tilde{f} = \varphi(f)$, $\varphi \in C_b^\infty(\mathbb{R})$, we can suppose that $|f| \leq 1$. Consider the sequence $\{f_n = E_{M_n} f\}$, $f_n \to f$ in $L_p(X, \mu)$. Due to Lemma 2.1 for every $h \in H$

$$D_{p,h}^G f_n = E_{M_n} D_{p, P_{M_n} h}^G f + \operatorname{cov}_{M_n}\big(f, (\rho, P_{M_n} h)\big). \tag{2.4}$$

The H-norm of the first term in the right-hand side of (2.4) is estimated by

$$\sup_{\|h\|_H = 1} \left| E_{M_n} D_{p, P_{M_n} h}^G f \right| \leq E_{M_n} \sup_{\|h\|_H = 1} \left| D_{p, P_{M_n} h}^G f \right| \leq E_{M_n} \left\| D_p^G f \right\|_H.$$

In order to estimate the second term we use inequalities

$$\operatorname{cov}_M(\varphi, \psi) \leq [\operatorname{cov}_M(\varphi, \varphi)]^{1/2} \cdot [\operatorname{cov}_M(\psi, \psi)]^{1/2},$$

valid for $\varphi, \psi \in L_2(X, \mu)$, and

$$\operatorname{cov}_M((\rho, h), (\rho, h)) \leq \left\| \tilde{R}_\mu^M \right\|_{\mathcal{L}(M)} \|h\|_H^2, \qquad h \in M.$$

This gives that

$$\sup_{\|h\|_H = 1} \left| \operatorname{cov}_{M_n}\big(f, (\rho, P_{M_n} h)\big) \right| \leq \left\| \tilde{R}_\mu^{M_n} \right\|_{\mathcal{L}(M_n)}^{1/2}$$

(note that $\operatorname{cov}_{M_n}(f, f) \leq 1$ since $|f| \leq 1$), and consequently the sequence $D_p^G f_n$ (which is equal to $D_p f_n$ due to supposition made at the beginning of the proof) is bounded in $L_p(X, \mu, H)$. Since the closed unit ball is weakly compact in $L_p(X, \mu, H)$, we can take a subsequence $\{n_k\}$ such that $D_p f_{n_k}$ weakly converges. From the elements of this sequence we can compose the sequence of convex linear combinations, strongly convergent to the same limit. Since the sequence of the same combinations composed from the sequence $\{f_{n_k}\}$ converge to f in $L_p(X, \mu)$, we obtain the needed statement.

Therefore in order to prove the theorem we need Markov uniqueness property for finite-dimensional projections. The question of finite-dimensional Markov uniqueness is studied in details (see [7, 13] and references there), but most of the existing results are hardly applicable in our situation, because they have some „structural" conditions on the measure. Say, in [13] the density of the measure has to be positive a. s. with respect to Lebesgue measure (note that the proof of implication $\mu \in HK_p \Rightarrow \mu \in MU_p$ in [1] refers to results of [13] and therefore formally is correct only for such measures that their finite-dimensional projections have a. s. positive densities). Fortunately, we have the following moment condition: logarithmic derivative of μ, and conseguently logarithmic derivatives of all μ_n, have moments of every order. Let us show finite-dimensional Markov uniqueness under this condition.

Let $n \geq 1$ be fixed, then μ_n considered as a measure on $\mathbb{R}^n$, has a density α_n (see [3]). Moreover (see [1], proof of Proposition 4.3.1) for every $p \geq 1$ $\alpha_n^{1/p} \in W_p^1(\mathbb{R}^n, \lambda^n)$, and by Sobolev inclusion theorem we have that $\alpha_n \in C(\mathbb{R}^n)$. Now let $f \in G_p^1(\mathbb{R}^n, \mu_n)$, then $\varphi_n = f\alpha_n^{1/p}$ belongs to $W_p^1(\mathbb{R}^n, d\lambda^n)$ and its derivative is equal to $D_p^G f \cdot \alpha_n^{1/p} + (1/p) f_n \rho_n \alpha_n^{1/p}$, where $\rho_n = \nabla(\ln \alpha_n)$ is the logarithmic derivative of μ_n. Let us take $\kappa \in C_0^\infty(\mathbb{R}^n)$, $\kappa \geq 0$, $\int_{\mathbb{R}^n} \kappa(x)\, dx = 1$, and put

$$\varphi_n^\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \varphi_n(x + y) \kappa\left(\frac{y}{\varepsilon}\right) dy,$$

$$r_n^{\varepsilon}(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} \alpha_n^{1/p}(x+y)\kappa\left(\frac{y}{\varepsilon}\right)dy.$$

Due to standard properties of convolution $\varphi_n^{\varepsilon} \to \varphi_n$, $\varepsilon \to 0$, in $W_p^1(\mathbb{R}^n, d\lambda^n)$ and $r_n^{\varepsilon} \to \alpha_n^{1/p}$, $\varepsilon \to 0$, uniformly on every compact. Multiplying, if necessary, f on a function from $C_0^{\infty}(\mathbb{R}^n)$, we can suppose that f has a compact support and therefore supports of all functions φ_n^{ε} belong to some compact K. Choose $\delta(\varepsilon)$ such that

$$r_n^{\varepsilon}(x) + \delta(\varepsilon) > \alpha_n^{1/p}(x), \quad x \in K, \quad \varepsilon > 0.$$

It is easy to see that it can be chosen in such a way that $\delta(\varepsilon) \to 0$, $\varepsilon \to 0+$. Now one can verify that the family of the functions

$$f_n^{\varepsilon} = \frac{\varphi_n^{\varepsilon}}{r_n^{\varepsilon} + \delta(\varepsilon)}$$

belongs to $W_p^1(\mathbb{R}^n, d\mu_n)$ and converge to f in $W_p^1(\mathbb{R}^n, d\mu_n)$ as $\varepsilon \to 0+$.

The theorem is proved.

3. Rademacher theorem under „successful filtration" condition. In this section we are going to discuss the infinite-dimensional analogue of the following famous theorem by Rademacher.

Proposition 3.1. *For every $n \in \mathbb{N}$, $p \geq 1$ the following two properties for a function $f \in L_p(\mathbb{R}^n, \lambda^n)$ are equivalent*:

i) *f has a λ^n-modification, which is a globally Lipschitz function with Lipschitz constant C*;

ii) *$f \in W_p^1(\mathbb{R}^n, \lambda^n)$ and $\left\| D_p f \right\|_{\mathbb{R}^d} \leq C\lambda^n$-a. s.*

The natural question is how to extend this result to the situation described in Section 1. In this case the Lipschitz property of the function has to be changed, since the space of differentiability is H but not X.

Definition 3.1. *The function $f \in L_p(X, \mu)$ is called to be H-Lipschitz with the constant C if*

$$|f(x+jh) - f(x)| \leq C\|h\|_H, \quad x \in X, \quad h \in H.$$

Considering the Rademacher theorem as consisting of two parts „i) $\Rightarrow$ ii)" and „ii) $\Rightarrow$ i)", one can see that the first part is much easier to give an analogue.

Proposition 3.2. *Let $f \in L_p^1(X, \mu)$ and f has an H-Lipschitz modification with the constant C. Then $f \in G_p^1(X, \mu)$ and $\left\| D_p^G f \right\|_H \leq C$ μ-a. s.*

This result was obtained by S. Kusuoka, see [14, 15]. One can easily prove it using Proposition 3.1 for $n = 1$ and statement 2 of Lemma 1.2.

The proof of the analogue of the second part in the infinite-dimensional case is more complicated and requires additional properties on the measure to hold true. It was proved in [16] for Gaussian measure and in [17] for measure satisfying some more strong version of condition $\mu \in HK_{\infty}$.

Our main result in this section is the following theorem

Theorem 3.1. *Let $\mu \in SF_{\infty}$ to satisfy the following condition: for every $h \in H$ there exists some $\varepsilon = \varepsilon(h) > 0$ such that $E\exp\{\varepsilon|(\rho, h)|\} < +\infty$.*

Then for every $f \in W_p^1(X, \mu)$, such that $\left\| D_p f \right\|_H \leq C$ a. s., there exists a modification $\tilde{f}$ of f, which is H-Lipschitz with constant C.

***Remark* 3.1.** Under condition of the theorem $W_p^1(X, \mu) = G_p^1(X, \mu)$.

Proof of the theorem. Let us fix a sequence $\{x_n^*\}$ from the definition of the class SF_∞, $f_n = E_{M_n} f \to f$, $n \to \infty$, a. s. We can suppose that $|f| \le 1$ a. s., which implies, analogously to the proof of Theorem 2.1, that

$$\sup_n \| D_p f_n \|_{L_\infty(X, \mu, H)} = C_1 < +\infty,$$

a. s. Due to exponential integrability condition on the logarithmic derivative, every projection μ_n of μ has a continuous positive density (see [1], Proposition 4.3.1), and therefore we can apply finite-dimensional Rademacher theorem to every f_n and obtain a sequence of functions $\{\tilde{f}_n\}$ such that $\tilde{f}_n = f_n$ a. s. and

$$\| \tilde{f}_n(x + jh) - \tilde{f}_n(x) \| \le C_1 \| h \|_H, \qquad x \in X, \qquad h \in H.$$

Denote

$$A_0 = \{x : \exists \lim_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)\}.$$

Since measure μ is quasiinvariant w. r. t. H (see [1], Proposition 4.3.1), $\forall h \in H$ $\mu(A_0 + jh) = 1$. Let us choose in H a dense countable subset H_0 and put

$$A_1 = \bigcap_{h \in H_0} (A_0 + jh).$$

Then $\mu(A_1) = 1$ and, by the construction,

$$\forall x \in A_1, \ h \in H_0 \ \exists \lim_{n\to\infty} \tilde{f}_n(x + jh) = \tilde{f}(x + jh).$$

The fact that the sequence $\{\tilde{f}_n\}$ is uniformly H-Lipschitz implies that there exists the limit $\lim_{n\to\infty} \tilde{f}_n(x + jh)$ for every $x \in A_1$, $h \in H$. Therefore the following definition is correct:

$$\hat{f}(x) = \begin{cases} 0, & x \notin A_1 + jH, \\ \lim_{n\to\infty} \tilde{f}_n(x), & x \in A_1 + jH, \end{cases}$$

one can see that $\hat{f}$ is a modification of f, which is H-Lipschitz with the constant C_1. Now we need to „correct" the modification $\hat{f}$ in order to make Lipschitz constant equal to C. Let us show that for every M_n one can choose another modification $\hat{f}_n$ such that

$$\left| \hat{f}_n(x + jh) - \hat{f}_n(x) \right| \le \begin{cases} C_1 \| h \|_H, & x \in X, \quad h \in H, \\ C \| h \|_H, & x \in X, \quad h \in M_n. \end{cases}$$

We choose a sequence $\{\varphi_\varepsilon\} \subset C(\mathbb{R}^n)$, convergent to δ_0, and put

$$\hat{f}_n^\varepsilon(x) = \int_{H_n} \hat{f}(x + \varepsilon jh) \varphi_\varepsilon(h) \lambda^n(dh), \qquad x \in X.$$

The family $\{\hat{f}_n^\varepsilon, \varepsilon > 0\}$ is convergent a. s. to $\hat{f}$ as $\varepsilon \to 0$ and satisfy condition

$$\left| \hat{f}_n^\varepsilon(x + jh) - \hat{f}_n^\varepsilon(x) \right| \le \begin{cases} C_1 \| h \|_H, & x \in X, \quad h \in H, \\ C \| h \|_H, & x \in X, \quad h \in M_n. \end{cases}$$

This gives, after the considerations analogous to given above, needed modification $\hat{f}_n$.

At last, taking $n \to \infty$ and making again the same considerations, we obtain the needed statement.

The theorem is proved.

Let us note that the exponential integrability condition, imposed in Theorem 3.1 on the logarithmic derivative, is crucial: if it fails, then the statement of the Rademacher theorem can fail even in the simplest one-dimensional case.

***Example* 3.1.** Let $X = H = \mathbb{R}^1$ and μ has the density of the form

$$p(x) = c_\alpha \exp\left\{-\frac{1}{|x|^\alpha} - |x|^\alpha\right\}, \qquad x \in \mathbb{R},$$

$\alpha > 0$ is some constant. Logarithmic derivative now is equal

$$\rho(x) = \alpha \operatorname{sign}(x)\left[\frac{1}{|x|^{\alpha+1}} - |x|^{\alpha-1}\right]$$

and has all moments. However, the Rademacher theorem is not true: the function $f = 1\!\!1_{\mathbb{R}^+}$ belongs to $W_p^1(\mathbb{R}, \mu)$ with $D_p f = 0$ for every $p \geq 1$ (see the proof of Theorem 1.1), but it has not any continuous modification. Note that taking α large enough we see that every condition of the type

$$E\exp\left\{\varepsilon|(\rho, h)|^{1/r}\right\} < +\infty, \qquad r > 1, \qquad h \in H, \qquad \varepsilon = \varepsilon(h) > 0$$

is not sufficient for Rademacher theorem to hold true.

4. Examples. In this section we give some examples of measures satisfying „successful filtration" condition. The first example is well known.

***Example* 4.1** (Gaussian measure). Let (X, H, μ) be a canonical Wiener space, then for every $M \subset H$ $\tilde{R}_\mu^M = 0$ and $\mu \in HK_\infty$. More general, if μ have a density p_μ from the class G_p^1 with respect to some (smooth) product measure ν and

$$\frac{D_p^G p_\mu}{p_\mu} \in L_p(X, H, \mu),$$

then $\mu \in HK_p$.

In the second example the condition $\mu \in SF_p$ is much harder to be proved. It shows, in particular, that classes B_p^P and B_p are in general different.

***Example* 4.2** (Stratified measure). Let μ be equal

$$\mu = \frac{1}{2}(\mu^1 + \mu^2),$$

where $\mu^{1,2}$ are Gaussian measures on X with covariation operators jj^*, $4jj^*$ correspondingly.

One can show that there exist H-invariant sets $A_{1,2}$ such that $A_1 \cap A_2 = \varnothing$, $A_1 \cup \cup A_2 = X$ and $\mu^i(A_j) = \delta_{ij}$, $i, j = 1, 2$. This gives that logarithmic derivative of μ is equal $(\rho, h)(\cdot) = -(\cdot, h)\cdot\eta$, where

$$\eta = 1\!\!1_{A_1} + \frac{1}{4}1\!\!1_{A_2}$$

and $(\cdot, h)$ is the measurable linear functional on X, correspondent to h. We will show that

1) $\mu \in B_p^P \cap HK_p$, $p \in (1, +\infty)$;

2) $\mu \in B_\infty$, $\mu \notin B_\infty^P$, $\mu \notin HK_\infty$.

First, note that η is H-invariant, and therefore $D_p^G \eta = 0$, $B_\mu h = \eta I_H$, which gives that $\mu \in B_\infty$.

It is a little bit easier to study the question whether $\mu \in B_p^P$, let us start with it. For every fixed basis corresponding projection μ_n of the measure μ on $\mathbb{R}^n$ has the density equal to

$$p_n(x) = \frac{1}{(\sqrt{2\pi})^n} f_n\left(\frac{\|x\|_n^2}{2}\right),$$

where $\|\cdot\|_n$ is Euclid norm in $\mathbb{R}^n$ and

$$f_n(u) = \frac{1}{2}\left(e^{-u} + \frac{1}{2^n} e^{-u/4}\right)$$

(further we omit subscript at f).

Straightforward computations give that

$$\left| \| B_{\mu_n} \|_{\mathcal{L}(M_n)} - \frac{f'\left(\frac{\|x\|_n^2}{2}\right)}{f\left(\frac{\|x\|_n^2}{2}\right)} \right| \le \|x\|_n^2 \left| \frac{f''\left(\frac{\|x\|_n^2}{2}\right)}{f\left(\frac{\|x\|_n^2}{2}\right)} - \left(\frac{f'\left(\frac{\|x\|_n^2}{2}\right)}{f\left(\frac{\|x\|_n^2}{2}\right)} \right)^2 \right|. \tag{4.1}$$

The functions $(f_n' / f_n)(\cdot)$ are uniformly bounded, and therefore the question whether μ belongs to B_p^P is determined by the right-hand side of (4.1). One has

$$\frac{f''(u)}{f(u)} - \left(\frac{f'(u)}{f(u)}\right)^2 = \frac{9}{16} \frac{\frac{1}{2^n} e^{-u-u/4}}{\left(e^{-u} + \frac{1}{2^n} e^{-u/4}\right)^2}, \qquad u \ge 0. \tag{4.2}$$

Since the function (4.2) at the point

$$u_n = \frac{8n}{7} \ln 2$$

takes value

$$\frac{9}{16} \frac{1}{2^2} = \text{const} > 0$$

and $u_n \to \infty$, $n \to \infty$, one has that $\operatorname{ess\,sup} \| B_{\mu_n}(\cdot) \|_{\mathcal{L}(M_n)} \to \infty$, $n \to \infty$. This gives that $\mu \notin B_\infty^P$. For $p < +\infty$

$$\int_{\mathbb{R}^n} \|x\|_n^{2p} \left(\frac{f''\left(\frac{\|x\|_n^2}{2}\right)}{f\left(\frac{\|x\|_n^2}{2}\right)} - \left(\frac{f'\left(\frac{\|x\|_n^2}{2}\right)}{f\left(\frac{\|x\|_n^2}{2}\right)} \right)^2 \right)^p p_n(x)\, dx =$$

$$= \left(\frac{9}{16}\right)^p \int_{\mathbb{R}^n} \frac{\|x\|_n^{2p}}{(\sqrt{2\pi})^n} \frac{\left(\frac{1}{2^n} e^{-\frac{\|x\|_n^2}{8}}\right)^p}{\left(e^{-\frac{\|x\|_n^2}{8}} + \frac{1}{2^n} e^{-\frac{\|x\|_n^2}{8}}\right)^{2p-1}}\, dx =$$

$$= \left(\frac{9}{16}\right)^p \int_{\mathbb{R}^n} \frac{\|x\|_n^{2p}}{(\sqrt{2\pi})^n} \frac{1}{(\sqrt{2})^n} e^{-\frac{5\|x\|_n^2}{16}} \frac{dx}{\left((\sqrt{2})^n e^{-\frac{3\|x\|_n^2}{16}} + \left(\frac{1}{\sqrt{2}}\right)^n e^{\frac{3\|x\|_n^2}{16}}\right)^{2p-1}} \le$$

$$\le \left(\frac{9}{16}\right)^p \frac{1}{2^{2p-1}} \left(\frac{8}{5}\right)^p \frac{1}{(\sqrt{2})^n} \left(\frac{2\sqrt{2}}{\sqrt{5}}\right)^n \int_{\mathbb{R}^n} \frac{\|y\|_n^{2p}}{(\sqrt{2\pi})^n} e^{-\frac{\|y\|_n^2}{2}} dy \to 0, \qquad n \to \infty,$$

since

$$\int_{\mathbb{R}^n} \frac{\|y\|_n^{2p}}{(\sqrt{2\pi})^n} e^{-\frac{\|y\|_n^2}{2}} dy = O(n^p), \qquad n \to \infty.$$

This gives that $\mu \in B_p^P$, $p < +\infty$.

In order to prove the Höegh-Krohn condition let us denote by r_{M_n} the random M_n-valued element such that $\left(r_{M_n}, h\right)_{M_n} = (\cdot, h)$, $h \in M_n$. Then operator $\tilde{R}_\mu^{M_n}$ can be represented in the form

$$\tilde{R}_\mu^{M_n} = \left[r_{M_n} \otimes r_{M_n}\right] \mathrm{cov}_{M_n}(\eta, \eta),$$

and its norms in $\mathcal{L}(M_n)$ and $\mathcal{L}_2(M_n)$ are both equal to $\left\| r_{M_n} \right\|_{M_n}^2 \mathrm{cov}_{M_n}(\eta, \eta)$. This implies that in the case under consideration the property $\mu \in SF_p$, which is already proved, implies $\mu \in HK_p$.

Note that in the considered example using „successful filtration" condition is not a proper way to prove Markov uniqueness or Rademacher theorem. In this case one can show that $\mathbb{1}_{A_i} \in W_p^1(X, \mu)$ with $D_p \mathbb{1}_{A_i} = 0$, $i = 1, 2$, and the both mentioned results can be proved separately on $A_{1,2}$ using result of Example 4.1. The given example is interesting from the point of view of explicit computations, demonstrating connections between classes of measures introduced in Section 2.

***Example* 4.3** (Convex measure). One can see from the previous example that the straightforward verification of the condition $\mu \in SF_p$ (or sufficient conditions $\mu \in HK_p$, $\mu \in B_p^P$) can be rather complicated since these conditions are imposed, in fact, on an infinite family of projections on initial measure. It occurs that these difficulties are greatly reduced in the case when μ is a *convex* measure.

Definition 4.1 [18]. *The measure* μ *is said to be convex if for every compacts* $A, B \subset X$ *and every* $\alpha \in [0, 1]$

$$\mu(\alpha A + (1-\alpha)B) \ge \mu^\alpha(A)\mu^{1-\alpha}(B).$$

Lemma 4.1. *Let* μ *be convex and satisfy assumptions of Section* 1. *Then* $\mu \in B_p^P$ *iff* $\mu \in B_p$.

Proof. We have to prove only implication $\mu \in B_p \Rightarrow \mu \in B_p^P$. Under conditions of the lemma every finite-dimensional projection of μ has a density of the type $\exp(-V)$, where V is a convex function (see [18] for a characterization of the class of convex measures in the terms of finite-dimensional projections). Then for every projection μ_n correspondent operator B_{μ_n} (which is well defined due to supposition $\mu \in B_p$ and Lemma 2.1) is positive, i. e., $\left(B_{\mu_n} h, h\right)_{M_n} \ge 0$ a. s. for every $h \in M_P$.

This is a corollary of the relation $B_{\mu_n} = \nabla^2 V$, which is to be understood in a generalized sence, and can be obtained from the definition of stochastic derivative. Then by (2.3) one has that

$$\operatorname{cov}_{M_n}((\rho, h), (\rho, h)) \le E_{M_n}(B_\mu h, h)_H, \qquad h \in M_n.$$

Now the needed statement holds true due to Cauchy inequality valid for cov_M.

The lemma is proved.

Let us give a concrete example of using of this result. Fix the equence $\{x_n^*\}$ (such that $\{e_n = x_n^*\}$ is ONB in H), and let us choose a symmetric operator $A \in \mathcal{L}(H)$ such that $A \ge \gamma I_H$ for some $\gamma > 0$. We suppose that

$$\sum_{k \ne j} \left(Ae_k, e_j\right)_H^2 = +\infty. \tag{4.3}$$

Also for notational convenience we demand (this limitation can be removed) that $Ae_n \in j^* X^*$, $n \ge 1$.

Let us fix an even function $\varphi \in C_b^1(\mathbb{R})$ such that $\inf_{\mathbb{R}} \varphi'(x) > -\gamma$, and consider measure μ, which have logarithmic derivative w. r. t. H, and on e_n this derivative is given by the formula

$$(\rho, e_n) = -\varphi\left(\left\langle x_n^*, \cdot\right\rangle\right) - \left\langle \left[j^*\right]^{-1} Ae_n, \cdot \right\rangle, \qquad n \ge 1. \tag{4.4}$$

We are not going to discuss the uniqueness problem since even for $\varphi = 0$ it is nontrivial (see, for instance, [1], Ch. 6.4). We take by μ *any* measure satisfying (4.4) and additional condition to be H-ergodic (or nonstratified):

$$D_p f = 0 \Leftrightarrow f = \text{const}, \qquad f \in W_p^1(X, \mu).$$

Note that every two such measures (if exist) are mutually singular and every measure satisfying (4.4) is a mixture of such measures.

The existence of H-ergodic measure satisfying (4.4) can be proved, for instance, using some limit procedure. One possible choice of the limiting measures is the finite-dimensional measures ν_n with the density ot the type $\exp[-V_n]$ with

$$\frac{\partial}{\partial x_k} V_n(x) = -\varphi(x_k) - \left(\tilde{A}_k^n, x\right)_{\mathbb{R}^n}, \qquad x \in \mathbb{R}^n, \qquad k = 1, \ldots, n,$$

where vectors $\tilde{A}_1^n, \ldots, \tilde{A}_n^n$ are given by equalities

$$\left(\tilde{A}_k^n, x\right)_{\mathbb{R}^n} = \left\langle \left[j^*\right]^{-1} Ae_k, \sum_{r=1}^n x_r j e_r \right\rangle, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Another possible choice is to take the Gaussian measure κ_0 with covariation operator equal jAj^* and consider the sequence $\{\kappa_n\}$ with

$$\frac{d\kappa_n}{d\kappa_0}(x) = C_n \exp\left[-\sum_{k=1}^n \psi\left(\left\langle x, x_k^*\right\rangle\right)\right],$$

$$x \in X, \qquad \psi(\cdot) = \int_0^{\cdot} \varphi(u)\, du.$$

By the construction there exist $\alpha > 0$, $\beta < +\infty$ such that $\nu_n \in C_{M_n}(\alpha, \beta)$,

$\kappa_n \in C_H(\alpha, \beta)$, $n \geq 1$ (by the definition, see [19], $\mu \in BC_H(\alpha, \beta) \Leftrightarrow \alpha I_H \leq B_\mu \leq \leq \beta I_H$ a. s.). This implies (see [19]) that for every $p > 1$ there exist constants $a_p = a(p, \alpha, \beta, H)$, $b_p = b(p, \alpha, \beta, H)$ such that

$$a_p \| j^* x^* \|_H^p \leq \int_X \left| \langle x, x^* \rangle - \int_X \langle y, x^* \rangle \kappa_n(dy) \right|^p \kappa_n(dx) \leq b_p \| j^* x^* \|_H^p, \qquad (4.5)$$

$$x^* \in X^*, \quad p > 1, \quad n \geq 1.$$

The same estimate is valid also for ν_n for every $x^* \in \langle x_1^*, \ldots, x_n^* \rangle$. This implies, that under additional condition that inclusion j is p-summating for some p, the sequences $\{\nu_n\}$, $\{\kappa_n\}$ are weakly compact. Let us take by μ some limiting point of one of these sequences. Since $\{\nu_n\}$, $\{\kappa_n\}$ are convex for every n, μ is also convex (see [18]). By the same reason we have that $\mu \in C_H(\alpha, \beta)$ (see [19]).

Due to Lemma 4.1, $\mu \in B_\infty^P$, and therefore $\mu \in SF_\infty$. Let us consider the question whether $\mu \in HK_p$. First let us consider the case when the sequence $\{x_n\}$ from the Definition 2.2 is the same with the sequence used in the construction of μ. Let us show that for every $n \geq 1$ and $y^* \in X^*$ such that $j^* y^* \perp \langle j^* x_1^*, \ldots, x_n^* \rangle$,

$$E\left(\langle \cdot, y^* \rangle - E_{M_n}\left[\langle \cdot, y^* \rangle\right]\right)^2 \geq \delta \| j^* y^* \|_H^2, \qquad (4.6)$$

where the constant $\delta > 0$ depends only on α, β, H. We take decomposition $X = \langle j j^* y^* \rangle + \tilde{X}$, where $\tilde{X} = \langle y^* \rangle^\perp$ and denote $\tilde{M} = \langle j^* y^* \rangle^\perp$. Now (4.6) follows from inequalities

$$E_{M^n}\left(\langle \cdot, y^* \rangle - E_{M_n}\left[\langle \cdot, y^* \rangle\right]\right)^2 \geq \operatorname{cov}_{\tilde{M}}\left(\langle \cdot, y^* \rangle, \langle \cdot, y^* \rangle\right) \geq \delta \| j^* y^* \|_H^2,$$

the last inequality is a corollary of (4.5) and the fact that almost all conditional measures on $x + \langle j j^* y^* \rangle$ belong to $B_{\langle j^* y^* \rangle}(\alpha, \beta)$. Due to (4.3), (4.6) we have that

$$\lim_{n \to \infty} \left\| \sum_{k=1}^n \left[(\rho, e_k) - E_{M_n}(\rho, e_k)\right]^2 \right\|_{L_{p/2 \vee 1}(X, \mu)} = +\infty \quad \text{for every} \quad p \geq 1.$$

The question whether condition of Definition 2.2 holds for *some* sequence $\{\tilde{x}_n^*\}$ is more difficult and rely on the form of the function φ. If $\varphi(u) = u^2$ (μ is Gaussian) and A has eigenbasis, then one should take by $\{\tilde{x}_n^*\}$ this basis. On the other hand, if $\varphi(u) = (\gamma/2)\sin u$, then such sequence does not exist and $\mu \notin HK_2$. The proof of this fact is analogous to the previous considerations (we omit details in order to shorten exposition), with inequality (4.6) combined with inequality

$$\inf_{a, b} \int_{\mathbb{R}} [\sin(cu) + au + b]^2 \mu(du) \geq K |c|^2, \quad |c| \leq 1, \quad K = K(\alpha, \beta) > 0,$$

valid for every one-dimensional measure $\mu \in C_{\mathbb{R}}(\alpha, \beta)$.

Let us end this example with a conclusive remark. Conditions (4.4) can be interpreted in such a way that μ is an invariant measure for the following elliptic operator

$$L_0 f(x) = -\Delta f(x) - \big([\Phi(x) + Ax], \nabla_H f\big)_H, \qquad f \in C^\infty_{0,\,\mathrm{cyl}}(X), \tag{4.7}$$

where generalized random elements Φ, Ax in H are given by formal series

$$\Phi(x) = \sum_n \varphi\big(\langle x, x_n^* \rangle\big) e_n,$$

$$Ax = \sum_n \Big\langle \big[j^*\big]^{-1} A e_n, x \Big\rangle e_n, \qquad x \in X.$$

We can formulate the result of our considerations in the following form.

Proposition 4.1. *Let* $\mathcal{M}(L_0)$ *be the family of the invariant measures for operator* (4.7), *then* $\mathcal{M}(L_0)$ *is nonemply and for every* $\mu \in \mathcal{M}(L_0)$ *both Rademacher theorem and Markov uniqueness for every* $p > 1$ *hold true.*

The statement of the proposition for $\mu \in \mathcal{M}(L_0)$, which is not H-ergodic, one can prove by stratifying μ into a mixture of mutually singular H-ergodic measures and then using previous results.

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