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NEWTON – KANTOROVICH ITERATIVE REGULARIZATION FOR NONLINEAR ILL-POSED EQUATIONS INVOLVING ACCRETIVE OPERATORS

ІТЕРАЦІЙНА РЕГУЛЯРИЗАЦІЯ НЬЮТОНА – КАНТОРОВИЧА ДЛЯ НЕЛІНІЙНИХ НЕКОРЕКТНИХ РІВНЯНЬ З АКРЕТИВНИМИ ОПЕРАТОРАМИ

The Newton – Kantorovich iterative regularization for nonlinear ill-posed equation involving monotone operator in Hilbert spaces is developed for the case of accretive operator in Banach spaces. Estimate for convergence rates of the method is established.

Розроблено ітераційну регуляризацію Ньютона – Канторовича для нелінійних некоректних рівнянь з монотонним оператором у гільбертових просторах для випадку акретивного оператора в банахових просторах. Встановлено оцінки швидкостей збіжності методу.

1. Introduction. Consider the operator equation of the first kind

$$A(x) = f, \quad f \in R(A) \subset X, \tag{1}$$

where the operator $A: D(A) = X \to X$ is a nonlinear and m-accretive operator, X together with X^* and the adjoint space of X are uniformly convex Banach spaces, X possesses the approximations (see [1]), and $D(A)$, $R(A)$ denote the domain and the range of A, respectively. For the sake of simplicity, norms of X and X^* will be denoted by the symbol $\|\cdot\|$, and we write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. If the normalized dual mapping J of X is continuous and sequential weak continuous, then the condition

$$\langle A(x) - f, J(x) \rangle > 0, \qquad \|x\| > \tilde{r}, \tag{2}$$

with some positive constant $\tilde{r}$ is sufficient for the existence of a solution of (1) (see [1]). It is well known that problem (1) is ill-posed (see [2]). For the case where A is a monotone operator in the Hilbert space X, the operator version of Tikhonov regularization method was considered in [3]. Later, the results were generalized for the equations involving monotone operator in Banach spaces (see [4]).

Meantimes, the Newton – Kantorovich iterative regularization

$$x_0 \in X, \qquad A(x_n) + \alpha_n x_n + \big(A'(x_n) + \alpha_n I\big)(x_{n+1} - x_n) = f \tag{3}$$

was investigated in [5] for the monotone operator A in the Hilbert space X, where I denotes the identity operator in X and $\{\alpha_n\}$ is the sequence of positive numbers. Then, algorithm (3) was developed for the case of the monotone operator A from Banach space X into X^* with some modification in the form

$$x_0 \in X, \qquad A(x_n) + A'(x_n)(x_{n+1} - x_n) + \alpha_n J^s(x_{n+1}) = f, \tag{4}$$

where $J^s: X \to X^*$ satisfies the condition (see [6])

$$\langle x, J^s(x) \rangle = \|x\|^s, \qquad \|J^s(x)\| = \|x\|^{s-1}, \quad s \geq 2.$$

On the other hand, it is well known that the theory of accretive operators is a direction of developing the theory of monotone operator in Hilbert space. Therefore, the consideration of Newton – Kantorovich iterative regularization for equations involving accretive operator is a necessary problem. The purpose of the paper is to fulfil the task, and to give an estimate for convergence rates of the method.

Below, the notation $a \sim b$ means that $a = O(b)$ and $b = O(a)$.

In the following section, we suppose that all above conditions are satisfied.

2. Main results. First, consider the operator equation

$$A_{\alpha_n}(x) = f, \quad A_{\alpha_n} = A + \alpha_n I. \tag{5}$$

Equation (5) for every fixed α_n possesses a unique solution denoted by x_{α_n}, because the operator A is m-accretive and $\alpha_n > 0$.

Theorem 1. *Assume that the following conditions hold*:

i) A *is Frechet differentiable with*

$$\left\| A(x) - A(\tilde{x}) - J^* A'(\tilde{x})^* J(x-\tilde{x}) \right\| \leq \tau \| A(x) - A(\tilde{x}) \| \quad \forall\, x \in X, \tag{6}$$

where $\tau > 0$ *is some constant,* $\tilde{x} \in S_0$ *is the set of solutions of* (1), *and* J^* *denotes the normalized dual mapping of* X^*;

ii) *there exists an element* $z \in X$ *such that* $A'(\tilde{x})z = -\tilde{x}$.

Then

$$\left\| x_{\alpha_n} - \tilde{x} \right\| = O\left(\sqrt{\alpha_n}\right).$$

Proof. Since

$$\begin{aligned} \left\| x_{\alpha_n} - \tilde{x} \right\|^2 &= \left\langle x_{\alpha_n} - \tilde{x}, J(x_{\alpha_n} - \tilde{x}) \right\rangle = \\ &= \frac{1}{\alpha_n}\left\langle f - A(x_{\alpha_n}), J(x_{\alpha_n} - \tilde{x}) \right\rangle - \left\langle \tilde{x}, J(x_{\alpha_n} - \tilde{x}) \right\rangle \leq \\ &\leq \left\langle z, A'(\tilde{x})^* J(x_{\alpha_n} - \tilde{x}) \right\rangle \leq \|z\| \left\| A'(\tilde{x})^* J(x_{\alpha_n} - \tilde{x}) \right\| = \\ &= \|z\| \left\| J^* A'(\tilde{x})^* J(x_{\alpha_n} - \tilde{x}) \right\| \leq (\tau + 1)\alpha_n \|z\| \left\| x_{\alpha_n} \right\|, \end{aligned}$$

we have the boundedness of $\{x_{\alpha_n}\}$. Therefore, we obtain from the last inequality that $\left\| x_{\alpha_n} - \tilde{x} \right\| = O\left(\sqrt{\alpha_n}\right)$. The theorem is proved.

The conclusion that the sequence $\{x_{\alpha_n}\}$ converges to $\tilde{x}$ as $n \to \infty$ if (1) has only unique solution and J is sequential weak continuous and continuous, was considered in [7]. Unfornately, the class of proper Banach spaces having the dual mapping J with sequential weak continuity is small (only the finite-dimensional Banach spaces and l_p). Moreover, the problem of convergence rates for $\{x_{\alpha_n}\}$ is still opened upto now. Hence, the results obtained in the present paper are essential has because they permit us to use the method in the spaces of type L_p, W_p, and, in particular, even if (1) possesses nonunique solution.

Note that the requirement of uniqueness for solution of (1) will be redundant, when we have (6) for some $\tilde{x} \in S_0$. This condition can be replaced by

$$\left\| A(x) - A(\tilde{x}) - J^* A'(\tilde{x})^* J(x-\tilde{x}) \right\| \leq \tau \| x - \tilde{x} \| \| A'(\tilde{x}) J(x-\tilde{x}) \| \quad \forall\, x \in X,$$

which is proposed in [8] for the case where X is a Hilbert space. Then it is applied for estimating the convergence rates of the regularized solutions for nonlinear ill-posed problems involving monotone operators in Banach spaces (see [9]).

We now return to the problem of convergence of the iteration sequence $\{x_n\}$ obtained by (3) if X is a Banach space. As in [5], assume that A is twice differentiable in the sense of Gateau, $\langle A''(x+th)h^2, J(h)\rangle$ is summable for each fixed x and h, and $\|A''(x)\| \le N \;\; \forall\, x \in X$.

Theorem 2. *Assume that the conditions of Theorem* 1 *are satisfied, the sequence* $\{\alpha_n\}$, $\alpha_n > 0$ *and* x_0 *satisfy the following conditions*:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad 1 \le \frac{\alpha_{n-1}}{\alpha_n} \le r, \quad n = 1, 2, \ldots,$$

$$\frac{N\|x_0 - x_{\alpha_0}\|}{2_{\alpha_0}} \le q < \frac{1}{r}, \qquad \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}^2} \le \frac{(1/r-q)q}{Nd},$$

$$d \ge \|\tilde{x}\|, \quad \tilde{x} \in S_0.$$

Then we have

$$\lim_{n\to\infty} \|x_n - \tilde{x}\| = 0.$$

Proof. Set

$$\tilde{A}_{\alpha_{n-1}}(x) \equiv A_{\alpha_{n-1}}(x_{n-1}) + \big(A'(x_{n-1}) + \alpha_{n-1} I\big)(x - x_{n-1}). \tag{7}$$

On the basis of (3) and the accretive property of $A'(x)$, $x \in X$ (see [1]), we obtain

$$\langle \tilde{A}_{\alpha_{n-1}}(x_{\alpha_{n-1}}) - A_{\alpha_{n-1}}(x_{\alpha_{n-1}}), J(x_{\alpha_{n-1}} - x_n)\rangle =$$
$$= \langle A_{\alpha_{n-1}}(x_{n-1}) - A_{\alpha_{n-1}}(x_{\alpha_{n-1}}), J(x_{\alpha_{n-1}} - x_n)\rangle +$$
$$+ \langle (A'(x_{n-1}) + \alpha_{n-1} I)(x_{\alpha_{n-1}} - x_{n-1}), J(x_{\alpha_{n-1}} - x_n)\rangle =$$
$$= \langle A_{\alpha_{n-1}}(x_{n-1}) + (A'(x_{n-1}) + \alpha_{n-1} I)(x_n - x_{n-1}) - f, J(x_{\alpha_{n-1}} - x_n)\rangle +$$
$$+ \langle (A'(x_{n-1}) + \alpha_{n-1} I)(x_{\alpha_{n-1}} - x_n), J(x_{\alpha_{n-1}} - x_n)\rangle \ge \alpha_{n-1}\|x_{\alpha_{n-1}} - x_n\|^2.$$

On the other hand, (7) and equality

$$A(x_{\alpha_{n-1}}) = A(x_{n-1}) + A'(x_{n-1})(x_{\alpha_{n-1}} - x_{n-1}) + \frac{A''(c)}{2!}(x_{\alpha_{n-1}} - x_{n-1})(x_{\alpha_{n-1}} - x_{n-1}),$$

where c is some element in X, imply that

$$\langle \tilde{A}_{\alpha_{n-1}}(x_{\alpha_{n-1}}) - A_{\alpha_{n-1}}(x_{\alpha_{n-1}}), J(x_{\alpha_{n-1}} - x_n)\rangle \le \frac{N}{2!}\|x_{\alpha_{n-1}} - x_{n-1}\|^2 \|x_{\alpha_{n-1}} - x_n\|.$$

Therefore,

$$\Delta_n := \|x_{\alpha_n} - x_n\| \le \|x_{\alpha_{n-1}} - x_n\| + \|x_{\alpha_{n-1}} - x_{\alpha_n}\| \le \frac{N}{2\alpha_{n-1}}\Delta_{n-1}^2 + \|x_{\alpha_{n-1}} - x_{\alpha_n}\|. \tag{8}$$

By virtue of (5), we have

$$\langle A(x_{\alpha_n}) - A(x_{\alpha_{n-1}}), J(x_{\alpha_n} - x_{\alpha_{n-1}})\rangle + \alpha_{n-1}\langle x_{\alpha_n} - x_{\alpha_{n-1}}, J(x_{\alpha_n} - x_{\alpha_{n-1}})\rangle =$$
$$= (\alpha_{n-1} - \alpha_n)\langle x_{\alpha_{n-1}}, J(x_{\alpha_n} - x_{\alpha_{n-1}})\rangle$$

or

$$\|x_{\alpha_{n-1}} - x_{\alpha_n}\| \le \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}}\|x_{\alpha_{n-1}}\| \le 2\frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}}\|\tilde{x}\|.$$

The last inequality follows from $\left\| x_{\alpha_n} - \tilde{x} \right\| \le \| \tilde{x} \| \ \forall\, n$ (see [7]). Now inequality (8) has the form

$$\Delta_n \le \frac{N}{2\alpha_{n-1}} \Delta_{n-1}^2 + 2\frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}} \| \tilde{x} \|.$$

As in [5], we have $N\Delta_n / 2\alpha_n \le q \ \ \forall\, n$. Thus,

$$\Delta_n = O\left(\left\| x_{\alpha_{n-1}} - x_{\alpha_n} \right\|\right) + O(q^n) = O\left(\left\| x_{\alpha_{n-1}} - \tilde{x} \right\|\right) + O(q^n) = O\left(\sqrt{\alpha_{n-1}}\right) + O(q^n).$$

The theorem is proved.

For an accretive operator A, algorithms (3) and (4) are alike because, in this case, we need to replace J in (4) by the identity operator I. Then (4) is completely transformed into (3) without any change.

Suppose that instead of (A, f) we have the approximations (A_n, f_n) such that

$$\| A_n(x) - A(x) \| \le h_n g(\| x \|), \qquad \| f_n - f \| \le \delta_n,$$

$h_n, \delta_n \to 0$ as $n \to +\infty$, where A_n is also *m*-accretive and continuous, and $g(t)$ is a continuous, bounded, and nonnegative function. The Newton – Kantorovich iterative regularization is defined as follows:

$$x_0 \in X, \qquad A_n(x_n) + \alpha_n x_n + \left(A_n'(x_n) + \alpha_n I\right)(x_{n+1} - x_n) = f_n. \tag{9}$$

In addition, suppose that A_n possesses the same properties as A has. For each fixed n, an element x_{n+1} is well defined, because $A_n'(x_n) + \alpha_n I$ is strongly accretive.

The convergence and convergence rates of the method defined by (9) will be established on the basis of the following result:

Theorem 3. *Assume that the conditions of Theorem* 1 *hold, and* $\delta_n / \alpha_n \to 0$, $h_n / \alpha_n \to 0$, $\alpha_n \to 0$ *as* $n \to 0$. *Then the solution* $\tilde{x}_n$ *of the equation*

$$\overline{A}_n(x) = f_n, \qquad \overline{A}_n = A_n + \alpha_n I, \tag{10}$$

converges to $\tilde{x}$. *Moreover, if* α_n *is chosen such that* $\alpha_n \sim (\delta_n + h_n)^p$, $0 < p < 1$, *then, for* $0 < \delta_n + h_n < 1$, *we have*

$$\| \tilde{x}_n - \tilde{x} \| = O\left((\delta_n + h_n)^\theta\right), \qquad \theta = \min\{1 - p, p/2\}.$$

Proof. Since A_n is m-accretive, then there exists a unique solution $\tilde{x}_n$ of (10). From (5), (10) and the property of J, we have

$$\| \tilde{x}_n - \tilde{x} \|^2 = \langle \tilde{x}_n - \tilde{x}, J(\tilde{x}_n - \tilde{x}) \rangle =$$

$$= \frac{1}{\alpha_n} \langle f_n - A_n(\tilde{x}_n), J(\tilde{x}_n - \tilde{x}) \rangle + \langle -\tilde{x}, J(\tilde{x}_n - \tilde{x}) \rangle \le$$

$$\le \frac{1}{\alpha_n} (\delta_n + h_n g(\| \tilde{x} \|)) \| \tilde{x}_n - \tilde{x} \| + \langle z, A'(\tilde{x})^* J(\tilde{x}_n - \tilde{x}) \rangle. \tag{11}$$

Hence, $\{\tilde{x}_n\}$ is bounded. On the other hand,

$$\langle z, A'(\tilde{x})^* J(\tilde{x}_n - \tilde{x}) \rangle \le \| z \| \left\| J^* A'(\tilde{x})^* J(\tilde{x}_n - \tilde{x}) \right\| \le \| z \| (\tau + 1) \| A(\tilde{x}_n) - f \| \le$$

$$\le \| z \| (\tau + 1)(\| A_n(\tilde{x}_n) - f_n \| + \delta_n + h_n g(\| \tilde{x}_n \|)) \le$$

$$\le \| z \| (\tau + 1)(\alpha_n \| \tilde{x}_n \| + \delta_n + h_n g(\| \tilde{x}_n \|)).$$

Therefore, (11) implies that

$$\|\tilde{x}_n - \tilde{x}\|^2 \le C_1 \frac{\delta_n + h_n}{\alpha_n} \|\tilde{x}_n - \tilde{x}\| + C_2(\delta_n + h_n)^p \le$$

$$\le C_1(\delta_n + h_n)^{1-p}\|\tilde{x}_n - \tilde{x}\| + C_2(\delta_n + h_n)^p, \quad 0 < \delta_n + h_n < 1,$$

where C_i are the positive constants. Hence, the conclusions of the theorem follow from the last two inequalities (see [9]).

The theorem is proved.

Theorem 4. *Assume that the following conditions hold*:

i) *conditions of Theorem* 2;

ii) *the sequence* $\{\alpha_n\}$, $\alpha_n > 0$, *is chosen such that*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad 1 \le \frac{\alpha_{n-1}}{\alpha_n} \le r, \quad n = 1, 2, \dots,$$

$$\frac{N\|x_0 - x_{\alpha_0}\|}{2\alpha_0} \le q < \frac{1}{r}, \qquad a_n \le 2\frac{(1/r - q)q}{N},$$

where

$$a_n = \left(\frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}^2} + \frac{h_n + h_{n-1}}{\alpha_{n-1}^2}\right)(2\|\tilde{x}\| + b_n) + \frac{\delta_n + \delta_{n-1}}{\alpha_{n-1}^2},$$

$$b_n = \max\left\{\frac{\delta_n + h_n g(\|\tilde{x}\|)}{\alpha_n}, \frac{\delta_{n-1} + h_{n-1} g(\|\tilde{x}\|)}{\alpha_{n-1}}\right\}.$$

Then, we have

$$\lim_{n\to+\infty} \|x_n - \tilde{x}\| = 0,$$

where x_n *is defined by* (9).

Proof. Setting

$$\tilde{A}_{n-1}(x) \equiv \overline{A}_{n-1}(x_{n-1}) + (A'_{n-1}(x_{n-1}) + \alpha_{n-1}I)(x - x_{n-1}),$$

we have

$$\left\langle \tilde{A}_{n-1}(\tilde{x}_{n-1}) - \overline{A}_{n-1}(\tilde{x}_{n-1}), J(\tilde{x}_{n-1} - x_n)\right\rangle \ge \alpha_{n-1}\|\tilde{x}_{n-1} - x_n\|^2.$$

Using the differential property of A_{n-1}, we can write

$$A_{n-1}(\tilde{x}_{n-1}) = A_{n-1}(x_{n-1}) + A'_n(x_{n-1})(\tilde{x}_{n-1} - x_{n-1}) +$$

$$+ \frac{A''_n(\tilde{c})}{2!}(\tilde{x}_{n-1} - x_{n-1})(\tilde{x}_{n-1} - x_{n-1}),$$

where $\tilde{c}$ is some element in X. Therefore,

$$\left\langle \tilde{A}_{n-1}(\tilde{x}_{n-1}) - \overline{A}_{n-1}(\tilde{x}_{n-1}), J(\tilde{x}_{n-1} - x_n)\right\rangle \le \frac{N}{2}\|\tilde{x}_{n-1} - x_{n-1}\|^2\|\tilde{x}_{n-1} - x_n\|.$$

Hence,

$$\tilde{\Delta}_n := \|\tilde{x}_n - x_n\| \le \frac{N}{2\alpha_{n-1}}\tilde{\Delta}_{n-1}^2 + \|\tilde{x}_{n-1} - \tilde{x}_n\|.$$

Relation (10) implies that

$$\langle A_n(\tilde{x}_n) - A_{n-1}(\tilde{x}_{n-1}), J(\tilde{x}_n - \tilde{x}_{n-1})\rangle + \alpha_{n-1}\langle \tilde{x}_n - \tilde{x}_{n-1}, J(\tilde{x}_n - \tilde{x}_{n-1})\rangle =$$

$$= \langle f_n - f_{n-1}, J(\tilde{x}_n - \tilde{x}_{n-1})\rangle + (\alpha_{n-1} - \alpha_n)\langle \tilde{x}_n, J(\tilde{x}_n - \tilde{x}_{n-1})\rangle$$

and

$$\| \tilde{x}_n \| \leq 2\| \tilde{x} \| + \frac{\delta_n}{\alpha_n} + \frac{h_n}{\alpha_n} g(\| \tilde{x} \|) \quad \forall\, n.$$

Thus,

$$\| \tilde{x}_{n-1} - \tilde{x}_n \| \leq \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}} \| \tilde{x}_n \| + \frac{h_n + h_{n-1}}{\alpha_{n-1}} \| \tilde{x}_{n-1} \| + \frac{\delta_n + \delta_{n-1}}{\alpha_{n-1}} \leq a_n \alpha_{n-1}.$$

Consequently,

$$\tilde{\Delta}_n \leq \frac{N}{2\alpha_{n-1}} \tilde{\Delta}_{n-1}^2 + a_n \alpha_{n-1}.$$

As in [5], we have $N\tilde{\Delta}_n / 2\alpha_n \leq q \ \ \forall n$. Thus,

$$\tilde{\Delta}_n = O(\| \tilde{x}_{n-1} - \tilde{x} \|) + O(q^n) = O\left(\sqrt{\alpha_{n-1}}\right) + O(q^n).$$

The theorem is proved.

Remark. For the given δ_n and h_n, we can chose $\alpha_n = \alpha(\delta_n, h_n)$ such that

$$\frac{\delta_n}{\alpha_n}, \quad \frac{h_n}{\alpha_n}, \quad \frac{\delta_n + \delta_{n-1}}{\alpha_{n-1}^2}, \quad \frac{h_n + h_{n-1}}{\alpha_{n-1}^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, for example, if $\delta_n = h_n = q^n$, then we can take $\alpha_n = q^{n/3}$. Consequently, for sufficiently large $n = N_0$, we have $\alpha_{N_0} \leq 2(1/r - q)q \,/\, N$. Then we can chose $\alpha_0 = \alpha_{N_0}$ such that

$$\frac{N \left\| x_0 - x_{\alpha_{N_0}} \right\|}{2\alpha_{N_0}} \leq q < \frac{1}{r}$$

and replace n in (9) by $n + N_0$.

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