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Dang Dinh Chau (Hanoi Univ. Sci., Vietnam),
Vu Tuan (Hanoi Pedagog. Univ., Vietnam)

ASYMPTOTICAL EQUIVALENCE OF TRIANGULAR DIFFERENTIAL EQUATION IN HILBERT SPACES

АСИМПТОТИЧНА ЕКВІВАЛЕНТНІСТЬ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ТРИКУТНОЇ ФОРМИ В ГІЛЬБЕРТОВИХ ПРОСТОРАХ

In this article, we study conditions for the asymptotic equivalence of differential equations in Hilbert spaces. Besides, we discuss the relation between properties of solutions of differential equations of triangular form and those of truncated differential equations.

Вивчено умови асимптотичної еквівалентності диференціальних рівнянь у гільбертових просторах. Розглянуто також зв'язок між властивостями розв'язків диференціальних рівнянь трикутної форми та неповних диференціальних рівнянь.

In a separable Hilbert space H, let us consider the differential equations

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

$$\frac{dy}{dt} = g(t, y), \tag{2}$$

where $f\colon R^+ \times H \to H$ and $g: R^+ \times H \to H$ are operators satisfying the conditions $f(t, 0) = 0$, $g(t, 0) = 0$ $\forall t \in R^+$ and all conditions of global theorem of existence and uniqueness of solution (see [1, p. 187 – 189]).

Definition 1 [2 – 4]. *Differential equations* (1) *and* (2) *are said to be asymptotically equivalent if there exists a bijection between a set of solutions* $\{x(t)\}$ *of* (1) *and the one of* $\{y(t)\}$ *of* (2) *such that*

$$\lim_{t\to\infty} \| x(t) - y(t) \| = 0.$$

Let $\{e_i\}_1^\infty$ be a basis of the separable Hilbert space H and let $x = \sum_{i=1}^\infty x_i e_i$ be an element of H. Then the operator $P_n\colon H \to H$ defined as

$$P_n x = \sum_{i=1}^{n} x_i e_i$$

is a projection on H. We introduce the notation $H_n = \operatorname{Im} P_n$.

Suppose that $J = \{n_1, n_2, \ldots, n_j, \ldots\}$ is a strictly increasing sequence of natural numbers ($n_j \to \infty$ as $j \to +\infty$). Together with system (1), (2), we consider the following systems of differential equations:

$$\begin{aligned} &\frac{dx}{dt} = P_m f(t, P_m x), \\ &(I - P_m)x = 0, \quad m \in J, \end{aligned} \tag{3}$$

$$\begin{aligned} &\frac{dy}{dt} = P_m g(t, P_m y), \\ &(I - P_m)y = 0, \quad m \in J. \end{aligned} \tag{4}$$

The stability of the solutions of differential equations (1) (or (2)) with the right-hand side satisfying the conditions

$$f(t, P_m x) \equiv P_m f(t, P_m x), \tag{5}$$

$$g(t, P_m x) \equiv P_m g(t, P_m x) \tag{6}$$

$$(\forall t \in R^+ \quad \forall m \in J \quad \forall x \in H)$$

was already studied in [5, 6]. In the present paper, we give some new definitions of asymptotical equivalence for these classes of differential equations and the corresponding results.

Definition 2. *Differential equations* (1) *and* (2) *are called asymptotically equivalent by part with respect to the set* J (*or* J*-asymptotically equivalent*) *if systems* (3) *and* (4) *are asymptotically equivalent for all* $m \in J$.

Since (5) we have the following lemma.

Lemma 1. *For any solution* $x(t) = x(t, t_0, P_m x_0)$, $x_0 \in H$, *of differential equation* (1) *the following relation*:

$$x(t, t_0, P_m x_0) = P_m x(t, t_0, P_m x_0)$$

will be held for all $t \in R^+$, $m \in J$, $x_0 \in H$.

Proof. For given $m \in J$, let us consider the differential equation

$$\frac{du}{dt} = f(t, P_m u), \qquad u \in H, \quad t \in R^+. \tag{7}$$

For $\xi_0 \in P_m H$, the solution $u(t) = x(t, t_0, \xi_0)$ of (7) is also a solution of the equation

$$u(t) = \xi_0 + \int_{t_0}^{t} f(\tau, P_m u(\tau))\, d\tau. \tag{8}$$

By virtue of (5) and the equation $P_m \xi_0 = \xi_0$, we have

$$u(t) = P_m \xi_0 + P_m \int_{t_0}^{t} f(\tau, P_m u(\tau))\, d\tau$$

or

$$u(t) = P_m \left\{ \xi_0 + \int_{t_0}^{t} f(\tau, P_m u(\tau))\, d\tau \right\}.$$

Hence,

$$u(t) = P_m u(t) \quad \forall t \in R^+.$$

Consequently, we can rewrite (8) as follows:

$$u(t) = \xi_0 + \int_{t_0}^{t} f(\tau, u(\tau))\, d\tau.$$

This shows that $u(t) = x(t, t_0, \xi_0)$ is a solution of (1), too.

Denoting by $x(t) = x(t, t_0, \xi_0)$ the solution of differential equation (1) satisfying the condition $x(t_0) = \xi_0$, by uniqueness of solution, we have

$$x(t) = u(t).$$

Hence, for $x_0 \in H$, any solution $x(t) = x(t, t_0, P_m x_0)$, $m \in J$, of differential equation (1) will satisfy the relation

$$x(t, t_0, P_m x_0) = P_m x(t, t_0, P_m x_0) \quad (\forall t \in R^+).$$

The lemma is proved.

***Remark* 1.** By virtue of Lemma 1, we can see that if conditions (5) and (6) are satisfied, then all solutions of equations (3), (4) are solutions of equations (1), (2), respectively. Therefore, from the asymptotical equivalence of system (1), (2), we can deduce their J-asymptotical equivalent.

Now we consider the following linear differential equations:

$$\frac{dx}{dt} = Ax, \tag{9}$$

$$\frac{dy}{dt} = [A + B(t)]y, \tag{10}$$

where $A \in \mathcal{L}(H)$, $B(t) \in \mathcal{L}(H)$ $\forall t \in [0, \infty)$ and

$$\int_0^\infty \| B(\tau) \| \, d\tau < \infty. \tag{11}$$

We assume that conditions (5), (6) are satisfied for these equations, that is

$$(A - P_m A)P_m x = 0, \tag{12}$$

$$\big(B(t) - P_m B(t)\big) P_m x = 0 \tag{13}$$

$$\forall m \in J \quad \forall x \in H.$$

Together with (9), (10), we consider also the sequences of truncated differential equations

$$\begin{aligned} &\frac{dx}{dt} = AP_m x, \\ &(I - P_m)x = 0, \quad m \in J, \end{aligned} \tag{14}$$

$$\begin{aligned} &\frac{dy}{dt} = [A + B(t)]P_m y, \\ &(I - P_m)y = 0, \quad m \in J. \end{aligned} \tag{15}$$

We denote by $X_m(t)$ the Cauchy operator of (14) satisfying $X_m(0) = E_m$ and by $Y_m(t)$ the Cauchy operator of (15) satisfying $Y_m(t_0) = E_m$, where E_m is a unit operator in H_m.

Lemma 2. *If all solutions of equation* (14) *are bounded, then*:

1. *The Cauchy operator* $X_m(t)$ *of* (14) *can be written in the form*:

$$X_m(t) = U_m(t) + V_m(t),$$

where $U_m(t)$ *and* $V_m(t)$: $H_m \to H_m$, *so that there exist positive constants* a_m, b_m, c_m *satisfying*

$$\| U_m(t) \| \le a_m e^{-b_m t} \quad \forall t \in R^+, \tag{16}$$

$$\| V_m(t) \| \le c_m \quad \forall t \in R. \tag{17}$$

2. *The operators* $F_m : H \to H$ *defined by*

$$F_m \xi = \int_{t_0}^{\infty} V_m(t_0 - \tau) B(\tau) Y_m(\tau) P_m \xi \, d\tau, \quad \xi \in H,$$

are bounded and moreover the following inequality is valid:

$$\| F_m \| \le \alpha_m < 1, \quad t_0 \ge \Delta > 0.$$

Proof. Using condition (12), we can write the Cauchy operator of (14) in the form $X_m(t) = e^{AP_m t}$. Since $\dim \operatorname{Im} P_m < \infty$ and $X_m(t)$ is bounded uniformly in t for every fixed m, using the same method in [2, p.160], we can prove Conclusion 1 of Lemma 2.

Denote by $Y_m(t)$ the Cauchy operator of equation (14) satisfying $Y_m(t_0) = E$. We see that $Y_m(t)$ satisfies the equations

$$Y_m(t) = X_m(t - t_0) + \int_{t_0}^{t} X_m(t-\tau)B(\tau)Y_m(\tau)\, d\tau \;\Rightarrow$$

$$\Rightarrow \; \|Y_m(t)\| \le \|X_m(t-t_0)\| + \int_{t_0}^{t} \|X_m(t-\tau)\| \, \|B(\tau)\| \, \|Y_m(\tau)\|\, d\tau.$$

By virtue of (16), (17), we have

$$\|Y_m(t)\| \le a_1 + a_1 \int_{t_0}^{t} \|B(\tau)\| \, \|Y_m(\tau)\|\, d\tau,$$

where $a_1 = 2\max(a_m, c_m)$.

Due to the Gronwall – Bellman lemma and condition (11), we have

$$\|Y_m(t)\| \le a_1 e^{\int_{t_0}^{t} \|B(\tau)\| d\tau} \le a_1 e^{\int_0^{\infty} \|B(\tau)\| d\tau}.$$

Hence, there exists a number K_m independent of t_0 so that

$$\| Y_m(t) \| \le K_m \quad \forall t \in R^+. \tag{18}$$

Moreover, for any $\alpha_m \le 1$, we can find a number $\Delta > 0$ so that

$$\int_{t_0}^{+\infty} \|B(\tau)\|\, d\tau \le \frac{\alpha_m}{c_m K_m} \quad \forall t_0 > \Delta.$$

This implies that

$$\|F_m\| \le \int_{t_0}^{\infty} \|V_m(t_0 - \tau)\| \|B(\tau)\| \|Y_m(\tau)\|\, d\tau \le$$

$$\le c_m K_m \int_{t_0}^{\infty} \|B(\tau)\|\, d\tau \le \alpha_m \le 1 \quad \forall t_0 > \Delta.$$

The lemma is proved.

Theorem 1. *Assume that, for any* $m \in J$, *the solutions of* (14) *are bounded. Then differential equations* (9) *and* (10) *are J-asymptotically equivalent.*

Proof. For each $m \in J$, we put

$$Q_m x = (I + F_m) P_m x, \quad x \in H.$$

Due to Lemma 2, the inequality $\| F_m \| < 1$ holds for $t_0 > \Delta$. Therefore, the operator Q_m is invertible.

Denoting $\eta_0 = Q_m^{-1}\xi_0$, $\xi_0 \in H_m$, $m \in J$, we consider the solutions $x(t) = x(t, t_0, \xi_0)$ of (14) and $y(t) = y(t, t_0, \eta_0)$ of (15).

It is clear that

$$x(t) = X_m(t-t_0)\xi_0$$

and

$$y(t) = X_m(t-t_0)\eta_0 + \int_{t_0}^{t} X_m(t-\tau)B(\tau)y(\tau)\,d\tau.$$

By analogy with Lemma 2, since H_m is a finite dimensional subspace of H and $X_m(t)$ is bounded uniformly in t for every fixed m, by using the same method in [2, p. 160], we can prove that

$$X_m(t-t_0) = U_m(t-t_0) + V_m(t-t_0),$$

$$V_m(t-\tau) = X_m(t-t_0)V_m(t_0-\tau).$$

From the definition of Q_m, we have

$$\xi_0 = Q_m\eta_0 = \eta_0 + \int_{t_0}^{\infty} V_m(t_0-\tau)B(\tau)Y_m(\tau)\eta_0\,d\tau.$$

Hence,

$$x(t) = X_m(t-t_0)\eta_0 + X_m(t-t_0)\int_{t_0}^{\infty} V_m(t_0-\tau)B(\tau)Y_m(\tau)\eta_0\,d\tau =$$

$$= X_m(t-t_0)\eta_0 + \int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(\tau)\eta_0\,d\tau.$$

Consequently,

$$\|y(t) - x(t)\| =$$

$$= \left\| -\int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(\tau)\eta_0\,d\tau + \int_{t_0}^{t} X_m(t-\tau)B(\tau)y(\tau)\,d\tau \right\| \Leftrightarrow$$

$$\Leftrightarrow \|y(t) - x(t)\| =$$

$$\Leftrightarrow \left\| -\int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(t)\eta_0\,d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)\,d\tau + \right.$$

$$\left. + \int_{t_0}^{t} V_m(t-\tau)B(\tau)y(\tau)\,d\tau \right\|.$$

Since $y(t) = Y_m(t)\eta_0$, we have

$$\|y(t) - x(t)\| =$$

$$= \left\| -\int_{t_0}^{\infty} V_m(t-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)\,d\tau + \right.$$

$$+ \int_{t_0}^{t} V_m(t-\tau)B(\tau)y(\tau)\,d\tau \Bigg\| \Leftrightarrow$$

$$\Leftrightarrow \|y(t) - x(t)\| =$$

$$= \left\| -\int_{t}^{\infty} V_m(t-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)\,d\tau \right\|.$$

Using (16) – (18) and taking into account $y(t) = Y_m(t)\eta_0$, we have

$$\|y(t) - x(t)\| \le a_m K_m \|\eta_0\| \int_{t_0}^{t} e^{-b_m(t-\tau)} \|B(\tau)\|\,d\tau + c_m K_m \|\eta_0\| \int_{t}^{\infty} \|B(\tau)\|\,d\tau$$

or

$$\|y(t) - x(t)\| \le M_1 \int_{t_0}^{t} e^{-b_m(t-\tau)} \|B(\tau)\|\,d\tau + M_2 \int_{t}^{\infty} \|B(\tau)\|\,d\tau \quad \forall\, t \ge t_0,$$

where

$$M_1 = a_m K_m \|\eta_0\|, \quad M_2 = c_m K_m \|\eta_0\|.$$

Then, for every positive number $\varepsilon > 0$, there exists a sufficiently large number t and $t > 2t_0$ such that the following inequalities are valid:

$$\int_{t_0}^{t/2} e^{-b_m(t-\tau)} \|B(\tau)\|\,d\tau \le e^{-\frac{b_m t}{2}} \int_{t}^{\infty} \|B(\tau)\|\,d\tau < \frac{\varepsilon}{3M_1},$$

$$\int_{t/2}^{t} \|B(\tau)\|\,d\tau < \frac{\varepsilon}{3M_1}, \qquad \int_{t}^{\infty} \|B(\tau)\|\,d\tau < \frac{\varepsilon}{3M_2}.$$

Hence,

$$\|y(t) - x(t)\| \le M_1 \left(\int_{t_0}^{t/2} e^{-b_m(t-\tau)} \|B(\tau)\|\,d\tau + \int_{t/2}^{t} e^{-b_m(t-\tau)} \|B(\tau)\|\,d\tau \right) +$$

$$+ M_2 \int_{t}^{\infty} \|B(\tau)\|\,d\tau < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This means that

$$\lim_{t\to\infty} \|y(t) - x(t)\| = 0.$$

By the uniqueness of solutions of differential equations (14) and (15), the map Q_m is bijective between two sets of solutions of equations (14) and (15).

The theorem is proved.

Lemma 3. *If all solutions of the differential equations* (9) *are bounded, then*:

1) *there exists a positive number* $\Delta = \Delta(\alpha)$ *such that*

$$\|F_m\| \le \alpha < 1 \quad \forall\, t_0 \ge \Delta \quad \forall\, m \in J;$$

2) $\{F_m\}$ *and* $\{Q_m\}$ *are convergent sequences of operators as* $m \to \infty$.

Proof. Due to the boundedness of all solutions of (9), there is a number $\beta_1 > 0$ such that the Cauchy operator $X(t)$ of (9) satisfies the relation

$$\|X(t)\| \leq \beta_1 \quad \forall t \in R^+.$$

If we denote by $Y(t)$ the Cauchy operator of (10) satisfying $Y(t_0) = E$, we see that $Y(t)$ satisfies the equation

$$Y(t) = X(t-t_0) + \int_{t_0}^{t} X(t-\tau)B(\tau)Y(\tau)\,d\tau \Rightarrow$$

$$\Rightarrow \|Y(t)\| \leq \|X(t-t_0)\| + \int_{t_0}^{t} \|X(t-\tau)\|\|B(\tau)\|\|Y(\tau)\|\,d\tau \Rightarrow$$

$$\Rightarrow \|Y(t)\| \leq \beta_1 + \beta_1 \int_{t_0}^{t} \|B(\tau)\|\|Y(\tau)\|\,d\tau.$$

Due to the Gronwall – Bellman lemma and condition (11), there exists a number β_2 independent of t_0 and of m such that

$$\|Y(t)\| \leq \beta_2 \quad \forall t \in R^+.$$

Consequently,

$$\|X_m(t)\| \leq \beta_1, \quad \|Y_m(t)\| \leq \beta_2 \quad \forall t \in R^+ \quad \forall m \in J.$$

On the other hand, for any $0 < \alpha < 1$, we can find a number $\Delta = \Delta(\alpha) > 0$ such that

$$\int_{t_0}^{\infty} \|B(t)\|\,d\tau \leq \frac{\alpha}{\beta_1\beta_2} < +\infty \quad \forall t_0 \geq \Delta.$$

Analogously, as in the proof of Lemma 2, we have

$$\|F_m\| \leq \int_{t_0}^{\infty} \|V_m(t_0-\tau)\|\|B(\tau)\|\|Y_m(\tau)\|\,d\tau \leq$$

$$\leq \beta_1\beta_2 \int_{t_0}^{\infty} \|B(\tau)\|\,d\tau \leq \alpha < 1 \quad \forall m \in J \quad \forall t_0 \geq \Delta.$$

By definition,

$$F_m\xi = \int_{t_0}^{+\infty} V_m(t_0-\tau)\,B(\tau)Y(\tau)P_m\xi\,d\tau.$$

From (12) and (13) we can show that for all $m, m+p \in J$, $p > 0$,

$$X_{m+p}(t-t_0)P_m\xi = X_m(t-t_0)P_m\xi \quad \forall \xi \in H,$$

$$Y_{m+p}(t)P_m\xi = Y_m(t)P_m\xi \quad \forall \xi \in H.$$

Hence,

$$F_{m+p}P_m\xi = F_mP_m\xi \quad \forall m, m+p \in J, \quad p > 0.$$

We now prove the convergence of $\{F_m\}$. In fact, for all $m, m+p \in J$, $p > 0$, we have

$$\|F_{m+p} - F_m\| = \|F_{m+p}P_{m+p} - F_mP_m\| = \|F_{m+p}(P_{m+p} - P_m) + (F_{m+p} - F_m)P_m\| =$$

$$= \|F_{m+p}(P_{m+p} - P_m)\| \le \|F_{m+p}\|\|P_{m+p} - P_m\|.$$

By definition, $\lim_{m\to\infty} P_m = I$. Hence, by the boundedness of F_m, the above yields that $\{F_m\}$ is a Cauchy sequence, so $\{F_m\}$ is convergent. This implies the convergence of $\{Q_m\}$.

Theorem 2. *If all solutions of the differential equation* (9) *are bounded, then equations* (9) *and* (10) *are asymptotically equivalent.*

Proof. By virtue of Lemma 3, we can put

$$F = \lim_{m\to\infty} F_m \quad \text{and} \quad Q = \lim_{m\to\infty} Q_m.$$

Hence, $Q = I + F$. Since $\|F_m\| \le \alpha < 1 \ \forall m \in J \ \forall t_0 \ge \Delta$, we have

$$\|F\| \le \alpha < 1 \quad \forall t_0 \ge \Delta.$$

Therefore, $Q: H \to H$ is an invertible operator.

By uniqueness of solutions of equations (9) and (10), we deduce that the map Q is also bijective between two sets of solutions $\{x(t)\}$ of (9) and $\{y(t)\}$ of (10).

Let $y_0 = Q^{-1}x_0$ and $x(t) = X(t - t_0)x_0$, $y(t) = Y(t)y_0$. Since

$$\lim_{m\to\infty} P_m y_0 = y_0, \qquad \lim_{m\to\infty} Q_m y_0 = Q y_0 = x_0,$$

we can deduce that, for any arbitrarily given, $\varepsilon < 0$ there exists sufficiently large $m_1 \in J$ such that, for all $m \ge m_1$, we have

$$\|y(t;t_0,y_0) - y(t;t_0,P_m y_0)\| < \frac{\varepsilon}{3},$$

$$\|x(t;t_0,y_0) - x(t;t_0,Q_m y_0)\| < \frac{\varepsilon}{3}$$

for all $t \ge t_0$.

By virtue of Theorem 1 and boundedness of all solutions of (9), we deduce that differential equations (9) and (10) are J-asymptotically equivalent. Consequently, there exists $\tau_0 \in (t_0, \infty)$ such that, for all $t \ge \tau_0$,

$$\|x(t;t_0,Q_{m_1}y_0) - y(t;t_0,P_{m_1}y_0)\| < \frac{\varepsilon}{3},$$

where t_0 is choosen sufficiently large such that

$$\|F_m\| \le \alpha < 1 \quad \forall m \in J.$$

Therefore,

$$\|y(t;t_0,y_0) - x(t;t_0,x_0)\| \le$$

$$\le \|y(t;t_0,y_0) - y(t;t_0,P_{m_1}y_0)\| + \|y(t;t_0,P_{m_1}y_0) - x(t;t_0,Q_{m_1}y_0)\| +$$

$$+ \|x(t;t_0,Q_{m_1}y_0) - x(t;t_0,x_0)\| \le$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall t \ge \tau_0.$$

This implies that

$$\lim_{t\to\infty} \| x(t;t_0,x_0) - y(t;t_0,y_0) \| = 0.$$

The theorem is proved.

By virtue of Lemma 1, we can immediately obtain the following corollaries:

***Corollary* 1.** *Assume that all solutions of the differential equation* (9) *are bounded. Then the differential equations* (9) *and* (10) *are asymptotically equivalent if and only if they are* J*-asymptotically equivalent.*

***Corollary* 2.** *If all solutions of differential equations* (14) *are uniformly bounded for all* $m \in J$, *then differential equations* (9) *and* (10) *are asymptotically equivalent.*

***Remark* 2.** In the case where the supposition of the boundedness of solutions of differential equation (9) is not satisfied, by similar way as in [4], we can consider ψ-asymptotical equivalence. Therefore, it is clear that, by choosing suitable in the Hilbert space H, we can get the broadeness of the Levisions theorem for linear differential equation with the operator A on the right-hand side of (9) being compact self-adjoint.

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