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RANDOM ATTRACTORS FOR STOCHASTIC 2D HYDRODYNAMICAL TYPE SYSTEMS*

ВИПАДКОВІ АТРАКТОРИ ДЛЯ СТОХАСТИЧНИХ ДВОВИМІРНИХ СИСТЕМ ГІДРОДИНАМІЧНОГО ТИПУ

We study the asymptotic behavior of solutions to a class of abstract nonlinear stochastic evolution equations with additive noise that covers numerous 2D hydrodynamical models, such as the 2D Navier – Stokes equations, 2D Boussinesq equations, 2D MHD equations, etc., and also some 3D models, like the 3D Leray α-model. We prove the existence of random attractors for the associated continuous random dynamical systems. Then we establish the upper semicontinuity of the random attractors as the parameter tends to zero.

Вивчається асимптотична поведінка розв’язків одного класу абстрактних нелінійних стохастичних рівнянь еволюції з адитивним шумом, що включає різноманітні двовимірні гідродинамічні моделі, такі як двовимірні рівняння Нав’є – Стокса, двовимірні рівняння Буссінеска, двовимірні рівняння магнітогідродинаміки тощо, а також деякі тривимірні моделі типу тривимірної α-моделі Лерея. Доведено існування випадкових атракторів для відповідних неперервних випадкових динамічних систем. Крім того, встановлено напівнеперервність зверху випадкових атракторів у випадку, коли параметр прямує до нуля.

1. Introduction. The study of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. In the deterministic case, the notion of global attractors, a compact invariant and attracting set, plays a central role (see, for example, [11, 22]). The concept of random attractors was introduced in [14, 15] as an extension to stochastic systems of the concept of global attractors for deterministic systems. The theory of random attractors has been shown to be very useful for the study of the long-time behavior of infinite-dimensional random dynamical systems, see the recent survey [16] and references therein. Up to now, the existence of random attractors has been proved for many classes of stochastic partial differential equations, see, e.g., [2, 4, 6 – 8, 17 – 20, 24 – 28].

In this paper, we study the long-time behavior of solutions to the following abstract nonlinear stochastic 2D hydrodynamical type system:

$$du + (Au + B(u,u) + Ru)dt = f dt + \varepsilon h d\omega. \tag{1.1}$$

As pointed out in [11, 12], with suitable choices of A, B and $R,$ this abstract model covers many 2D hydrodynamical models such as 2D Navier – Stokes equations, 2D Boussinesq equations, 2D MHD equations, 2D magnetic Bénard equations, and also some 3D models such as 3D Leray-α model, the shell models of turbulence. In the paper [12], the authors proved the existence and uniqueness of weak solutions, and more importantly, the Wentzell – Freidlin type large deviation principle for small multiplicative noise to this equation. The support of distribution of solutions to this abstract

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model was described later in [13]. The stability and stabilization of solutions to the abstract model (1.1) was investigated recently in [3]. It is also noticed that the existence and long-time behavior of solutions in terms of existence of global attractors to the corresponding deterministic version of this model (i.e., equation (1.1) without the stochastic term) was given in the monograph [11].

In this paper we consider the abstract equation (1.1) with additive noise. We first prove the existence of a random attractor for the continuous random dynamical system generated by the equation. Then we establish the upper semicontinuity of the random attractor at $\varepsilon = 0$, that is, we compare the random attractor of the stochastic equation (1.1) and the global attractor of the limit deterministic equation, which is formally obtained when $\varepsilon = 0$. Under the assumptions in the paper (see Subsection 2.2 for details), the associated random dynamical system is not necessary compact, and therefore the pullback asymptotic compactness of the dynamical system cannot be obtained directly by constructing random absorbing sets in a more regular space and using some compact embeddings. In order to overcome this essential difficulty, we exploit the energy equations method to prove the pullback asymptotic compactness. This method was first introduced by Ball in [5] for the deterministic wave equation, and then extensively used by many authors for weakly dissipative equations or equations in unbounded domains, both in deterministic and stochastic cases (see, for instance, [6, 10, 21, 24] and references therein). It is worthy noticing that, as a direct consequence of the abstract results obtained in this paper, we get the existence and upper semicontinuity of random attractors for many 2D models in fluid mechanics, in both bounded domains and unbounded domains satisfying the Poincaré inequality (see Remark 4.1 below).

The outline of this paper is as follows. In Section 2, we recall the theory of random attractors and give a description of the problem. The existence of a random attractor for the associated random dynamical system is proved in Section 3, while its upper semicontinuity is investigated in Section 4.

2. Preliminaries.

2.1. Random attractors.

In this subsection, we recall some concepts and results on theories of random dynamical systems and random attractors in [1, 9, 14, 18, 23].

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 2.1. *$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta\colon \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t\theta_s$ for all $s, t \in \mathbb{R}$, and $\theta_t(P) = P$ for all $t \in \mathbb{R}$.*

Definition 2.2. *A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping*

$$\Phi\colon \mathbb{R}^+ \times \Omega \times X \to X, (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$, the following conditions:

(i) $\Phi(0, \omega, \cdot)$ *is the identity of X;*

(ii) $\Phi(t + s, \omega, x) = \Phi(t, \theta_s\omega, \Phi(s, \omega, x))$ *for all $t,\ s \in \mathbb{R}^+, x \in X$;*

(iii) $\Phi(t, \omega, \cdot)\colon X \to X$ *is continuous for all $t \in \mathbb{R}^+$.*

Definition 2.3. *A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,*

$$\lim_{t\to\infty} e^{-\beta t}\|B(\theta_{-t}\omega)\| = 0 \qquad \textit{for all} \qquad \beta > 0,$$

where $\|B\|_X = \sup_{x\in B} \|x\|_X$.

Hereafter, we always assume that Φ is a continuous RDS over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ and denote by $\mathcal{D}$ a collection of random subsets $\{B(\omega)\}_{\omega\in\Omega}$ of X.

Definition 2.4. *A random set* $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ *is said to be a random absorbing set for* Φ *in* $\mathcal{D}$ *if for every* $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ *and* P*-a.e.* $\omega \in \Omega$, *there exists* $T_B(\omega) > 0$ *such that*

$$\Phi\left(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\right) \subseteq K(\omega) \qquad \textit{for all} \qquad t \geq T_B(\omega).$$

Definition 2.5. *A random dynamical system* Φ *is called* $\mathcal{D}$*-pullback asymptotically compact in* X *if for* P*-a.e.* $\omega \in \Omega, \{\Phi\left(t_n, \theta_{-t_n}\omega, x_n\right)\}_{n\geq 1}$ *has a convergent subsequence in* X *whenever* $t_n \to \infty$, *and* $x_n \in B(\theta_{-t_n}\omega)$, *where* $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.6. *A random set* $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *of* X *is called a* $\mathcal{D}$*-random attractor for* Φ *if the following conditions are satisfied, for* P*-a.e.* $\omega \in \Omega$:

(i) $\mathcal{A}(\omega)$ *is compact, and* $\omega \mapsto d(x, \mathcal{A}(\omega))$ *is measurable for every* $x \in X$;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *is invariant, that is,* $\Phi\left(t, \omega, \mathcal{A}(\omega)\right) = \mathcal{A}(\theta_t\omega)$ *for all* $t \geq 0$;

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *attracts every set in* $\mathcal{D}$, *that is, for every* $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to+\infty} \operatorname{dist}\left(\Phi\left(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\right), \mathcal{A}\left(\omega\right)\right) = 0,$$

where dist *is the Hausdorff semidistance*

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_X \qquad \textit{for all} \qquad A, B \subset X.$$

The following existence result for the random attractor for a continuous RDS can be found in [7, 18, 23].

Theorem 2.1. *Let* $\mathcal{D}$ *be an inclusion-closed collection of random subsets of* X *and assume that* Φ *is a continuous RDS which has a random absorbing set* $\{K(\omega)\}_{\omega\in\Omega}$. *If* Φ *is* $\mathcal{D}$*-pullback asymptotically compact in* X, *then it has a unique* $\mathcal{D}$*-random attractor* $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi\left(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\right)}.$$

Let Φ_0 be an autonomous dynamical system defined on the Banach space X. Given $\varepsilon \in (0, 1]$, suppose Φ_ε is a random dynamical system over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ which has a random absorbing set $K_\varepsilon = \{K_\varepsilon(\omega)\}_{\omega\in\Omega}$ and a random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$. We suppose that the autonomous dynamical system $\Phi_0 : \mathbb{R}^+ \times X \to X$ has a global attractor $\mathcal{A}_0$, which means that $\mathcal{A}_0$ is compact, invariant and attracts every bounded subset of X uniformly (see, e.g., [11] for the theory of global attractors).

Definition 2.7. *The family of random attractors* $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,1]}$ *is said to be upper semicontinuous at* $\varepsilon = 0$ *if*

$$\lim_{\varepsilon\to 0} \operatorname{dist}\left(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0\right) = 0 \qquad \textit{for} \qquad P\textit{-a.e.} \quad \omega \in \Omega.$$

Theorem 2.2 [23]. *Suppose that the following conditions hold for* P*-a.e.* $\omega \in \Omega$:

(i) $\Phi_{\varepsilon_n}\left(t, \omega, x_n\right) \to \Phi(t)x$ *for all* $t \geq 0$, *provided* $\varepsilon_n \to 0$ *and* $x_n \to x$ *in* X;

(ii) $\limsup_{\varepsilon\to 0} \|K_\varepsilon(\omega)\|_X \le M,$ *where* $\|K_\varepsilon(\omega)\|_X = \sup_{x\in K_\varepsilon(\omega)} \|x\|_X;$

(iii) $\bigcup_{0<\varepsilon\le 1} \mathcal{A}_\varepsilon(\omega)$ *is precompact in* X.

Then the family of random attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,1]}$ *is upper semicontinuous at* $\varepsilon = 0$.

2.2. Model description. Let H be a separable Hilbert space with the the norm $|\cdot|$ and the inner product $(\cdot,\cdot)$, let A be an (unbounded) self-adjoint positive linear operator on H. Set $V = \mathrm{Dom}(A^{1/2})$. For $v \in V$ set $\|v\| = |A^{1/2}v|$. Let V' denote the dual of V (with respect to the inner product $(.,.)$ of H). Then we have the triple $V \subset H \subset V'$. Let $\langle u, v\rangle$ denote the duality between $u \in V$ and $v \in V'$ such that $\langle u, v\rangle = (u, v)$ for $u \in V,\ v \in H$.

We assume $B\colon V \times V \to V'$ and $R\colon H \to H$ are continuous mappings satisfying the following conditions:

Main assumptions:

$B\colon V \times V \to V'$ is a bilinear continuous mapping.

For all $u, v, w \in V$,

$$\langle B(u, v), w\rangle = -\langle B(u, w), v\rangle. \tag{2.1}$$

There exists a Banach (interpolation) space $\mathcal{H}$ possessing the properties:

(i) $V \subset \mathcal{H} \subset H$;

(ii) there exists a constant $a_0 > 0$ such that

$$\|v\|^2_{\mathcal{H}} \le a_0|v|\|v\| \qquad \text{for any} \qquad v \in V; \tag{2.2}$$

(iii) there exists a constant $C > 0$ such that

$$|\langle B(u, v), w\rangle| \le C\|u\|_{\mathcal{H}}\|v\|\|w\|_{\mathcal{H}} \qquad \forall u, v, w \in V. \tag{2.3}$$

$R\colon H \to H$ is a bounded linear operator such that

$$\|R\|_{\mathrm{op}} < \lambda, \tag{2.4}$$

where $\lambda > 0$ is the best constant in the inequality

$$\|u\|^2 \ge \lambda|u|^2 \qquad \forall u \in V. \tag{2.5}$$

From (2.1) – (2.3), one can see that

for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u, v), w\rangle| \le \eta\|w\|^2 + C_\eta\|u\|^2_{\mathcal{H}}\|v\|^2_{\mathcal{H}} \qquad \text{for} \qquad u, v, w \in V; \tag{2.6}$$

there exist a positive constant C_0 such that

$$\begin{gathered} |\langle B(u, v), w\rangle| \le C_0\|u\|\|v\|\|w\| \qquad \forall u, v, w \in V, \\ \|B(u, u)\|_{V'} \le C_0\|u\|^2 \qquad \forall u \in V. \end{gathered} \tag{2.7}$$

Let $f \in H$ and $h \in D(A)$. We consider a small random perturbation of the 2D hydrodynamical type systems given by

$$du + (Au + B(u,u) + Ru)dt = fdt + \varepsilon h d\omega, \tag{2.8}$$

where $\omega(t)$ is a two-sided real-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the Wiener distribution, Ω is a subset of

$$\{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

with $P(\Omega) = 1$, $\mathcal{F}$ is a σ-algebra. In addition, the space $(\Omega, \mathcal{F}, P)$ is invariant under the Wiener shift

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad \omega \in \Omega, \qquad t \in \mathbb{R}.$$

This means that $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Let σ be a fixed positive constant such that

$$\sigma > \frac{2\lambda C^2 a_0^2}{(\lambda - \|R\|_{\mathrm{op}})^2}\|h\|^2, \tag{2.9}$$

where a_0, C and λ are the constants in (2.2), (2.3) and (2.5), respectively. Consider the one-dimensional Ornstein – Uhlenbeck equation

$$dy + \sigma y dt = d\omega(t).$$

One can check that a solution of this equation is given by

$$y(\theta_t \omega) = -\sigma \int\limits_{-\infty}^{0} e^{\sigma s} (\theta_t \omega)(s) ds.$$

Note that the random variable $|y(\omega)|$ is tempered and $y(\theta_t \omega)$ is P-a.e. continuous. Therefore, it follows from [1] (Proposition 4.3.3) that there exists a tempered function $r(\omega) > 0$ such that

$$|y(\omega)|^2 + |y(\omega)|^4 \le r(\omega) \qquad \forall \omega \in \Omega, \tag{2.10}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$, $r(\theta_t w) \le e^{\frac{\sigma}{2}|t|} r(\omega)$, $t \in \mathbb{R}$.

Now, we need to transfer the stochastic equation (2.8) into a deterministic one with random parameters. Let $z(\theta_t \omega) = h y(\theta_t \omega)$ and $v(t,\omega) = u(t,\omega) - \varepsilon z(\theta_t \omega)$, then v is a solution of the equation

$$\frac{dv}{dt} + Av + B(v,v) + Rv + \varepsilon B(v,z) + \varepsilon B(z,v) = f - \varepsilon Az - \varepsilon Rz - \varepsilon^2 B(z,z) + \varepsilon \sigma z \tag{2.11}$$

with $v_0(\omega) = u_0(\omega) - \varepsilon z(\omega)$.

Let $\omega \in \Omega$ and $v_0 \in H$. A mapping $v(\cdot, \omega, v_0) : [0, +\infty) \to H$ is called a solution of problem (2.11) if, for every $T > 0$,

$$v(\cdot, \omega, v_0) \in C([0,T]; H) \cap L^2(0,T; V),$$

and v satisfies

$$(v(t),\xi)+\int\limits_0^t\langle Av,\xi\rangle ds+\int\limits_0^t\langle\tilde{B}(v),\xi\rangle ds+\int\limits_0^t(Rv,\xi)ds=\int\limits_0^t(\tilde{F},\xi)ds, \qquad (2.12)$$

where $\tilde{B}(v)=\varepsilon B(v,z)+\varepsilon B(z,v)$ and $\tilde{F}=f-\varepsilon Az-\varepsilon Rz-\varepsilon^2B(z,z)+\varepsilon\sigma z$ for every $t\geq 0$ and $\xi\in V$. Since (2.11) is a deterministic equation, it follows from [11] (Section 4.4) that for every $\omega\in\Omega$ and $v_0\in H$ given, problem (2.11) has a unique solution v in the sense of (2.12) which continuously depends on v_0 with the respect to the norm of H. Moreover, the solution v is $(\mathcal{F},\mathcal{B}(H))$-measurable in $\omega\in\Omega$. This enables us to define a mapping $\Phi\colon\mathbb{R}^+\times\Omega\times H\to H$ by

$$\Phi(t,\omega,u_0(\omega))=u(t,\omega,u_0(\omega))=v(t,\omega,v_0(\omega))+\varepsilon z(\theta_t\omega), \qquad (2.13)$$

then we see that Φ is a continuous RDS associated with the stochastic equation (2.8).

Given a bounded nonempty subset B of H, we write $|B|=\sup_{\phi\in B}|\phi|_H$. We denote by $\mathcal{D}$ the collection of random sets $\{B(\omega\}_{\omega\in\Omega}$ of H, which satisfy for P-a.e. $\omega\in\Omega$,

$$\lim_{t\to+\infty}e^{-\frac{\nu}{2}t}|B(\theta_{-t}\omega)|_H=0,$$

where $\nu:=\lambda-\|R\|_{\mathrm{op}}>0$.

3. Existence of a random attractor. ***3.1. Uniform estimates of solutions and existence of a random absorbing set in H.*** In this subsection, we first establish the uniform estimates on the solutions to problem (2.11), then we will show that the RDS Φ associated with the stochastic equation (2.8) has a random absorbing set in H.

Lemma 3.1. *Let $0<\varepsilon\leq 1$, $f\in H$, $h\in D(A)$, and* (2.9) *hold. Then for any $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and for P-a.e. ω, there exists $T=T(B,\omega)>0$ independent of ε such that for all $t\geq T$ and $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, the solution v of* (2.11) *satisfies*

$$|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_H^2\leq 1+c\int\limits_{-\infty}^0\exp\left(\nu\tau+\beta\int\limits_\tau^0|y(\theta_r\omega)|^2dr\right)g(\theta_\tau\omega)d\tau, \qquad (3.1)$$

where

$$\nu=\lambda-\|R\|_{\mathrm{op}}>0,$$

$$\beta=\frac{\lambda C^2a_0^2}{\nu}\|h\|^2,$$

$$c=\frac{2(3+\sigma^2)}{\nu|f|^2}\max\left\{C_0^2\|h\|^4,(\|R\|_{\mathrm{op}}^2+1)\|h\|_{D(A)}^2\right\}, \qquad (3.2)$$

$$g(\theta_\tau\omega)=|y(\theta_\tau\omega)|^4+|y(\theta_\tau\omega)|^2+1.$$

Proof. It follows from (2.11) that, for each $\omega \in \Omega$ and $v \in V$,

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 \leq -\varepsilon\langle B(v,z),v\rangle + \|R\|_{\text{op}}|v|^2 + \varepsilon(Rz,v) + (F,v) \leq$$

$$\leq -\varepsilon\langle B(v,z),v\rangle + \frac{\|R\|_{\text{op}}}{\lambda}\|v\|^2 + \|R\|_{\text{op}}|z||v| + (F,v),$$

where $F = f - \varepsilon Az - \varepsilon^2 B(z,z) + \varepsilon\sigma z$. Therefore,

$$\frac{d}{dt}|v|^2 + 2\beta_1\|v\|^2 \leq -2\varepsilon\langle B(v,z),v\rangle + 2\|R\|_{\text{op}}|z||v| + 2(F,v), \tag{3.3}$$

where $\beta_1 = \dfrac{\lambda - \|R\|_{\text{op}}}{\lambda} > 0$. The right-hand side of (3.3) is bounded by

$$2|\varepsilon\langle B(v,z),v\rangle| + 2\|R\|_{\text{op}}|z|\|v\| + 2(F,v) \leq$$

$$\leq \frac{C^2a_0^2}{\beta_1}\|z\|^2|v|^2 + \frac{2}{\nu}(\|R\|^2_{\text{op}}|z|^2 + |F|^2) + \beta_1\|v\|^2. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\frac{d}{dt}|v|^2 + \left(\nu - \frac{\lambda C^2 a_0^2}{\nu}\|z\|^2\right)|v|^2 \leq \frac{2}{\nu}(\|R\|^2_{\text{op}}|z|^2 + |F|^2),$$

where $\nu = \lambda\beta_1 = \lambda - \|R\|_{\text{op}}$. On the other hand, using Schwarz's inequality and noting that $0 < \varepsilon \leq 1$, we have

$$|F|^2 = |f - \varepsilon Az - \varepsilon^2 B(z,z) + \varepsilon\sigma z|^2 \leq$$

$$\leq (3+\sigma^2)\left[C_0^2\|z\|^4 + |Az|^2 + |f|^2 + |z|^2\right].$$

Therefore,

$$|F(\theta_t\omega|^2 \leq (3+\sigma^2)[C_0^2\|h\|^4|y(\theta_t\omega)|^4 + |Ah|^2|y(\theta_t\omega)|^2 + |f|^2 + |h|^2|y(\theta_t\omega)|^2].$$

Hence, let $\beta = \dfrac{\lambda C^2 a_0^2}{\nu}\|h\|^2$, we get

$$\frac{d}{dt}|v|^2 + (\nu - \beta|y(\theta_t\omega|^2)|v|^2 \leq c\left(|y(\theta_t\omega)|^4 + |y(\theta_t\omega)|^2 + 1\right),$$

where

$$c = \frac{2(3+\sigma^2)}{\nu|f|^2}\max\left\{C_0^2\|h\|^4, (\|R\|^2_{\text{op}} + 1)\|h\|^2_{D(A)}\right\}.$$

Multiplying (3.3) by $\exp\left(\nu t - \beta\displaystyle\int_0^t |y(\theta_r\omega|^2 dr\right)$ and integrating the resulting inequality on $[0,s]$, we obtain

$$|v(s,\omega,v_0(\omega)|_H^2 \le \exp\left(-\nu s + \beta\int\limits_0^s |y(\theta_r\omega|^2 dr\right)|v_0(\omega)|^2 +$$

$$+c\int\limits_0^s \exp\left(\nu(\tau - s) - \beta\int\limits_s^\tau |y(\theta_r\omega|^2 dr\right) g(\theta_\tau\omega)d\tau, \tag{3.5}$$

where $g(\theta_\tau\omega) = |y(\theta_\tau\omega)|^4 + |y(\theta_\tau\omega)|^2 + 1$.

We now estimate the last term on the right-hand side of (3.5). To do this, in (3.5), we replace s and ω by t and $\theta_{-t}\omega$, respectively,

$$|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)|_H^2 \le \exp\left(-\nu t + \beta\int\limits_0^t |y(\theta_{r-t}\omega|^2 dr\right)|v_0(\theta_{-t}\omega)|_H^2 +$$

$$+c\int\limits_0^t \exp\left(\nu(\tau - t) - \beta\int\limits_t^\tau |y(\theta_{r-t}\omega|^2 dr\right) g(\theta_{\tau-t}\omega)dr =$$

$$= \exp\left(-\nu t + \beta\int\limits_{-t}^0 |y(\theta_r\omega|^2 dr\right)|v_0(\theta_{-t}\omega)|^2 +$$

$$+c\int\limits_{-t}^0 \exp\left(\nu\tau + \beta\int\limits_\tau^0 |y(\theta_r\omega|^2 dr\right) g(\theta_\tau\omega)d\tau. \tag{3.6}$$

Thanks to the Ergodic theorem, for any $\omega \in \Omega$,

$$\lim_{t\to+\infty} \frac{1}{t}\int\limits_{-t}^0 |y(\theta_r\omega|^2 dr = E(|y(\omega)|^2).$$

On the other hand,

$$E(|y(\omega)|^2) = \frac{\Gamma\left(\frac{3}{2}\right)}{\sigma\sqrt{\pi}} = \frac{\sqrt{\pi}}{2\sigma\sqrt{\pi}} = \frac{1}{2\sigma},$$

where $\Gamma(\cdot)$ is the Gamma function. Thus, there exists a number $T_1 = T_1(B,\omega) > 0$ such that, for all $t \ge T_1$,

$$\beta\int\limits_{-t}^0 |y(\theta_r\omega|^2 dr < \frac{\beta}{\sigma}t \le \frac{\nu t}{2}, \tag{3.7}$$

where we have used the condition (2.9). By (3.6) and (3.7), we have

$$|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)|_H^2 \le e^{-\frac{\nu}{2}t}|v_0(\theta_{-t}\omega)|_H^2+$$

$$+c\int\limits_{-t}^{0}\exp\left(\nu\tau+\beta\int\limits_{\tau}^{0}|y(\theta_r\omega|^2dr\right)g(\theta_\tau\omega)d\tau. \tag{3.8}$$

Note that $|y(\theta_r\omega|$ is tempered, by using (2.10) and a few simple calculations, we can find that the integrand of the second term on the right-hand side of (3.8) converges to zero exponentially when $t\to+\infty$. Thus, the integral

$$R_0=c\int\limits_{-\infty}^{0}\exp\left(\nu\tau+\beta\int\limits_{\tau}^{0}|y(\theta_r\omega|^2dr\right)g(\theta_\tau\omega)dr$$

is convergent. Moreover, since $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$, for the first term on the right-hand side of (3.8), we get

$$e^{-\frac{\nu}{2}t}|v_0(\theta_{-t}\omega)|_H^2 \le e^{-\frac{\nu}{2}t}|B(\theta_{-t}\omega)|_H^2\to 0 \qquad \text{as} t\to+\infty.$$

This shows that there exists $T_2=T_2(B,\omega)>0$ such that

$$e^{-\frac{\nu}{2}t}|v_0(\theta_{-t}\omega)|_H^2\le 1 \qquad \text{for all} \qquad t\ge T_2. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_H^2\le 1+c\int\limits_{-\infty}^{0}\exp\left(\nu\tau+\beta\int\limits_{\tau}^{0}|y(\theta_r\omega)|^2dr\right)g(\theta_\tau\omega)d\tau$$

for all $t\ge T=\max\{T_1,T_2\}$.

Lemma 3.1 is proved.

Lemma 3.2. *The random dynamical system Φ has a random absorbing set $K=\{K(\omega)\}_{\omega\in\Omega}$ in H, where K is independent of ε.*

Proof. Let $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ be fixed. For given $u_0(\omega)\in H$, let v be the solution of (2.11) with the initial condition $v(0)=u_0(\omega)-\varepsilon z(\omega)$. Then we have

$$|v_0(\omega)|_H^2=|u_0(\omega)-\varepsilon z(\omega)|_H^2\le 2(|u_0(\omega)|_H^2+\varepsilon^2|z(\omega)|_H^2)\le$$

$$\le 2(|B(\omega)|_H^2+|z(\omega)|_H^2).$$

This implies that $v_0(\omega)\in\tilde{B}(\omega)$ for all $\omega\in\Omega$, where

$$\tilde{B}(\omega)=\left\{u\in H: |u|^2\le 2(|B(\omega)|_H^2+|z(\omega)|_H^2)\right\}. \tag{3.10}$$

Moreover, since $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $|z(\omega)|$ is tempered, this follows $\left\{\tilde{B}(\omega)\right\}_{\omega\in\Omega}\in\mathcal{D}$. Then, by (3.1), we obtain

$$|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_H^2 \le 1 + c\int\limits_{-\infty}^{0} \exp\left(\nu\tau + \beta\int\limits_{\tau}^{0}|y(\theta_r\omega)|^2 dr\right) g(\theta_\tau\omega)d\tau, \tag{3.11}$$

provided $t \ge T$ and $v_0(\theta_{-t}\omega) \in \tilde{B}(\theta_{-t}\omega)$. On the other hand,

$$\varPhi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)) = v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)) + \varepsilon z(\omega). \tag{3.12}$$

From (3.11) and (3.12), we get, for all $t \ge T$ and $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$|\varPhi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|_H^2 \le 2(r_0(\omega) + |z(\omega)|_H^2),$$

where $r_0(\omega) = 1 + c\int_{-\infty}^{0} \exp\left(\nu\tau + \beta\int_{\tau}^{0}|y(\theta_r\omega)|^2 dr\right) g(\theta_\tau\omega)d\tau$. This implies that $\varPhi$ possesses a random absorbing set in H, which is independent of ε.

3.2. Pullback asymptotic compactness. In this subsection, we prove the $\mathcal{D}$-pullback asymptotic compactness of solutions to problem (2.11). For this purpose, we need the following weak continuity of solutions with respect to initial data, which can be established by the standard method as in [6, 21].

Lemma 3.3. *Let $\omega \in \Omega$ and $x_0 \in H$. If $x_n \rightharpoonup x_0$ weakly in H, then the solution v of problem* (2.11) *has the following properties:*

$$v(t,\omega,x_n) \rightharpoonup v(t,\omega,x_0) \text{ weakly in } H \text{ for all } t \ge 0,$$

$$v(\cdot,\omega,x_n) \rightharpoonup v(\cdot,\omega,x_0) \text{ weakly in } L^2(0,T;V) \text{ for all } T > 0,$$

$$u(\cdot,\omega,x_n - \varepsilon z(\omega)) \rightharpoonup u(\cdot,\omega,x_0 - \varepsilon z(\omega)) \text{ weakly in } L^2(0,T;V) \text{ for all } T > 0.$$

The next lemma is concerned with the pullback asymptotic compactness of problem (2.11).

Lemma 3.4. *For every $\omega \in \Omega, B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $t_n \to +\infty, x_n \in B(\theta_{-t_n}\omega)$, the sequence of solutions $\{v(t_n,\theta_{-t_n}\omega,x_n)\}$ to* (2.11) *has a convergent subsequence in H.*

Proof. Since $t_n \to +\infty$, there exists $N_0 \in \mathbb{N}$ such that $t_n \ge T$ for all $n \ge N_0$. Note that $x_n \in B(\theta_{-t_n}\omega)$, we get from (3.1) that, for all $n \ge N_0$,

$$|v(t_n,\theta_{-t_n}\omega,x_n)|_H^2 \le 1 + c\int\limits_{-\infty}^{0} \exp\left(\nu\tau + \beta\int\limits_{\tau}^{0}|y(\theta_r\omega)|^2 dr\right) g(\theta_\tau\omega)d\tau.$$

Hence, there exist $\tilde{v} \in H$ and a subsequence (which is not relabeled) such that

$$v(t_n,\theta_{-t_n}\omega,x_n) \rightharpoonup \tilde{v} \qquad \text{in } H. \tag{3.13}$$

We now prove that the weak convergence of (3.13) is actually a strong convergence, which will complete the proof. Note that (3.13) implies that

$$\liminf_{n\to\infty} |v(t_n,\theta_{-t_n}\omega,x_n)|_H \ge |\tilde{v}|_H.$$

So we only need to show that

$$\limsup_{n\to\infty} |v(t_n, \theta_{-t_n}\omega, x_n)|_H \le |\tilde{v}|_H. \tag{3.14}$$

We will establish (3.14) by the method of energy equations due to Ball [5]. Given $k \in \mathbb{N}$, we have

$$v(t_n, \theta_{-t_n}\omega, x_n) = v(k + t_n - k, \theta_{-t_n}\omega, x_n) = v(k, \theta_{-k}\omega, v(t_n - k, \theta_{-t_n}\omega, x_n)). \tag{3.15}$$

Since $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$, for each k, let N_k be large enough such that $t_n \ge T + k$ for all $n \ge N_k$. Then it follows from Lemma 3.1 that, for $n \ge N_k$,

$$|v(t_n - k, \theta_{-t_n}\omega, x_n)|_H^2 \le$$

$$\le\ e^{\nu k}\left[e^{-\frac{\nu}{2}t_n}|x_n|_H^2 + c\int\limits_{-\infty}^{0} \exp\left(\nu\tau + \beta\int\limits_{\tau}^{0}|y(\theta_r\omega|^2 dr\right) g(\theta_\tau\omega)d\tau\right] \le$$

$$\le\ e^{\nu k}\left(1 + c\int\limits_{-\infty}^{0} \exp\left(\nu\tau + \beta\int\limits_{\tau}^{0}|y(\theta_r\omega|^2 dr\right) g(\theta_\tau\omega)d\tau\right), \tag{3.16}$$

which shows that, for each fixed $k \in \mathbb{N}$, the sequence $v(t_n - k, \theta_{-t_n}\omega, x_n)$ is bounded in H. By a diagonal process, one can find a subsequence (which we do not relabel) and a point $\tilde{v}_k \in H$ for each $k \in \mathbb{N}$ such that

$$v(t_n - k, \theta_{-t_n}\omega, x_n) \rightharpoonup \tilde{v}_k \quad \text{in } H. \tag{3.17}$$

By (3.15)–(3.17) and Lemma 3.3, we get that, for each $k \in \mathbb{N}$,

$$v(t_n, \theta_{-t_n}\omega, x_n) \rightharpoonup v(k, \theta_{-k}\omega, \tilde{v}_k) \quad \text{in } H, \tag{3.18}$$

and

$$v(\cdot, \theta_{-t_n}\omega, v(t_n, \theta_{-t_n}\omega, x_n)) \rightharpoonup v(\cdot, \theta_{-k}\omega, \tilde{v}_k) \ \text{ in } L^2(0, k; V).$$

From (3.13) and (3.18), we obtain

$$v(k, \theta_{-k}\omega, \tilde{v}_k) = \tilde{v} \quad \text{for all } k > 0. \tag{3.19}$$

Denote $\phi(v) = 2\dfrac{\nu}{\lambda}\|v\|^2 - \dfrac{3}{2}\nu|v|^2$, we have

$$\frac{\nu}{2\lambda}\|v\|^2 \le \phi(v) \le 2\frac{\nu}{\lambda}\|v\|^2 \quad \text{for all } v \in V,$$

this indicates that $\phi(\cdot)$ is an equivalent norm of V. On the other hand, (2.11) implies that

$$\frac{d}{dt}|v|^2 + \frac{3}{2}\nu|v|^2 + \phi(v) + 2\langle B(u,u), v\rangle \le 2(\tilde{F}, v), \tag{3.20}$$

where ν is defined in (3.2) and $\tilde{F} = f - \varepsilon Az - \varepsilon Rz + \varepsilon\sigma z$.

Multiplying (3.20) by $e^{\eta t}$ with $\eta = \dfrac{3\nu}{2}$ and integrating the resulting equation on $(0,t)$, we obtain

$$|v\left(t,\omega,v_0(\omega)\right)|_H^2 + 2\int_0^t e^{-\eta(t-s)}\langle B(u\left(s,\omega,u_0(\omega)\right), u(s,\omega,u_0(\omega))), v(s,\omega,v_0(\omega))\rangle ds =$$

$$= e^{-\eta t}|v_0(\omega)|_H^2 + 2\int_0^t e^{-\eta(t-s)}(f + \varepsilon\sigma z(\theta_s\omega) - \varepsilon Az(\theta_s\omega) - \varepsilon Rz(\theta_s\omega), v\left(s,\omega,v_0(\omega)\right) ds +$$

$$+ \int_0^t e^{-\nu(t-s)}\phi(s,\omega,v_0(\omega))ds. \tag{3.21}$$

Replacing t,ω in (3.21) by k and $\theta_{-t}\omega$, respectively, and by (3.19), we find

$$|\tilde{v}|_H^2 = |v(k,\theta_{-k}\omega,\tilde{v}_k)|_H^2 =$$

$$= 2\int_0^k e^{-\eta(k-s)}\langle B(u\left(s,\theta_{-k}\omega,u_{n,k}\right), u(s,\theta_{-k}\omega,u_{n,k})), v(s,\theta_{-k}\omega,\tilde{v}_k)\rangle ds +$$

$$+2\int_0^k e^{-\eta(k-s)}(f + \varepsilon\sigma z(\theta_{s-k}\omega) - \varepsilon Az(\theta_{s-k}\omega) - \varepsilon Rz(\theta_{s-k}\omega), v\left(s,\theta_{-k}\omega,\tilde{v}_k\right))ds +$$

$$+e^{-\eta k}|\tilde{v}_k|_H^2 + \int_0^k e^{-\eta(k-s)}\phi(s,\theta_{-k}\omega,\tilde{v}_k)ds, \tag{3.22}$$

where $u_{n,k} = \tilde{v}_k + \varepsilon z(\theta_{-k}\omega)$. Similarly, by (3.15) and (3.21), we get

$$|v(k,\theta_{-k},v_{n,k})|_H^2 =$$

$$= 2\int_0^k e^{-\eta(k-s)}\langle B(u\left(s,\theta_{-k}\omega,u_{n,k}\right), u(s,\theta_{-k}\omega,u_{n,k})), v(s,\theta_{-k}\omega,v_{n,k})\rangle ds +$$

$$+2\int_0^k e^{-\eta(k-s)}(f + \varepsilon\sigma z(\theta_{s-k}\omega) - \varepsilon Az(\theta_{s-k}\omega) - \varepsilon Rz(\theta_{s-k}\omega), v\left(s,\theta_{-k}\omega,v_{n,k}\right))ds +$$

$$+e^{-\eta k}|v_{n,k}|_H^2+\int\limits_0^k e^{-\eta(k-s)}\phi(s,\theta_{-k}\omega,v_{n,k})ds, \tag{3.23}$$

where

$$v_{n,k}=v(t_n-k,\theta_{-t_n}\omega,x_n),\quad u_{n,k}=v_{n,k}+\varepsilon z(\theta_{-k}\omega).$$

We now consider the limit of each term on the right-hand side of (3.23) as $n\to\infty$. For the first term, by Lemma 3.2 and [6] (Corollary 5.3),

$$\limsup_{n\to\infty}\int\limits_0^k e^{-\eta(k-s)}\langle B(u\,(s,\theta_{-k}\omega,u_{n,k})\,,u(s,\theta_{-k}\omega,u_{n,k})),v(s,\theta_{-k}\omega,v_{n,k})\rangle ds=$$

$$=\int\limits_0^k e^{-\nu(k-s)}\langle B(u\,(s,\theta_{-k}\omega,\tilde u_k)\,,u(s,\theta_{-k}\omega,\tilde u_k)),v(s,\theta_{-k}\omega,\tilde v_k)\rangle ds, \tag{3.24}$$

where $\tilde u_k=\tilde v_k+\varepsilon z(\theta_{-k}\omega)$. For the second term, note that

$$e^{-\eta(k-\cdot)}(f+\varepsilon\sigma z(\theta_{\cdot-k}\omega)-\varepsilon Az(\theta_{\cdot-k}\omega)-\varepsilon Rz(\theta_{\cdot-k}\omega))\in L^2(0,k;V'), \tag{3.25}$$

thus, we find

$$\limsup_{n\to\infty}\int\limits_0^k e^{-\eta(k-s)}(f+\varepsilon\sigma z(\theta_{s-k}\omega)-\varepsilon Az(\theta_{s-k}\omega)-\varepsilon Rz(\theta_{s-k}\omega),v\,(s,\theta_{-k}\omega,v_{n,k}))ds=$$

$$=\int\limits_0^k e^{-\eta(k-s)}(f+\varepsilon\sigma z(\theta_{s-k}\omega)-\varepsilon Az(\theta_{s-k}\omega)-\varepsilon Rz(\theta_{s-k}\omega),v\,(s,\theta_{-k}\omega,\tilde v_k))ds. \tag{3.26}$$

Moreover, $\int_0^k e^{-\eta(k-s)}\phi(\cdot)ds$ defines a norm in $L^2(0,k;V)$ which is equivalent to the usual one, thus, by (3.17), we obtain

$$\liminf_{n\to\infty}\int\limits_0^k e^{-\eta(k-s)}\phi(s,\theta_{-k}\omega,v_{n,k})ds=\int\limits_0^k e^{-\eta(k-s)}\phi(s,\theta_{-k}\omega,\tilde v_k)ds. \tag{3.27}$$

Finally, by (3.16) and $\eta=\dfrac{3\nu}{2}$, we get

$$e^{-\eta k}|v_{n,k}|_H^2\le e^{-(\eta-\nu)k}\left(1+c\int\limits_{-\infty}^0\exp\left(\nu\tau+\beta\int\limits_\tau^0|y(\theta_r\omega|^2dr\right)g(\theta_\tau\omega)d\tau\right)=$$

$$= e^{-\frac{\nu}{2}k} \left(1 + c \int\limits_{-\infty}^{0} \exp \left(\nu\tau + \beta \int\limits_{\tau}^{0} |y(\theta_r\omega|^2 dr \right) g(\theta_\tau\omega) d\tau \right). \tag{3.28}$$

By (3.15) and (3.22) – (3.28), we have

$$\limsup_{n\to\infty} |v(t_n, \theta_{-t_n}\omega, x_n)|_H^2 - |\tilde{v}|_H^2 \le e^{-\frac{\nu}{2}k} r_0(\omega) - e^{-\nu k}|\tilde{v}_k|_H^2 \le$$

$$\le e^{-\frac{\nu}{2}k} r_0(\omega) \to 0 \qquad \text{as} \quad k \to +\infty,$$

where $r_0(\omega) = 1 + c \int_{-\infty}^{0} \exp\left(\nu\tau + \beta \int_{\tau}^{0} |y(\theta_r\omega|^2 dr \right) g(\theta_\tau\omega) d\tau < +\infty$. This implies that

$$\limsup_{n\to\infty} |v(t_n, \theta_{-t_n}\omega, x_n)|_H^2 \le |\tilde{v}|_H^2,$$

whence (3.14) follows.

Lemma 3.4 is proved.

Lemma 3.5. *The RDS Φ is pullback $\mathcal{D}$-asymptotically compact in H, that is, for every $\omega \in \Omega$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, and $t_n \to +\infty$, $x_n \in B(\theta_{-t_n}\omega)$, the sequence $\Phi(t_n, \theta_{-t_n}\omega, x_n)$ has a convergent subsequence in H.*

Proof. Since $B \in \mathcal{D}$ and $x_n \in B(\theta_{-t_n}\omega)$, by the proof of Lemma 3.2, we find, for each $n \in \mathbb{N}$, $y_n = x_n - \varepsilon z(\omega) \in \tilde{B}$, where $\tilde{B} \in \mathcal{D}$ is the family defined by (3.10). Then it follows from Lemma 3.4 that the sequence $v(t_n, \theta_{-t_n}\omega, y_n)$ of solutions to problem (2.11) has a convergent subsequence in H. On the other hand, by (2.13), we have

$$u(t_n, \theta_{-t_n}\omega, x_n) = v(t_n, \theta_{-t_n}\omega, y_n) + \varepsilon z(\omega),$$

and hence, the sequence $u(t_n, \theta_{-t_n}\omega, x_n)$ has a convergent subsequence in H. This implies that $\Phi(t_n, \theta_{-t_n}\omega, x_n)$ has a convergent subsequence in H.

Lemma 3.5 is proved.

3.3. Existence of a random attractor.

Theorem 3.1. *For each $\varepsilon > 0$, the continuous RDS Φ associated with problem* (2.8) *has a unique $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$ in H.*

Proof. By Lemma 3.2, we know that the continuous RDS Φ has a family of $\mathcal{D}$-random absorbing sets $\{K_\varepsilon(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$. On the other hand, by Lemma 3.5, we find that RDS Φ is $\mathcal{D}$-pullback asymptotically compact. Then it follows from Theorem 2.1 that Φ has a unique $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon$ in H and the structure of $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$ is given by

$$\mathcal{A}_\varepsilon(\omega) = \bigcap_{\tau\ge 0} \overline{\bigcup_{t\ge\tau} \Phi\left(t, \theta_{-t}\omega, K_\varepsilon(\theta_{-t}\omega)\right)}.$$

4. Upper semicontinuity of the random attractor. In this section, we prove the upper semicontinuity of random attractors for the 2D hydrodynamical type systems when the stochastic perturbations approach zero. The existence of the global attractor $\mathcal{A}_0$ for the (deterministic) dynamical system associated to (1.1) when $\varepsilon = 0$ has been proved in [11] (Section 4.5). To prove the upper semicontinuity result, we first establish the convergence of solutions to problem (2.11) when $\varepsilon \to 0$, and then show that the union of all perturbed random attractors is precompact in H.

Lemma 4.1. *For every $t \in \mathbb{R}$ and $\omega \in \Omega$, if $\varepsilon_n \to 0$ and $x_n \to x_0$ in H, then*

$$v_{\varepsilon_n}(t, \omega, x_n) \to v_0(t, x_0).$$

Proof. Let $v_0(\cdot) = v_0(\cdot, x_0)$, for every $n \in \mathbb{N}$, we denote $v_n(\cdot) = v_{\varepsilon_n}(\cdot, \omega, x_n)$ a solution of equation (2.11) with ε is replaced by ε_n. Denote $w_n = v_n - v_0$, we obtain

$$\partial_t w_n + Aw_n + Rw_n + B(v_n + \varepsilon_n z, v_n + \varepsilon_n z) - B(v_0, v_0) = -\varepsilon_n Az - \varepsilon_n Rz + \varepsilon_n \sigma z,$$

with $w_n(0) = x_n - x_0$. Hence, we have

$$\frac{1}{2}\frac{d}{dt}|w_n|^2 + \|w_n\|^2 = -(Rw_n, w_n) - \langle B(v_n + \varepsilon_n z, v_n + \varepsilon_n z) - B(v_0, v_0), w_n\rangle -$$

$$-(\varepsilon_n Az + \varepsilon_n \sigma z, w_n) + (\varepsilon_n Rz, w_n) \leq$$

$$\leq \frac{\|R\|_{\mathrm{op}}}{\lambda}\|w_n\|^2 - \langle B(v_n + \varepsilon_n z, v_n + \varepsilon_n z) - B(v_0, v_0), w_n\rangle -$$

$$-(\varepsilon_n Az + \varepsilon_n \sigma z, w_n) + (\varepsilon_n Rz, w_n),$$

thus,

$$\frac{d}{dt}|w_n|^2 + \frac{2\nu}{\lambda}\|w_n\|^2 \leq -2\langle B(v_n + \varepsilon_n z, v_n + \varepsilon_n z) - B(v_0, v_0), w_n\rangle +$$

$$+2(-\varepsilon_n Az + \varepsilon_n \sigma z, w_n) + 2(-\varepsilon_n Rz, w_n). \tag{4.1}$$

Now we will estimate the terms on the right-hand side of (4.1). For the first term, we get

$$|-2\langle B(v_n + \varepsilon_n z, v_n + \varepsilon_n z) - B(v_0, v_0), w_n\rangle| \leq$$

$$\leq 2|\langle B(w_n, w_n), v_0\rangle| + \varepsilon_n|\langle B(v_n, z), w_n\rangle| + \varepsilon_n|\langle B(z, v_n), w_n\rangle| + \varepsilon_n^2|\langle B(z, z), w_n\rangle|, \tag{4.2}$$

and by (2.2)–(2.5), we obtain

$$
\begin{aligned}
&2|\langle B(w_n,w_n),v_0\rangle| \le 2Ca_0|w_n|\|v_0\|\|w_n\| \le \\
&\le \frac{4\lambda C^2a_0^2}{\nu}|w_n|^2\|v_0\|^2 + \frac{\nu}{4\lambda}\|w_n\|^2, \\
&2\varepsilon_n|\langle B(v_n,z),w_n\rangle| \le 2\varepsilon_n C_0\|v_n\|\|z\|\|w_n\| \le \\
&\le \frac{4\lambda C_0^2}{\nu}\varepsilon_n^2\|z\|^2\|v_n\|^2 + \frac{\nu}{4\lambda}\|w_n\|^2, \\
&2\varepsilon_n|\langle B(z,v_n),w_n\rangle| \le 2C_0\varepsilon_n\|v_n\|\|z\|\|w_n\| \le \\
&\le \frac{4\lambda C_0^2}{\nu}\varepsilon_n^2\|z\|^2\|v_n\|^2 + \frac{\nu}{4\lambda}\|w_n\|^2, \\
&2\varepsilon_n^2|\langle B(z,z),w_n\rangle| \le 2C_0\varepsilon_n^2\|z\|^2\|w_n\| \le \\
&\le \frac{4\lambda C_0^2}{\nu}\varepsilon_n^4\|z\|^4 + \frac{\nu}{4\lambda}\|w_n\|^2.
\end{aligned} \tag{4.3}
$$

For the second term, after a few simple computations, we get

$$
2|(\varepsilon_n Az + \varepsilon_n\sigma z, w_n)| \le \frac{2}{\nu}(1+\sigma^2)\varepsilon_n^2(|Az|^2+|z|^2) + \frac{\nu}{2\lambda}\|w_n\|^2 =
$$

$$
= c_1\varepsilon_n^2(|Az|^2+|z|^2) + \frac{\nu}{2\lambda}\|w_n\|^2, \tag{4.4}
$$

where $c_1 = \dfrac{2}{\nu}(1+\sigma^2)$. Moreover,

$$
2|(-\varepsilon_n Rz, w_n)| = 2\varepsilon_n|(Rz,w_n)| \le \frac{2}{\nu}\varepsilon_n^2\|R\|_{\text{op}}^2|z|^2 + \frac{\nu}{2\lambda}\|w_n\|^2 \le
$$

$$
\le c_2\varepsilon_n^2|z|^2 + \frac{\nu}{2\lambda}\|w_n\|^2, \tag{4.5}
$$

where $c_2 = \dfrac{2\|R\|_{\text{op}}^2}{\nu}$. From (4.1)–(4.5), we have

$$
\frac{d}{dt}|w_n|^2 \le c_3\rho^2|w_n|^2 + \left(c_4\rho^2 + (c_1+c_2)\|h\|_{D(A)}^2\right)\varepsilon_n^2|y(\theta_t\omega)|^2 + c_5\varepsilon_n^4|y(\theta_t\omega)|^4,
$$

where

$$
c_3 = \frac{4\lambda C^2a_0^2}{\nu}\rho^2,
$$

$$c_4 = \frac{8\lambda C_0^2}{\nu}\|h\|^2,$$

$$c_5 = \frac{4\lambda C_0^2}{\nu}\|h\|^4,$$

$$\rho = \rho(t) = \sup_{s\in[0,t]} \|v_n(s)\|.$$

By the Gronwall lemma, we deduce that

$$|w_n(t)|_H^2 \leq (|w_n(0)|_H^2 + \varepsilon_n^2 r(t))e^{c_3\rho^2 t} \to 0 \text{ as } n\to\infty,$$

where

$$r(t) = \int\limits_0^t \Big[(c_4\rho^2 + (c_1+c_2)\|h\|_{D(A)}^2)|y(\theta_s\omega)|^2 + c_5|y(\theta_s\omega)|^4\Big]\,ds < +\infty.$$

This implies that $\lim_{n\to\infty}|w_n(t)|_H = 0$.

Lemma 4.1 is proved.

The above lemma enables us to obtain the second condition in Theorem 2.2.

Lemma 4.2. *For P-a.e. $\omega\in\Omega$,*

$$\Phi_{\varepsilon_n}(t,\omega,x_n) \to \Phi(t)x \quad \textit{for all} \quad t\geq 0, \quad \textit{provided} \quad \varepsilon_n\to 0 \quad \textit{and} \quad x_n\to x \quad \textit{in} \quad H.$$

Proof. By (2.13), we have

$$\Phi_{\varepsilon_n}(t,\omega,x_n) = v_{\varepsilon_n}(t,\omega,x_n - \varepsilon_n z(\omega)) + \varepsilon_n z(\theta_t\omega).$$

Note that $x_n\to x_0$ in H and $\varepsilon_n\to 0$, we find

$$|x_n - \varepsilon_n z(\omega) - x_0|_H \leq |x_n - x_0|_H + \varepsilon_n|z(\omega)|_H \to 0.$$

Therefore, by Lemma 4.1, we obtain

$$|v_{\varepsilon_n}(t,\omega,x_n-\varepsilon_n z(\omega)) - v_0(t,x_0)|_H \to 0 \qquad \text{as} \qquad n\to\infty.$$

But $\Phi(t,x_0) = v_0(t,x_0)$, then

$$|\Phi_{\varepsilon_n}(t,\omega,x_n) - \Phi(t,x_0)|_H = |v_{\varepsilon_n}(t,\omega,x_n-\varepsilon_n z(\omega)) + \varepsilon_n z(\theta_t\omega) - v_0(t,x_0)|_H \leq$$

$$\leq |v_{\varepsilon_n}(t,\omega,x_n-\varepsilon_n z(\omega)) - v_0(t,x_0)|_H + \varepsilon_n|z(\theta_t\omega)|_H \to 0.$$

Lemma 4.2 is proved.

The proof of the following lemma is similar to that given in [6], so we only state the result.

Lemma 4.3. *Let $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Suppose that $\varepsilon_n\to\varepsilon_0, t_n\to+\infty$ and $y_n\in B(\theta_{-t_n}\omega)$, then*

$$\{\Phi_{\varepsilon_n}(t_n,\theta_{-t_n}\omega,y_n)\}$$

is precompact in H.

Now, given $0 < \varepsilon \leq 1$, it follows from Lemma 3.2 that, for every $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $T = T(B,\omega) > 0$, independent of ε, such that, for all $t \geq T$,

$$|\Phi_\varepsilon(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|_H^2 = |v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)) + \varepsilon z(\omega)|_H^2 \leq$$

$$\leq 2(|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_H^2 + \varepsilon^2|z(\omega)|_H^2) \leq$$

$$\leq 2(r_0(\omega) + \varepsilon^2|z(\omega)|_H^2),$$

where $r_0(\omega) = 1 + c\displaystyle\int_{-\infty}^{0} \exp\left(\nu\tau + \beta\int_{\tau}^{0} |y(\theta_r\omega)|^2 dr\right) g(\theta_\tau\omega)d\tau$. Denote

$$K_\varepsilon(\omega) = \left\{u \in H : |u|^2 \leq 2(r_0(\omega) + \varepsilon^2|z(\omega)|_H^2)\right\} \tag{4.6}$$

and

$$K(\omega) = \left\{u \in H : |u|^2 \leq 2(r_0(\omega) + |z(\omega)|_H^2)\right\}.$$

Then, for every $0 < \varepsilon \leq 1$, $\{K_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a closed absorbing set for Φ_ε in $\mathcal{D}$ and

$$\bigcup_{0<\varepsilon\leq 1} K_\varepsilon(\omega) \subset K(\omega). \tag{4.7}$$

It follows from the invariance of the random attractor $\{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and (4.7) that

$$\bigcup_{0<\varepsilon\leq 1} \mathcal{A}_\varepsilon(\omega) \subset \bigcup_{0<\varepsilon\leq 1} K_\varepsilon(\omega) \subset K(\omega). \tag{4.8}$$

Lemma 4.4. *For every $\omega \in \Omega$, the union $\bigcup_{0<\varepsilon\leq 1} \mathcal{A}_\varepsilon(\omega)$ is precompact in H.*

Proof. First, we take an sequence $\{x_n\} \subset \bigcup_{\varepsilon\in(0,1]} \mathcal{A}_\varepsilon(\omega)$, then there exists ε_0 such that $\mathcal{A}_{\varepsilon_0}$ contains infinitely many elements of x_n. From the compactness of $\mathcal{A}_{\varepsilon_0}$, we find that $\{x_n\}$ has a convergent subsequence. On the other hand, we can assume that $x_n \in \mathcal{A}_{\varepsilon_n}(\omega)$ with $\varepsilon_n \in (0,1]$ and $\varepsilon_n \neq \varepsilon_m$ when $m \neq n$. Due to $\{\varepsilon_n\} \subset (0,1]$, without loss of generality, we can assume that $\varepsilon_n \to \varepsilon_0 \in [0,1]$ as $n \to +\infty$. Fix a sequence t_n such that $t_n \to +\infty$. By the invariance of $\mathcal{A}_{\varepsilon_n}$, we find that

$$\mathcal{A}_{\varepsilon_n}(\omega) = \Phi_{\varepsilon_n}\left(t_n, \theta_{-t_n}\omega, \mathcal{A}_{\varepsilon_n}\left(\theta_{-t_n}\omega\right)\right).$$

Since $x_n \in \Phi_{\varepsilon_n}\left(t_n, \theta_{-t_n}\omega, \mathcal{A}_{\varepsilon_n}\left(\theta_{-t_n}\omega\right)\right)$, there exists an element $y_n \in \mathcal{A}_{\varepsilon_n}\left(\theta_{-t_n}\omega\right)$ such that $x_n = \Phi_{\varepsilon_n}\left(t_n, \theta_{-t_n}\omega, y_n\right)$. Note that by (4.8), we have $\varepsilon_n \to \varepsilon_0, t_n \to +\infty$, and $y_n \in K(\theta_{-t_n}\omega)$, thus applying Lemma 4.3, we obtain $\{\Phi_{\varepsilon_n}\left(t_n, \theta_{-t_n}w, y_n\right)\}$ is precompact in H, i.e., $\{x_n\}$ has a convergent subsequence.

Theorem 4.1. *For P-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0} \operatorname{dist}\left(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0\right) = 0. \tag{4.9}$$

Proof. Recall that $\{K_\varepsilon(\omega)\}_{\omega\in\Omega}$ is a closed absorbing set for Φ_ε in $\mathcal{D}$, where $K_\varepsilon(\omega)$ is given by (4.6). By (4.6) we find that

$$\limsup_{\varepsilon\to 0} |K_\varepsilon(\omega)| \leq \sqrt{2r_0(\omega)} := M. \tag{4.10}$$

Let $\varepsilon_n \to 0$ and $x_n \to x_0$ in H, then by Lemma 4.1 we find that, for P-a.e. $\omega \in \Omega$ and $t \geq 0$,

$$\Phi_{\varepsilon_n}(t,\omega,x_n) \to \Phi(t,x_0). \tag{4.11}$$

Note that (4.10), (4.11) and Lemma 4.2 indicate all conditions in Theorem 2.2 are satisfied, and hence, (4.9) follows.

***Remark* 4.1.** As a direct consequence of the abstract results obtained in the paper, we get the existence and upper semicontinuity of random attractors for many 2D partial differential equations in fluid mechanics with additive noise in bounded domains or unbounded domains satisfying the Poincaré inequality, including 2D Navier – Stokes equations, 2D MHD equations, 2D Boussinesq equations, 2D magnetic Bénard equations, and also some 3D models such as 3D Leray-α model, the shell models of turbulence. To do this, it only need to verify the abstract conditions for each concrete model (see [11] (Section 4.6) or [12] for details).

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