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JACOBI MATRIX PAIR AND DUAL ALTERNATIVE q-CHARLIER POLYNOMIALS*

ПАРА МАТРИЦЬ ЯКОБІ ТА ДУАЛЬНІ АЛЬТЕРНАТИВНІ q-МНОГОЧЛЕНИ ШАРЛЬЄ

By using two operators, representable by Jacobi matrices, we introduce a family of q-orthogonal polynomials, which turn out to be dual with respect to alternative q-Charlier polynomials. A discrete orthogonality relation and completeness property for these polynomials are obtained.

За допомогою двох операторів, зображуваних матрицями Якобі, введено сім'ю q-ортогональних многочленів, що є дуальними по відношенню до альтернативних q-многочленів Шарльє. Для цих многочленів отримано дискретне співвідношення ортогональності та властивість повноти.

1. Introduction. It is well known that the characteristic properties of orthogonal polynomials are interwoven with spectral properties of symmetric operators, which can be represented in some basis by a Jacobi matrix, and with the classical moment problem (see, for example, [1 – 3]). Namely, the spectrum of an operator, represented by a Jacobi matrix, is determined by an orthogonality measure for corresponding orthogonal polynomials. If orthogonal polynomials admit many orthogonality relations, then the corresponding symmetric operator is not self-adjoint and it has infinitely many self-adjoint extensions. These extensions are determined by orthogonality measures for the appropriate orthogonal polynomials.

Contrary to orthogonal polynomials of the hypergeometric type (Wilson, Jacobi, Laguerre and so on), basic hypergeometric polynomials (or q-orthogonal polynomials) are not so deeply understood yet. These polynomials are collected in the q-analogue of the Askey-scheme of orthogonal polynomials (see, for example, [4]), which starts with Askey – Wilson polynomials and q-Racah polynomials, introduced in [5] and [6], respectively. The importance of these polynomials is magnified by the fact that they are closely related to the theory of quantum groups. As an instance of such connection we refer to a paper [7], in which Al-Salam – Chihara q-orthogonal polynomials have been employed to construct locally compact quantum group $SU_q(1, 1)$. Another application of q-orthogonal polynomials is related to the theory of q-difference equations, which often surface in contemporary theoretical and mathematical physics.

The purpose of the present paper is to study the duality properties of alternative q-Charlier polynomials. We shall show below that this leads to novel type of q-orthogonal polynomials (expressed in terms of the basic hypergeometric function ${}_3\phi_0$) and a discrete orthogonality relation for them. To achieve this, we essentially use two operators I_1 and J, which are certain representation operators for the quantum algebra $U_q(\mathrm{su}_{1,1})$ with a lowest weight (however, we do not employ explicitly the theory of representations in what follows). An orthogonality relation for a set of orthogonal polynomials and the spectral measure for the associated symmetric (self-adjoint) operator, representable by a Jacobi matrix, are closely interrelated. But the problem usually arises how to find this orthogonality relation (or spectral measure). We propose to use for this purpose a second operator, also representable (with respect to another basis) by a Jacobi matrix. By employing these two operators, one is able to find orthogonality relations for both alternative q-Charlier polynomials and their duals.

The first operator I_1 (which is a Hilbert – Schmidt operator and, therefore, has the

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discrete spectrum) is related to the three-term recurrence relation for alternative q-Charlier polynomials. We diagonalize the operator I_1 and obtain two bases in the Hilbert space: an initial basis and a basis of eigenvectors of I_1. The initial basis is orthonormalized. It is a problem to normalize the second basis. We normalize it by means of the second operator J. These orthonormalized bases are connected by an orthogonal matrix A (since its matrix elements are real). Now the orthogonality relations for rows and columns of this matrix lead to the orthogonality relations for alternative q-Charlier polynomials and for the functions, which are dual to these polynomials (note that the orthogonality relation for alternative q-Charlier polynomials is given in [4], Section 3.22, but no proof of it was published; as is written in [4], no other references to these polynomials are known). We extract from the latter functions a dual set of polynomials and obtain a discrete orthogonality relation for them. The unitarity of the matrix A, $AA^{-1} = A^{-1}A = E$, proves the completeness property of dual alternative q-Charlier polynomials in the corresponding Hilbert space L^2.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions and the standard q-analysis (see, for example, [8] or [9]).

2. Pair of operators (I_1, J). Let $\mathcal{H}$ be a separable complex Hilbert space with an orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$. We define on $\mathcal{H}$ two operators. The first one, denoted as q^{J_0} and taken from the theory of representations of quantum groups, acts upon the basis elements as $q^{J_0}|n\rangle = q^n|n\rangle$. The second operator, denoted as I_1, is given by the formula

$$I_1|n\rangle = a_n|n+1\rangle + a_{n-1}|n-1\rangle + b_n|n\rangle, \tag{1}$$

$$a_n = -(aq^{3n+1})^{1/2} \frac{\sqrt{(1-q^{n+1})(1+aq^n)}}{(1+aq^{2n+1})\sqrt{(1+aq^{2n})(1+aq^{2n+2})}},$$

$$b_n = q^n\left(\frac{1+aq^n}{(1+aq^{2n})(1+aq^{2n+1})} + aq^{n-1}\frac{1-q^n}{(1+aq^{2n-1})(1+aq^{2n})}\right),$$

where a is any fixed positive number. Clearly, I_1 is a symmetric operator.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator I_1 is bounded. Therefore, we assume that it is defined on the whole Hilbert space $\mathcal{H}$. For this reason, I_1 is a self-adjoint operator. Let us show that I_1 is a Hilbert – Schmidt operator. For the coefficients a_n and b_n from (1), we have $a_{n+1}/a_n \to q^{3/2}$ and $b_{n+1}/b_n \to q$ when $n \to \infty$. Since $0 < q < 1$, for the sum of all matrix elements of the operator I_1 in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, we have $\sum_n (2a_n + b_n) < \infty$ and, therefore, $\sum_n (2a_n^2 + b_n^2) < \infty$. Thus, I_1 is a Hilbert – Schmidt operator. This means that the spectrum of I_1 is discrete and has a single accumulation point at 0. Moreover, the spectrum of I_1 is simple, since I_1 is representable by a Jacobi matrix with $a_n \neq 0$ (see [2], Chapter VII).

To find eigenfunctions ξ_λ of the operator I_1, $I_1\xi_\lambda = \lambda\xi_\lambda$, we set $\xi_\lambda = \sum_n \beta_n(\lambda)|n\rangle$, where $\beta_n(\lambda)$ are appropriate numerical coefficients. Acting by the operator I_1 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda)\big(a_n|n+1\rangle + a_{n-1}|n-1\rangle + b_n|n\rangle\big) = \lambda \sum_{n=0}^{\infty} \beta_n(\lambda)|n\rangle,$$

where a_n and b_n are the same as in (1). Collecting in this identity all factors, which multiply $|n\rangle$ with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)a_n + \beta_{n-1}(\lambda)a_{n-1} + \beta_n(\lambda)b_n = \lambda\beta_n(\lambda).$$

The substitution

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n(1+aq^{2n})}{(q;q)_n(1+a)(a/q)^n}\right)^{1/2} q^{-n(n+3)/4}\beta'_n(\lambda)$$

reduces this relation to the following one

$$-A_n\beta'_{n+1}(\lambda) - C_n\beta'_{n-1}(\lambda) + (A_n+C_n)\beta'_n(\lambda) = \lambda\beta'_n(\lambda),$$

$$A_n = q^n\frac{1+aq^n}{(1+aq^{2n})(1+aq^{2n+1})}, \qquad C_n = aq^{2n-1}\frac{1-q^n}{(1+aq^{2n-1})(1+aq^{2n})}.$$

This is the recurrence relation for the alternative q-Charlier polynomials

$$K_n(\lambda;a;q) := {}_2\phi_1(q^{-n},-aq^n;0;q,q\lambda)$$

(see formulas (3.22.1) and (3.22.2) in [4]). Therefore, $\beta'_n(\lambda) = K_n(\lambda;a;q)$ and

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n(1+aq^{2n})}{(q;q)_n(1+a)a}\right)^{1/2} q^{-n(n+1)/4}K_n(\lambda;a;q). \tag{2}$$

For the eigenvectors ξ_λ we have the expression

$$\xi_\lambda = \sum_n \beta_n(\lambda)|n\rangle, \tag{3}$$

where $\beta_n(\lambda)$ is given by (2). Since the spectrum of the operator I_1 is discrete, only for a discrete set of values of λ these vectors may belong to the Hilbert space $\mathcal{H}$. This discrete set of eigenvectors determines the spectrum of I_1.

Now we look for the spectrum of I_1 and for a set of polynomials, dual to alternative q-Charlier polynomials. To this end we use the action of the operator

$$J := q^{-J_0} - aq^{J_0}$$

upon the eigenvectors ξ_λ, which belong to the Hilbert space $\mathcal{H}$. In order to find how this operator acts upon these vectors, one can use the q-difference equation

$$(q^{-n}-aq^n)K_n(\lambda) = -aK_n(q\lambda) + \lambda^{-1}K_n(\lambda) - \lambda^{-1}(1-\lambda)K_n(q^{-1}\lambda) \tag{4}$$

for the alternative q-Charlier polynomials $K_n(\lambda) \equiv K_n(\lambda;a;q)$ (see formula (3.22.5) in [4]; observe that (4) can be also derived from the explicit expressions for $K_n(\lambda)$). Multiply both sides of (4) by $d_n|n\rangle$ and sum up over n, where d_n are the factors in front of $K_n(\lambda;a;q)$ in expression (2) for the $\beta_n(\lambda)$. Taking into account formula (3) and the fact that $J|n\rangle = (q^{-n}-aq^n)|n\rangle$, one obtains the relation

$$J\xi_\lambda = -a\xi_{q\lambda} + \lambda^{-1}\xi_\lambda - \lambda^{-1}(1-\lambda)\xi_{q^{-1}\lambda}. \tag{5}$$

We shall see in the next section that the spectrum of the operator I_1 consists of the points q^n, $n = 0, 1, 2, \ldots$. This means that J has the form of a Jacobi matrix in the basis of eigenvectors of I_1; that is, the pair of the operators I_1 and J form an infinite dimensional analogue of a Leonard pair (see [10] for its definition).

3. Orthogonal matrix A. The aim of this section is to find, by using the operators I_1 and J, a basis in the Hilbert space $\mathcal{H}$, which consists of eigenvectors of the operator I_1 in a normalized form, and to derive explicitly the unitary matrix A, connecting this basis with the initial basis $|n\rangle$, $n = 0, 1, 2, \ldots$, in $\mathcal{H}$. First we have to find the spectrum of I_1.

Let us first look at a form of the spectrum of I_1 from the point of view of the spectral theory of Hilbert – Schmidt operators (a detailed calculation is given below this paragraph). If λ is a spectral point of the operator I_1, then (as it is easy to see from (5)) a successive action by the operator J upon the vector ξ_λ (eigenvector of I_1) leads to the eigenvectors $\xi_{q^m\lambda}$, $m = 0, \pm 1, \pm 2, \ldots$. However, since I_1 is a Hilbert – Schmidt operator, not all of these points belong to the spectrum of I_1, since $q^{-m}\lambda \to \infty$ when $m \to +\infty$ if $\lambda \neq 0$. This means that the coefficient $1 - \lambda'$ of $\xi_{q^{-1}\lambda'}$ in (5) must vanish for some eigenvalue λ'. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for the coefficient of $\xi_{q^{-1}\lambda'}$ in (5) to vanish, that is, the point $\lambda = 1$ is a spectral point for the operator I_1. Let us show that the corresponding eigenfunction $\xi_1 \equiv \xi_{q^0}$ belongs to the Hilbert space $\mathcal{H}$.

By formula (II.6) from Appendix II in [9], one has the following equality $K_n(1; a; q) = {}_2\phi_1(q^{-n}, -aq^n; 0; q, q) = (-a)^n q^{n^2}$. Therefore, for the scalar product $\langle \xi_1, \xi_1 \rangle$ in $\mathcal{H}$ we have

$$\langle \xi_1, \xi_1 \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n (1 + aq^{2n})}{(1+a)(q; q)_n a^n q^{n(n+1)/2}} K_n^2(1; a; q) = \sum_{n=0}^{\infty} \frac{(-a; q)_n (1 + aq^{2n}) a^n}{(1+q)(q; q)_n q^{-n(3n-1)/2}}. \quad (6)$$

In order to calculate this sum, we take the limit $d, e \to \infty$ in the formula of Exercise 2.12, Chapter 2, in [9]. Since

$$\lim_{d,e \to \infty} (d; q)_n (e; q)_n (aq/de)^n = q^{n(n-1)} (aq)^n,$$

we obtain that the sum in (6) is equal to $(-aq; q)_\infty$, that is, $\langle \xi_1, \xi_1 \rangle < \infty$ and ξ_1 belongs to the Hilbert space $\mathcal{H}$. Thus, the point $\lambda = 1$ does belong to the spectrum of I_1.

Let us find other spectral points of the operator I_1 (recall that the spectrum of I_1 is discrete). Setting $\lambda = 1$ in (5), we see that the operator J transforms ξ_{q^0} into a linear combination of the vectors ξ_q and ξ_{q^0}. Moreover, ξ_q belongs to the Hilbert space $\mathcal{H}$, since the series

$$\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n (1 + aq^{2n})}{(1+a)(q; q)_n a^n} q^{-n(n+1)/2} K_n^2(q; a; q)$$

is majorized by the corresponding series (6) for ξ_{q^0}. Therefore, ξ_q belongs to the Hilbert space $\mathcal{H}$ and the point q is an eigenvalue of the operator I_1. Similarly, setting $\lambda = q$ in (5), one finds likewise that ξ_{q^2} is an eigenvector of I_1 and the point q^2 belongs to the spectrum of I_1. Repeating this procedure, we find that all ξ_{q^n}, $n = 0, 1, 2, \ldots$, are eigenvectors of I_1 and the set q^n, $n = 0, 1, 2, \ldots$, belongs to the spectrum of I_1. So far, we do not know yet whether other spectral points exist or not.

The vectors ξ_{q^n}, $n = 0, 1, 2, \ldots$, are linearly independent elements of the Hilbert space $\mathcal{H}$ (since they correspond to different eigenvalues of the self-adjoint operator I_1). Suppose that values q^n, $n = 0, 1, 2, \ldots$, constitute the whole spectrum of I_1. Then the set of vectors ξ_{q^n}, $n = 0, 1, 2, \ldots$, is a basis in the Hilbert space $\mathcal{H}$. Introducing the notation $\Xi_k := \xi_{q^k}$, $k = 0, 1, 2, \ldots$, we find from (5) that

$$J\,\Xi_k = -a\,\Xi_{k+1} + q^{-k}\,\Xi_k - q^{-k}(1-q^k)\Xi_{k-1}.$$

As we see, the matrix of the operator J in the basis Ξ_k, $k = 0, 1, 2, \ldots$, is not symmetric, although in the initial basis $|n\rangle$, $n = 0, 1, 2, \ldots$, it was symmetric. The reason is that the matrix (a_{mn}) with entries $a_{mn} := \beta_m(q^n)$, $m, n = 0, 1, 2, \ldots$, where $\beta_m(q^n)$ are the coefficients (2) in the expansion $\xi_{q^n} = \sum_m \beta_m(q^n)|n\rangle$ (see (3)), is not unitary. This fact is equivalent to the statement that the basis $\Xi_n = \xi_{q^n}$, $n = 0, 1, 2, \ldots$, is not normalized. To normalize it, one has to multiply Ξ_n by appropriate numbers c_n (which are defined below). Let $\hat{\Xi}_n = c_n\Xi_n$, $n = 0, 1, 2, \ldots$, be a normalized basis. Then the matrix of the operator J is symmetric in this basis. It follows from (5) that J has in the basis $\{\hat{\Xi}_n\}$ the form

$$J\,\hat{\Xi}_n = -c_{n+1}^{-1}c_n a\,\hat{\Xi}_{n+1} + q^{-n}\,\hat{\Xi}_n - c_{n-1}^{-1}c_n q^{-n}(1-q^n)\hat{\Xi}_{n-1}.$$

The symmetricity of J in the basis $\{\hat{\Xi}_n\}$ means that $c_{n+1}^{-1}c_n a = c_n^{-1}c_{n+1}q^{-n-1}(1-q^{n+1})$, that is, $c_n/c_{n-1} = \sqrt{aq^n/(1-q^n)}$. Therefore, the coefficients c_n are equal to

$$c_n = c\left[a^n q^{n(n+1)/2}/(q;q)_n\right]^{1/2},$$

where c is a constant.

The expansions

$$\hat{\xi}_{q^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\beta_m(q^n)|m\rangle \equiv \sum_m \hat{a}_{mn}|m\rangle \tag{7}$$

interrelate two orthonormal bases in the Hilbert space $\mathcal{H}$. This means that the matrix $(\hat{a}_{mn})$, $m, n = 0, 1, 2, \ldots$, with entries

$$\hat{a}_{mn} = c_n\beta_m(q^n) = c\left(\frac{a^n q^{n(n+1)/2}}{(q;q)_n}\,\frac{(-a;q)_m(1+aq^{2m})}{(1+a)(q;q)_m a^m q^{m(m+1)/2}}\right)^{1/2} K_m(q^n;a;q), \tag{8}$$

is unitary, provided that the constant c is appropriately chosen. In order to calculate this constant, we use the relation

$$\langle\hat{\Xi}_0,\hat{\Xi}_0\rangle = \sum_{m=0}^{\infty}|\hat{a}_{m0}|^2 = \sum_{m=0}^{\infty}c_0^2\beta_m^2(q^0) = 1.$$

The last sum is a multiple of the sum in (6) and, consequently, $c = (-aq;q)_\infty^{-1/2}$.

The matrix $A := (\hat{a}_{mn})$ is real and orthogonal. Thus, if $\hat{\Xi}_n$, $n = 0, 1, 2, \ldots$, is a complete basis in $\mathcal{H}$, then $AA^{-1} = A^{-1}A = E$, that is,

$$\sum_n \hat{a}_{mn}\hat{a}_{m'n} = \delta_{mm'}, \qquad \sum_m \hat{a}_{mn}\hat{a}_{mn'} = \delta_{nn'}. \tag{9}$$

Substituting into the first sum over n in (9) the expressions for $\hat{a}_{mn}$ from (8), we obtain the identity

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} K_m(q^n;a;q)\, K_{m'}(q^n;a;q) = \frac{(-aq^m;q)_\infty\, a^m (q;q)_m}{(1+aq^{2m})} q^{m(m+1)/2} \delta_{mm'}, \quad (10)$$

which must yield the orthogonality relation for alternative q-Charlier polynomials. An only gap, which remains to be clarified, is the following. We have assumed that the points q^n, $n = 0, 1, 2, \ldots$, exhaust the whole spectrum of I_1. Let us show that this is the case.

Recall that the self-adjoint operator I_1 is represented by a Jacobi matrix in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$. According to the theory of operators of such type (see, for example, [2], Chapter VII), eigenvectors ξ_λ of I_1 are expanded into series in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some positive measure $d\mu(\lambda)$ (moreover, for self-adjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which these polynomials are orthogonal, coincides with the spectrum of the operator under discussion and this spectrum is simple.

We have found that the spectrum of I_1 contains the points q^n, $n = 0, 1, 2, \ldots$. If the operator I_1 had other spectral points x_k, then the left-hand side of (10) would contain other terms $\mu_{x_k} K_m(x_k;a;q) K_{m'}(x_k;a;q)$ with positive μ_{x_k}, corresponding to these additional points. Let us show that these additional summands do not appear. To this end we set $m = m' = 0$ in relation (10) with the additional summands. Since $K_0(x;a;q) = 1$, we have then the equality

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} + \sum_k \mu_{x_k} = (-aq;q)_\infty .$$

According to the definition of the q-exponential function $E_q(a)$ (see formula (II.2) from Appendix II in [9]), we have

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} = (-aq;q)_\infty .$$

Hence, $\sum_k \mu_{x_k} = 0$ and all μ_{x_k} have to vanish. This means that additional summands do not appear in (10) and hence (10) does represent the orthogonality relation for alternative q-Charlier polynomials. Thus, the following proposition is true:

Proposition 1. *The spectrum of the operator* I_1 *coincides with the set of points* q^n, $n = 0, 1, 2, \ldots$. *This spectrum is simple and the vectors* $\hat{\Xi}_n$, $n = 0, 1, 2, \ldots$, *form a complete set of eigenvectors of* I_1. *The matrix* $(\hat{a}_{mn})$ *with entries* (8) *relates the initial basis* $\{|n\rangle\}$ *with the normalized basis* $\{\hat{\Xi}_n\}$.

4. Dual alternative *q*-Charlier polynomials. Now we consider the second identity in (9), which gives the orthogonality relation for the matrix elements $\hat{a}_{mn}$, considered as functions of m. Due to the expression for alternative q-Charlier polynomials from Section 2, these functions coincide (up to multiplicative factors) with the functions

$$F_n(x;a|q) = {}_2\phi_1(x, -a/x; 0; q, q^{n+1}), \quad (11)$$

considered on the set $x \in \{q^{-m} | m = 0, 1, 2, \ldots\}$. Consequently,

$$\hat{a}_{mn} = c\left(\frac{a^n q^{n(n+1)/2}}{(q;q)_n} \frac{(1+aq^{2m})}{(-aq^m;q)_\infty (q;q)_m a^m q^{m(m+1)/2}} \right)^{1/2} F_n(q^{-m};a|q)$$

and the second identity in (9) gives the orthogonality relation for $F_n(q^{-m}; a|q)$:

$$\sum_{m=0}^{\infty} \frac{(1+aq^{2m})q^{-m(m+1)/2}}{a^m(-aq^m; q)_\infty (q; q)_m} F_n(q^{-m}; a|q) F_{n'}(q^{-m}; a|q) = \frac{(q; q)_n}{a^n} q^{-n(n+1)/2} \delta_{nn'}. \quad (12)$$

The functions $F_n(x; a|q)$ can be represented in another form. Indeed, taking the limit $c \to \infty$ in transformation (III.8) from Appendix III in [9], one derives the relation

$${}_2\phi_1(q^{-m}, -aq^m; 0; q, q^{n+1}) = (-a)^m q^{m^2} {}_3\phi_0(q^{-m}, -aq^m, q^{-n}; -; q, -q^n/a).$$

Therefore, we have

$$F_n(q^{-m}; a|q) = (-a)^m q^{m^2} {}_3\phi_0(q^{-m}, -aq^m, q^{-n}; -; q, -q^n/a). \quad (13)$$

The basic hypergeometric function ${}_3\phi_0$ in (13) is a polynomial of degree n in the variable $\mu(m) := q^{-m} - aq^m$, which represents a q-quadratic lattice; we denote it by

$$d_n(\mu(m); a; q) := {}_3\phi_0(q^{-m}, -aq^m, q^{-n}; -; q, -q^n/a). \quad (14)$$

Then formula (12) yields the orthogonality relation

$$\sum_{m=0}^{\infty} \frac{(1+aq^{2m})a^m q^{m(3m-1)/2}}{(-aq^m; q)_\infty (q; q)_m} d_n(\mu(m)) d_{n'}(\mu(m)) = \frac{(q; q)_n}{a^n} q^{-n(n+1)/2} \delta_{nn'} \quad (15)$$

for the polynomials (14) when $a > 0$. As far as we know this orthogonality relation is new. We call the polynomials $d_n(\mu(m); a; q)$ *dual alternative q-Charlier polynomials.* Thus, we proved the following theorem.

Theorem. *The polynomials* $d_n(\mu(m); a; q)$, *given by formula* (14), *are orthogonal on the set of points* $\mu(m) := q^{-m} - aq^m$, $m = 0, 1, 2, \ldots$, *and the orthogonality relation for them is given by formula* (15).

Remark. The duality of polynomials is a well-known notion in the case of polynomials, orthogonal with respect to a finite number of points (see, for example, [9], Chapter 7). It reflects the simple fact that a finite-dimensional matrix, orthogonal by rows, is also orthogonal by its columns. In the case when polynomials are orthogonal on a countable set of points, the situation is more complicated (see, for example, [11 – 13]). Usually a dual set with respect to such orthogonal polynomials is given in terms of functions (for instance, their explicit form (11) for the case of alternative q-Charlier polynomials is given above). One therefore needs to make one step further in order to extract an appropriate family of dual polynomials from these functions (of course, in all those cases when functions admit such a separation). Observe that in our approach to the duality of q-polynomials it is not assumed that an orthogonality relation for the initial set of polynomials is known; this orthogonality is straightforwardly derived. Besides, one naturally extracts from a dual set of orthogonal functions an appropriate dual family of q-polynomials. Other instances of the similar approach to the duality can be found in [14] and [15].

Let l_a^2 be the Hilbert space of functions on the set $\{m = 0, 1, 2, \ldots\}$ with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \frac{(1+aq^{2m})a^m}{(-aq^m; q)_\infty (q; q)_m} q^{m(3m-1)/2} f_1(m) \overline{f_2(m)}, \quad (16)$$

where the weight function is taken from (15). The polynomials (14) are in one-to-one correspondence with the columns of the orthogonal matrix $(\hat{a}_{mn})$ and the orthogonality relation (15) is equivalent to the orthogonality of these columns. Due to (9) the columns of the matrix $(\hat{a}_{mn})$ form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$ of sequences $\mathbf{a} = \{a_n \,|\, n = 0, 1, 2, \ldots\}$ with the scalar product $\langle \mathbf{a}, \mathbf{a}' \rangle = \sum_n a_n \overline{a'_n}$. This scalar product is equivalent to the scalar product (16) for the polynomials $d_n(\mu(m); a; q)$. The fact that the set of all columns of the matrix $(\hat{a}_{mn})$ is a basis in $\mathfrak{l}^2$ means that the set of all polynomials (14) forms a basis in $\mathfrak{l}^2_a$. Thus, the following proposition is true:

Proposition 2. *The set of polynomials* $d_n(\mu(m); a; q)$, $n = 0, 1, 2, \ldots$, *form an orthogonal basis in the Hilbert space* $\mathfrak{l}^2_a$ *that is, this set is complete in* $\mathfrak{l}^2_a$.

We have been unable to clarify yet whether the moment problem, connected with the polynomials (14), is determinate or indeterminate. Nevertheless, Proposition 2 means that if this moment problem is indeterminate, then the measure in (15) is extremal.

A recurrence relation for the polynomials $d_n(\mu(m); a; q)$,

$$(q^{-m} - aq^m) d_n(\mu(m)) = -a\, d_{n+1}(\mu(m)) + q^{-n} d_n(\mu(m)) - q^{-n}(1 - q^n) d_{n-1}(\mu(m)),$$

where $d_n(\mu(m)) \equiv d_n(\mu(m); a; q)$, is readily derived from (4). A q-difference equation for $d_n(\mu(m))$ can be obtained from the three-term recurrence relation for alternative q-Charlier polynomials $K_n(x; a; q)$.

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