UDC 517.9

S. O. Kuzhel'* (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv),
L. V. Matsyuk (Univ. "KROK", Kyiv)

ON AN APPLICATION OF THE LAX – PHILLIPS SCATTERING APPROACH IN THEORY OF SINGULAR PERTURBATIONS

ЗАСТОСУВАННЯ РОЗСІЯННЯ ЛАКСА – ФІЛЛІПСА В ТЕОРІЇ СИНГУЛЯРНИХ ЗБУРЕНЬ

For a singular perturbation $A = A_0 + \sum_{i,j=1}^{n} t_{ij} \langle \psi_j, \cdot \rangle \psi_i$, $n \le \infty$, of a positive self-adjoint operator A_0 with Lebesgue spectrum, the spectral analysis of the corresponding self-adjoint operator realizations $A_{\mathbf{T}}$ is carried out and the scattering matrix $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ is calculated in terms of parameters t_{ij} under some additional restrictions on singular elements ψ_j that provides the possibility of application of the Lax – Phillips approach in the scattering theory.

Для сингулярного збурення $A = A_0 + \sum_{i,j=1}^{n} t_{ij} \langle \psi_j, \cdot \rangle \psi_i$, $n \le \infty$, додатного самоспряженого оператора A_0 із спектром Лебега проведено спектральний аналіз відповідних самоспряжених реалізацій $A_{\mathbf{T}}$. Крім того, обчислено матрицю розсіяння $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ через параметри t_{ij} при деяких додаткових обмеженнях на сингулярні елементи ψ_j. Одержані результати дозволяють застосовувати схему Лакса – Філліпса в теорії розсіяння.

1. Statement of the problem. Let A_0 be a positive self-adjoint operator acting in a Hilbert space $\mathfrak{H}$ and let

$$\mathfrak{H}_2(A_0) \subset \mathfrak{H}_1(A_0) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A_0) \subset \mathfrak{H}_{-2}(A_0)$$

be the standard scale of Hilbert spaces associated with A_0 [1]. Precisely, $\mathfrak{H}_2(A_0) = \mathcal{D}(A_0)$, $\mathfrak{H}_1(A_0) = \mathcal{D}\left(A_0^{1/2}\right)$ with the norms $\|u\|_k = \left\|(A_0 + I)^{k/2} u\right\|$, $k = 1, 2$, and the conjugated spaces $\mathfrak{H}_{-k}(A_0)$ can be defined as the completions of $\mathfrak{H}$ with respect to the norms

$$\|u\|_{-k} = \left\|(A_0 + I)^{-k/2} u\right\| \quad \forall\, u \in \mathfrak{H}. \tag{1}$$

By (1), the operator $(A_0 + I)^{-1}$ can be continuously extended to an isometric mapping $(\mathbb{A}_0 + I)^{-1}$ of $\mathfrak{H}_{-2}(A_0)$ onto $\mathfrak{H}$. Thus, for any $\psi \in \mathfrak{H}_{-2}(A_0)$, the element $(\mathbb{A}_0 + I)^{-1}\psi$ belongs to $\mathfrak{H}$ and the relation

$$\langle \psi, u \rangle = \left((A_0 + I)u, (\mathbb{A}_0 + I)^{-1}\psi\right) \quad \forall\, u \in \mathfrak{H}_2(A_0) \tag{2}$$

enables one to consider any element $\psi \in \mathfrak{H}_{-2}(A_0)$ as a linear continuous functional on $\mathfrak{H}_2(A_0)$.

Let us fix an orthonormal system $\{\psi_j\}_{j=1}^{n}$ $(n \in \{\mathbb{N}, \infty\})$ in $\mathfrak{H}_{-2}(A_0)$ and consider the formal expression

$$A_0 + \sum_{i,j=1}^{n} t_{ij} \langle \psi_j, \cdot \rangle \psi_i, \quad t_{ij} = \bar{t}_{ji}, \quad n \in \{\mathbb{N}, \infty\}. \tag{3}$$

---

* Partially supported by CRDF (grant No. UM1-2567-OD-03) and DFFD of Ukraine (project No. 01.07/027).

In what follows we suppose that the linear subspace $\mathcal{X}$ of $\mathfrak{H}_{-2}(A_0)$ generated by the basis $\{\psi_j\}_{j=1}^n$ satisfies the condition $\mathcal{X} \cap \mathfrak{H} = \{0\}$ (i.e., elements ψ_j are $\mathfrak{H}$-independent). In this case, the potential $V = \sum_{i,j=1}^n t_{ij}\langle\psi_j,\cdot\rangle\psi_i$ is singular; the symmetric operator

$$A_{\text{sym}} := A_0\restriction_{\mathcal{D}(A_{\text{sym}})}, \qquad \mathcal{D}(A_{\text{sym}}) = \{u \in D(A_0) \mid \langle\psi_j, u\rangle = 0,\ 1 \le j \le n\} \tag{4}$$

is closed densely defined in $\mathfrak{H}$ and the deficiency indices of A_{sym} are equal to n.

Using approaches developed in the theory of singular perturbations [2], we can associate with the formal expression (3) its self-adjoint operator realization $A_{\mathbf{T}}$ acting in $\mathfrak{H}$ (see Theorem 1) and, as a result, to reduce the scattering problem for (3) to the study of the scattering operator

$$S(A_{\mathbf{T}}, A_0) = W_+^*(A_{\mathbf{T}}, A_0)\, W_-(A_{\mathbf{T}}, A_0) \tag{5}$$

for perturbed $A_{\mathbf{T}}$ and unperturbed A_0 operator realizations of (3), where the wave operators $W_\pm(A_{\mathbf{B}}, A_0)$ are defined as follows:

$$W_\pm(A_{\mathbf{T}}, A_0) := s - \lim_{t\to\pm\infty} e^{iA_{\mathbf{T}}t} e^{-iA_0 t}.$$

In the case $n < \infty$, operators $A_{\mathbf{T}}$ and A_0 are different self-adjoint extensions of the symmetric operator A_{sym} with finite deficiency indices. On the basis of this fact, the scattering matrix $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ (in other words, the image of the scattering operator $S(A_{\mathbf{T}}, A_0)$ in the spectral representation of A_0) was expressed in terms of parameters of the Krein's resolvent formula with the use of the stationary approach in the scattering theory (see [3])*.

In the present paper, we apply one of the well-developed nonstationary scattering approaches (the Lax – Phillips approach) for the study of spectral and scattering properties of operator realizations $A_{\mathbf{T}}$ of the formal expression (3), where, in general, the singular perturbation is not assumed to be of finite rank. In particular, for finite rank singular perturbations, we obtain a representation of the scattering matrix $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ directly in terms of parameters t_{ij} of the singular potential V.

Of course, in order to employ the Lax – Phillips approach we have to impose some restrictions on the unperturbed operator A_0 and singular elements ψ_j. An example of such restrictions and the spectral analysis of the corresponding operator realizations of (3) are contained in Section 3. In Section 2, we discuss the problem of realization of the heuristic expression (3) as a self-adjoint operator in $\mathfrak{H}$ and present a simple description of such realizations in terms of parameters t_{ij}. The expression of the scattering matrices $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ for nonnegative self-adjoint operator realization $A_{\mathbf{T}}$ of (3) is presented in Section 4. Section 5 contains an application of the obtained results to the case of one-dimensional Schrödinger operator with symmetric zero-range potentials.

Let us make a remark about notations. In what follows, any Hilbert space is assumed to be separable. $\mathcal{D}(A)$ and $\ker A$ denote the domain and the null-space of a linear operator A, respectively. $A\restriction_{\mathcal{D}}$ means the restriction of A onto a set $\mathcal{D}$.

2. Operator realizations of singular perturbations. To define a self-adjoint operator realization of (3) in $\mathfrak{H}$ with a given singular perturbation

* A survey of further development of the stationary scattering approach in the theory of singular perturbations can be found in [2].

$$V = \sum_{i,j=1}^{n} t_{ij} \langle \psi_j, \cdot \rangle \psi_i,$$

we use an approach suggested initially in [4] (see also [2]) for the case of finite rank singular perturbations and its generalization to the infinite dimensional case [5]. The main idea consists in the construction of some regularization

$$\mathbb{A}_{\mathbf{T}} := A_0 + \sum_{i,j=1}^{n} t_{ij} \langle \psi_j^{\mathrm{ex}}, \cdot \rangle \psi_i, \tag{6}$$

of (3) that is well defined as an operator from $\mathcal{D}(A_{\mathrm{sym}}^*)$ to $\mathfrak{H}_{-2}(A_0)$. In this case, the corresponding operator realization $A_{\mathbf{T}}$ of (3) is determined by the formula

$$A_{\mathbf{T}} = \mathbb{A}_{\mathbf{T}} \restriction_{\mathcal{D}(A_{\mathbf{T}})}, \qquad \mathcal{D}(A_{\mathbf{T}}) = \left\{ f \in \mathcal{D}(A_{\mathrm{sym}}^*) \,\middle|\, \mathbb{A}_{\mathbf{T}} f \in \mathfrak{H} \right\}. \tag{7}$$

Let us clarify the meaning of components $\mathbb{A}_0$ and ψ_j^{ex} in (6). First of all we observe that $\mathbb{A}_0$ is the continuation of A_0 as a bounded linear operator acting from $\mathfrak{H}$ into $\mathfrak{H}_{-2}(A_0)$ and this continuation is determined by the formula

$$\mathbb{A}_0 f := \left[(\mathbb{A}_0 + I)^{-1} \right]^{-1} f - f \quad \forall\, f \in \mathfrak{H}. \tag{8}$$

Next, the linear functionals ψ_j^{ex} are extensions of the functionals ψ_j onto $\mathcal{D}(A_{\mathrm{sym}}^*)$. Using the well-known relation

$$\mathcal{D}(A_{\mathrm{sym}}^*) = \mathcal{D}(A_0) \dotplus \mathcal{H}, \quad \text{where} \quad \mathcal{H} = \ker(A_{\mathrm{sym}}^* + I), \tag{9}$$

we arrive at the conclusion that ψ_j can be extended onto $\mathcal{D}(A_{\mathrm{sym}}^*)$ if we know their values on $\mathcal{H}$.

It follows from (1), (2), and (4) that the vectors

$$h_j = (\mathbb{A}_0 + I)^{-1} \psi_j, \quad j = 1, \ldots, n, \tag{10}$$

form an orthonormal basis of the Hilbert space $\mathcal{H}$. Hence, ψ_j^{ex}, $1 \le j \le n$, are well-defined by the formula

$$\langle \psi_j^{\mathrm{ex}}, f \rangle := \langle \psi_j, u \rangle + \sum_{p=1}^{n} \alpha_p r_{jp} \tag{11}$$

for all elements $f = u + \sum_{p=1}^{n} \alpha_p h_p$, $u \in \mathcal{D}(A_0)$, $\alpha_p \in \mathbb{C}$, from $\mathcal{D}(A_{\mathrm{sym}}^*)$ if we determine the entries

$$r_{jp} := \langle \psi_j, (\mathbb{A}_0 + I)^{-1} \psi_p \rangle = \langle \psi_j, h_p \rangle$$

of a (in general, infinite-dimensional) matrix $\mathbf{R} = (r_{jp})_{j,p=1}^{n}$.

If all $\psi_j \in \mathfrak{H}_{-1}(A_0)$, then r_{jp} are well defined and $\mathbf{R}$ is defined uniquely (see [2]). In other cases, the most appropriate choice of $\mathbf{R}$ has to be determined by imposing additional requirements related to the nature of a perturbation (see, e.g., [2]).

We recall (see, e.g., [6]) that a matrix $\mathbf{X} = \left(x_{ij}\right)_{i,j=1}^{n}$ is called *the matrix decomposition* with respect to the basis $\left\{h_j\right\}_{j=1}^{n}$ of a bounded operator X acting in $\mathcal{H}$ if its entries x_{ij} are defined by the expansions

$$Xh_j = \sum_{i=1}^{n} x_{ij} h_i, \quad 1 \le j \le n.$$

In what follows we assume that $\mathbf{R}$ *is already chosen as a matrix decomposition* (*with respect to the basis* $\left\{h_j\right\}_1^n$) *of a bounded self-adjoint operator* R *acting in* $\mathcal{H}$.

Our aim now is to describe operator realizations $A_{\mathbf{T}}$ of (3) in terms of parameters t_{ij} of the singular perturbation V. To do this, the method of boundary triplets (see [7] and references therein) can be used.

We recall that a triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where Γ_0, Γ_1 are linear mappings of $\mathcal{D}\left(A_{\text{sym}}^*\right)$ into $\mathcal{H}$, is called *a boundary triplet* of A_{sym}^* if

$$\left(A_{\text{sym}}^* f, g\right) - \left(f, A_{\text{sym}}^* g\right) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathcal{D}\left(A_{\text{sym}}^*\right), \tag{12}$$

and for any F_0, $F_1 \in \mathcal{H}$ there exists an element $f \in \mathcal{D}\left(A_{\text{sym}}^*\right)$ such that $\Gamma_0 f = F_0$, $\Gamma_1 f = F_1$.

Denote

$$\hat{\Gamma}_0 f = P_{A_0} f, \qquad \hat{\Gamma}_1 f = P_{\mathcal{H}}\left(A_{\text{sym}}^* + I\right) f, \quad f \in \mathcal{D}\left(A_{\text{sym}}^*\right), \tag{13}$$

where $P_{\mathcal{H}}$ is the orthogonal projector onto $\mathcal{H}$ in $\mathfrak{H}$ and P_{A_0} is the projector onto $\mathcal{H}$ with respect to the decomposition (9). The triplet $\left(\mathcal{H}, \hat{\Gamma}_0, \hat{\Gamma}_1\right)$ is an example of the well-known boundary triplet that is used widely in the Krein – Birman – Vishik extension theory.

Lemma 1. *The triplet* $(\mathcal{H}, \Gamma_0, \Gamma_1)$, *where*

$$\Gamma_0 f = \hat{\Gamma}_1 f + R\hat{\Gamma}_0 f, \qquad \Gamma_1 f = -\hat{\Gamma}_0 f, \quad f \in \mathcal{D}\left(A_{\text{sym}}^*\right), \tag{14}$$

forms a boundary triplet of A_{sym}^*.

Proof. Since $\left(\mathcal{H}, \hat{\Gamma}_0, \hat{\Gamma}_1\right)$ is a boundary triplet, we get

$$\left(A_{\text{sym}}^* f, g\right) - \left(f, A_{\text{sym}}^* g\right) = \left(\hat{\Gamma}_1 f, \hat{\Gamma}_0 g\right) - \left(\hat{\Gamma}_0 f, \hat{\Gamma}_1 g\right). \tag{15}$$

Expressing $\hat{\Gamma}_i$ in terms of Γ_i with the use of (14), substituting the obtained expressions into (15), and taking into account that R is a self-adjoint operator in $\mathcal{H}$, we establish (12) for Γ_i.

Let F_0, F_1 be arbitrary elements of $\mathcal{H}$. Since, $\left(\mathcal{H}, \hat{\Gamma}_0, \hat{\Gamma}_1\right)$ is a boundary triplet, there exists $f \in \mathcal{D}\left(A_{\text{sym}}^*\right)$ such that $\hat{\Gamma}_0 f = -F_1$ and $\hat{\Gamma}_1 f = F_0 + RF_1$. Comparing these relations with (14), we get $\Gamma_0 f = F_0$ and $\Gamma_1 f = F_1$. Thus, $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is a boundary triplet.

Lemma 1 is proved.

Theorem 1. *Let the coefficient matrix* $\mathbf{T} = (t_{ij})_{i,j=1}^{n}$ *of the singular perturbation* $V = \sum_{i,j=1}^{n} t_{ij}\langle \psi_j, \cdot\rangle \psi_i$ *in* (3) *be a matrix decomposition* (*with respect to the basis* $\{h_j\}_1^n$) *of a bounded self-adjoint operator* T *acting in* $\mathcal{H}$, *then the corresponding self-adjoint operator realization* $A_{\mathbf{T}}$ *of* (3) *is defined as follows:*

$$A_{\mathbf{T}}\restriction_{\mathcal{D}(A_{\mathbf{T}})}, \quad \mathcal{D}(A_{\mathbf{T}}) = \left\{ f \in \mathcal{D}\left(A_{\text{sym}}^*\right) \Big| \, T\Gamma_0 f = \Gamma_1 f \right\}. \tag{16}$$

Proof. Representing $f \in \mathcal{D}\left(A_{\text{sym}}^*\right)$ in the form $f = u + \sum_{i=1}\alpha_i h_i$, where $u \in \mathcal{D}(A_0)$, $h_i \in \mathcal{H}$, $\alpha_i \in \mathbb{C}$ and employing (2), (10), (11), (13), (14), we get

$$T\Gamma_0 f = T\left(\hat{\Gamma}_1 + R\hat{\Gamma}_0\right) f = \sum_{i,j,p=1}^{n} t_{ij}\left(\langle \psi_j, u\rangle + \alpha_p r_{jp}\right) h_i = \sum_{i,j=1}^{n} t_{ij}\left\langle \psi_j^{\text{ex}}, f\right\rangle h_i$$

and $\Gamma_1 f = -\sum_{i=1}^{n}\alpha_i h_i$. Using these relations and taking (6), (8), and (10) into account, we obtain

$$\mathbb{A}_{\mathbf{T}} f = A_0 u - \sum_{i=1}^{n}\alpha_i h_i + \sum_{i,j=1}^{n} t_{ij}\left\langle \psi_j^{\text{ex}}, f\right\rangle \psi_i + \sum_{i=1}^{n}\alpha_i \psi_i =$$

$$= A_{\text{sym}}^* f + \left[(\mathbb{A}_0 + I)^{-1}\right]^{-1}(T\Gamma_0 - \Gamma_1) f. \tag{17}$$

Since $\left[(\mathbb{A}_0 + I)^{-1}\right]^{-1}$ maps $\mathcal{H}$ onto the subspace $\mathcal{X}$ of $\mathfrak{H}_{-2}(A_0)$ generated by the basis $\{\psi_j\}_{j=1}^{n}$ and such that $\mathcal{X} \cap \mathcal{H} = \{0\}$, equalities (7) and (18) imply that $f \in \mathcal{D}(A)$ if and only if $T\Gamma_0 f - \Gamma_1 f = 0$. Therefore, the operator realization $A_{\mathbf{T}}$ of (3) is determined by (16). The self-adjointness of $A_{\mathbf{T}}$ follows from the fact that T is self-adjoint and the general properties of boundary triplets [7].

Theorem 1 is proved.

***Remark* 1.** For the case of finite rank singular perturbations, Lemma 1 and Theorem 1 were proved in [8].

3. A sufficient condition of the applicability of the Lax – Phillips approach and spectral analysis of $A_{\mathbf{T}}$. In what follows we suppose that the unperturbed operator A_0 in (3) has absolutely continuous spectrum on $\mathbb{R}_+ = [0, \infty)$ with the same multiplicity $m \le \infty$ at each point of $\mathbb{R}_+$ (i.e., the spectrum $\sigma(A_0)$ is Lebesgue and $\sigma(A_0) = \mathbb{R}_+$). This condition is equivalent (see [9]) to *the existence of a simple maximal symmetric operator* B *in* $\mathfrak{H}$ *such that* A_0 *is a self-adjoint extension of the symmetric operator* B^2 *and*

$$(A_0 u, u) = \left\| B^* u \right\|^2 \quad \forall\, u \in \mathcal{D}(A_0). \tag{18}$$

(Note that the non-zero deficiency index of B coincides with m and B^2 is a densely defined symmetric operator with deficiency indices m.)

We also suppose that the symmetric operator A_{sym} defined by (4) coincides with B^2 (if $n = m$) or A_{sym} is a symmetric extension of B^2 (if $n < m$).

It should be noted that such a situation is typical for Schrödinger operators with point interactions and for cases where singular elements ψ_j in (3) possess the homogeneity property with respect to the scaling transformations in $L_2(\mathbb{R}^p)$ (see, e.g., [10, 11]).

It has been established in [10, 12] that the restrictions imposed above on A_0 and A_{sym} are sufficient for the applicability of the well-developed methods of the Lax – Phillips scattering approach to the spectral analysis of $A_{\mathbf{T}}$.

Note also that the case $n < m$ can be reduced to the case $n = m$ by the supplement of elements ψ_j of the basis $\{\psi_j\}_{j=n+1}^m$ of $\ker(B^{*2} + I) \ominus \mathcal{H}$ with zero entries b_{ij} in (3). Thus, in what follows, without loss of generality, we suppose that $n = m$ and, hence, $A_{\text{sym}} = B^2$ and $\mathcal{H} = \ker(B^{*2} + I)$.

Assume that $-1 \in \rho(A_{\mathbf{T}})$ and denote

$$C_0 := (A_0 + I)^{-1} - \left(B^* B + I\right)^{-1}, \qquad C_{\mathbf{T}} := (A_{\mathbf{T}} + I)^{-1} - (A_0 + I)^{-1}. \tag{19}$$

Since A_0, B^*B, and $A_{\mathbf{T}}$ are self-adjoint extensions of B^2, the operators C_0 and $C_{\mathbf{T}}$ are self-adjoint in the Hilbert space $\mathcal{H}$. Moreover, taking (18) into account and using Lemma 3.5 in [12] (Chapter 4), we get that the spectrum $\sigma(C_0)$ of C_0 is a pure point (i.e., $\sigma(C_0) = \sigma_p(C_0)$) and it may consists of only points 0 and $1/2$.

Theorem 2. *For any self-adjoint operator realization* $A_{\mathbf{T}}$ *of* (3) *defined by* (7) *the following statements are true:*

1. *The point spectrum* $\sigma_\rho(A_{\mathbf{T}})$ *has empty intersection with* $\mathbb{R}_+$.

2. *If* $A_{\mathbf{T}}$ *is nonnegative, then the wave operators* $W_\pm(A_{\mathbf{T}}, A_0)$ *exist and are unitary operators in* $\mathfrak{H}$.

3. *For the case of finite rank singular perturbations* ($n < \infty$), $A_{\mathbf{T}}$ *is nonnegative if and only if* $\ker(I + RT) = \{0\}$ *and*

$$0 \le C_0 - T(I + RT)^{-1} \le \frac{1}{2} I.$$

Proof. Statement 1 follows from Corollary 3.3 in [12] (Chapter 4) and statement 2 is a particular case of Proposition 2 in [10].

Let us prove statement 3. Recalling the well-known result [13] on extremal properties of the Friedrichs B^*B and Krein – von Neumann BB^* extensions of B^2, we arrive at the conclusion that $A_{\mathbf{T}}$ is nonnegative if and only if $-1 \in \rho(A_{\mathbf{T}})$ and

$$\left(B^* B + I\right)^{-1} \le (A_{\mathbf{T}} + I)^{-1} \le \left(BB^* + I\right)^{-1}.$$

Using (19), we rewrite this relation as follows:

$$0 \le C_{\mathbf{T}} + C_0 \le C_N = \left(BB^* + I\right)^{-1} - \left(B^* B + I\right)^{-1}.$$

It follows from Lemma 3.5 in [12] (Chapter 4) that $C_N = \frac{1}{2} I$. Hence,

$$A_{\mathbf{T}} \ge 0 \iff -1 \in \rho(A_{\mathbf{T}}) \quad \text{and} \quad 0 \le C_{\mathbf{T}} + C_0 \le \frac{1}{2} I. \tag{20}$$

Let us show that conditions $-1 \in \rho(A_{\mathbf{T}})$ and $\ker(I + RT) = \{0\}$ are equivalent. Since $A_{\mathbf{T}}$ is a finite rank self-adjoint extension of A_{sym}, condition $-1 \in \sigma(A_{\mathbf{T}})$ is equivalent to the existence of an element $f \in \mathcal{D}(A_{\mathbf{T}}) \cap \mathcal{H}$. By virtue of (13), $\hat{\Gamma}_0 f \neq$

$\neq 0$ and $\hat{\Gamma}_1 f = 0$. Using (14) and (16), it is easy to establish that the existence of such f means that $\hat{\Gamma}_0 f \in \ker(I + TR)$ and, hence, $\ker(I + RT)$ also is a nontrivial subspace of $\mathfrak{H}$. Thus, $-1 \in \rho(A_{\mathbf{T}}) \Leftrightarrow \ker(I + RT) = \{0\}$.

To calculate $C_{\mathbf{T}}$ in (20), we observe that condition $-1 \in \rho(A_{\mathbf{T}})$ is equivalent (see, e.g., [7]) to the presentation of $\mathcal{D}(A_{\mathbf{T}})$ in the form

$$\mathcal{D}(A_{\mathbf{T}}) = \left\{ f \in \mathcal{D}\left(A_{\text{sym}}^*\right) \,\middle|\, C_{\mathbf{T}} \hat{\Gamma}_1 f = \hat{\Gamma}_0 f \right\}.$$

Comparing this relation with (16) and taking Lemma 1 into account, we get $C_{\mathbf{T}} = -T(I + RT)^{-1}$. Substituting the obtained expression into (20), we establish statement 3.

Theorem 2 is proved.

***Remark* 2.** Another description of nonnegative self-adjoint extensions of a nonnegative symmetric operator has been obtained recently in [14].

4. Scattering matrices. Since A_0 has a Lebesgue spectrum on $\mathbb{R}_+$, there exists an isometric mapping $\mathfrak{P} : \mathfrak{H} \overset{\text{onto}}{\to} L_2(\mathbb{R}_+, N)$ (N is an auxiliary Hilbert space and its dimension is equal to the multiplicity m of $\sigma(A_0)$) such that

$$(\mathfrak{P} A_0 u)(\delta) = \delta^2 (\mathfrak{P} u)(\delta), \quad \delta > 0, \quad \forall\, u \in \mathcal{D}(A_0).$$

The mapping $\mathfrak{P}$ determines a modified spectral representation of the unperturbed operator A_0 in which the action of A_0 corresponds to the multiplication by δ^2 in $L_2(\mathbb{R}_+, N)$. This representation is determined uniquely up to isometrics of N. Since the dimensionalities of N and $\mathcal{H}$ coincide and are equal to m, without loss of generality, we can choose $N = \mathcal{H}$.

Let us consider a nonnegative operator realization $A_{\mathbf{T}}$ of (3) defined by (16). By Theorem 2, the wave operators $W_\pm(A_{\mathbf{T}}, A_0)$ are complete and the image

$$\mathfrak{S}_{(A_{\mathbf{T}}, A_0)} := \mathfrak{P} S_{(A_{\mathbf{T}}, A_0)} \mathfrak{P}^{-1}$$

of the scattering operator $S_{(A_{\mathbf{T}}, A_0)}$ is a unitary operator in the modified spectral representation $L_2(\mathbb{R}_+, \mathcal{H})$.

Denote $J_{A_0} = P_{\ker C_0} - P_{\ker(C_0 - I/2)}$, where P_M is the orthogonal projector onto M in $\mathcal{H}$ and C_0 is defined by (19). It has been proved [10] that the operator $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}$ coincides with an operator of multiplication by the boundary value* $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ of the contraction operator-valued function

$$\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\lambda) = J_{A_0} [I - 2(1 - i\lambda) C] [I - 2(1 + i\lambda) C]^{-1}, \quad \lambda \in \mathbb{C}_+ \tag{21}$$

(here $C = (A_{\mathbf{T}} + I)^{-1} - \left(B^* B + I\right)^{-1}$) analytic in the upper half-plane.

In the case of finite rank singular perturbations ($n < \infty$), Theorem 2 yields that $C = C_0 - T(I + RT)^{-1}$. Thus, formula (21) provides a representation of the analytic continuation of the scattering matrix $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\delta)$ in terms of coefficients t_{ij} of the singular perturbation. This formula becomes especially simple if A_0 coincides with the Friedrichs $B^* B$ or with the Krein – von Neumann BB^* extensions of A_{sym}. In

* In the sense of strong convergence.

particular, if $A_0 = B^*B$, then $J_{A_0} = I$, $C_0 = 0$ and, after simple transformations, we get

$$\mathfrak{S}_{\left(A_T,\, B^*B\right)}(\lambda) = [I + (R + 2(1 - i\lambda)I)T][I + (R + 2(1 + i\lambda)I)T]^{-1}, \quad \lambda \in \mathbb{C}_+.$$

Similarly, if $A_0 = BB^*$, then $J_{A_0} = -I$, $C_0 = I/2$ and

$$\mathfrak{S}_{\left(A_{\mathbf{T}},\, BB^*\right)}(\lambda) = [i\lambda I + (i\lambda R + 2(1 - i\lambda)I)T][i\lambda I + (i\lambda R - 2(1 + i\lambda)I)T]^{-1}.$$

5. One-dimensional Schrödinger operator with symmetric zero-range potentials. A one-dimensional Schrödinger operator corresponding to a general zero-range symmetric potential at the point $x = 0$ can be given by the expression

$$A_0 + a\langle\delta, \cdot\rangle\delta + b\langle\delta', \cdot\rangle\delta + c\langle\delta, \cdot\rangle\delta' + d\langle\delta', \cdot\rangle\delta', \tag{22}$$

where $A_0 = -\dfrac{d^2}{dx^2}$ $\left(\mathcal{D}(A_0) = W_2^2(\mathbb{R})\right)$ acts in $\mathfrak{H} = L_2(\mathbb{R})$, δ' is the derivative of the Dirac δ-function (with support at 0), the parameters a, d are real, and $b = -\bar{c}$.

In this case, the singular elements $\psi_1 = 2\delta$ and $\psi_2 = 2\delta'$ form an orthonormal system in $\mathfrak{H}_{-2}(A_0) = W_2^{-2}(\mathbb{R})$ and the functions

$$(\mathbb{A}_0 + I)^{-1}\psi_1 = h_1(x) = \begin{cases} e^{-x}, & x > 0, \\ e^{x}, & x < 0, \end{cases}$$

$$(\mathbb{A}_0 + I)^{-1}\psi_2 = h_2(x) = \begin{cases} -e^{-x}, & x > 0, \\ e^{x}, & x < 0, \end{cases}$$

form an orthonormal basis of $\mathcal{H} = \ker\left(A_{\mathrm{sym}}^* + I\right)$, where $A_{\mathrm{sym}}^* = -\dfrac{d^2}{dx^2}$, $\mathcal{D}\left(A_{\mathrm{sym}}^*\right) = W_2^{-2}(\mathbb{R}\setminus\{0\})$ and $A_{\mathrm{sym}} = -\dfrac{d^2}{dx^2}$, $\mathcal{D}\left(A_{\mathrm{sym}}\right) = \{\, u(x) \in W_2^2(\mathbb{R}) \mid u(0) = u'(0) = 0\}$.

Representing (22) in the form (3), we get

$$-\frac{d^2}{dx^2} + \sum_{i,j=1}^{n} t_{ij}\langle\psi_j, \cdot\rangle\psi_i, \tag{23}$$

where coefficients $t_{11} = \dfrac{a}{4}$, $t_{12} = \dfrac{b}{4}$, $t_{21} = \dfrac{c}{4}$, $t_{22} = \dfrac{d}{4}$ form the Hermitian matrix $T = \{t_{ij}\}_{i,j=1}^{2}$.

To obtain the regularization $\mathbb{A}_{\mathbf{T}}$ of (23) it suffices to extend the distributions δ and δ' onto $W_2^2(\mathbb{R}\setminus\{0\})$. The most reasonable way (based on preserving of initial homogeneity of δ and δ' with respect to scaling transformations, see, for details, [2]) leads to the following definition:

$$\langle\delta_{\mathrm{ex}}, f\rangle = \frac{f(+0) + f(-0)}{2}, \quad \langle\delta'_{\mathrm{ex}}, f\rangle = -\frac{f'(+0) + f'(-0)}{2} \tag{24}$$

for all $f(x) \in W_2^2(\mathbb{R}\setminus\{0\})$. In this case, applying Theorem 1, we immediately obtain the following description of self-adjoint operator realizations $A_{\mathbf{T}}$ of (23) that has been obtained for the first time in [15]: $A_{\mathbf{T}} = -\frac{d^2}{dx^2}\upharpoonright_{\mathcal{D}(A_{\mathbf{T}})}$,

$$\mathcal{D}(A_{\mathbf{T}}) = \left\{ f(x) \in W_2^2(\mathbb{R}\setminus\{0\}) \,\middle|\, \mathbf{T}\Gamma_0 f = \Gamma_1 f \right\}, \qquad \mathbf{T} = \frac{1}{4}\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}, \tag{25}$$

where

$$\Gamma_0 f = \begin{pmatrix} f(+0) + f(-0) \\ -f'(+0) - f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \frac{1}{2}\begin{pmatrix} f'(+0) - f'(-0) \\ f(+0) - f(-0) \end{pmatrix}.$$

Let us consider the following simple maximal symmetric operator B in the space $L_2(\mathbb{R})$:

$$B = i(\operatorname{sign} x)\frac{d}{dx}, \qquad \mathcal{D}(B) = \left\{ u(x) \in W_2^1(\mathbb{R}\setminus\{0\}) \,\middle|\, u(-0) = u(+0) = 0 \right\}.$$

It is easy to verify that A_0 satisfies (18) and $A_{\text{sym}} = B^2$ for such a choice of B. Thus, we can apply the Lax – Phillips scattering approach to the investigation of operator realizations of (22). In our case, $\mathbf{R} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and the matrix decomposition $\mathbf{C}_0$ (with respect to the basis $\{h_i(x)\}_{i=1}^2$) of the operator C_0 defined by (19) has the form $\mathbf{C}_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$. Using Theorem 2, it is easy to prove the following statement.

Proposition 1. *A self-adjoint operator realizations* $A_{\mathbf{T}}$ *of* (23) *is nonnegative if and only if* $p := |b|^2 - ad + 2(a-d) + 4 \neq 0$ *and*

$$0 \leq \frac{1}{p}\begin{pmatrix} 4-2d & -2b \\ -2\bar{b} & p-4-2a \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case the spectrum $\sigma(A_{\mathbf{T}})$ *is Lebesgue and it coincides with* $[0, \infty)$. *In the opposite case,* $A_{\mathbf{T}}$ *possesses at least one negative eigenvalue.*

By virtue of the definition of C_0, the matrix decomposition $\mathbf{J}_{A_0}$ of J_{A_0} has the form $\mathbf{J}_{A_0} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Taking this fact into account and going over to the matrix decomposition in (21), we get that the matrix decomposition of the analytic continuation $\mathfrak{S}_{(A_{\mathbf{T}}, A_0)}(\lambda)$ of the scattering matrix into the upper half-plane has the form

$$\mathbf{S}_{(A_{\mathbf{T}}, A_0)}(\lambda) =$$

$$= \begin{pmatrix} -p + (1-i\lambda)\alpha_{11} & (1-i\lambda)\alpha_{12} \\ -(1-i\lambda)\alpha_{21} & p-(1-i\lambda)\alpha_{22} \end{pmatrix}\begin{pmatrix} p-(1+i\lambda)\alpha_{11} & -(1+i\lambda)\alpha_{12} \\ -(1+i\lambda)\alpha_{21} & p-(1+i\lambda)\alpha_{22} \end{pmatrix}^{-1},$$

where $\alpha_{11} = 4-2d$, $\alpha_{12} = -2b$, $\alpha_{21} = -2\bar{b}$, $\alpha_{22} = p-4-2a$.

1. *Berezansky Yu. M.* Expansion in eigenfunctions of self-adjoint operators. – Kiev: Naukova Dumka, 1965 (in Russian). Engl. transl.: Providence, R. I.: AMS, 1968.

2. *Albeverio S., Kurasov P.* Singular perturbations of differential operators // Solvable Schrödinger type operators: Lect. Notes Ser. – Cambridge: Cambridge Univ. Press, 2000. – **271**.
3. *Adamyan V., Pavlov B.* Zero-radius potentials and M. G. Krein's formula for generalized resolvents // Zap. Nauchn. Sem. LOMI. – 1986. – **149**. – P. 7 – 23.
4. *Albeverio S., Kurasov P.* Finite rank perturbations and distribution theory // Proc. Amer. Math. Soc. – 1999. – **127**, № 4. – P. 1151 – 1161.
5. *Arlinskii Yu. M., Tsekanovskii E. R.* Some remarks on singular perturbations of self-adjoint operators // Meth. Funct. Anal. and Top. – 2003. – **9**. – P. 287 – 308.
6. *Berezansky Yu. M., Us G. F., Sheftel Z. G.* Functional analysis. – Kiev: Vyshcha Shkola, 1990 (in Russian). Engl. transl.: Basel: Birkhäuser, 1996.
7. *Gorbachuk M. L., Gorbachuk V. I.* Boundary-value problems for operator-differential equations. – Kiev: Naukova Dumka, 1984 (in Russian). Engl. transl.: Dordrecht: Kluwer, 1991.
8. *Albeverio S., Kuzhel S.* η-Hermitian operators and previously unnoticed symmetries in the theory of singular perturbations. – Bonn, 2004. – 26 p. – (Preprint / Univ. Bonn).
9. *Kuzhel S.* On the determination of free evolution in the Lax – Phillips scattering scheme for second-order operator-differential equations // Math. Notes. – 2000. – **68**. – P. 724 – 729.
10. *Kuzhel S.* On elements of scattering theory for abstract Schrödinger equation Lax – Phillips approach // Meth. Funct. Anal. and Top. – 2001. – **7**, № 2. – P. 13 – 22.
11. *Kuzhel S., Moskalyova Yu.* The Lax – Phillips scattering approach and singular perturbations of Schrödinger operator homogeneous with respect to scaling transformations (submitted to J. Math. Kyoto Univ.).
12. *Kuzhel A., Kuzhel S.* Regular extensions of Hermitian operators. – Dordrecht: VSP, 1998.
13. *Krein M. G.* Theory of self-adjoint extensions of semibounded Hermitian operators and its applications. I // Math. Trans. – 1947. – **20**. – P. 431 – 495.
14. *Arlinskii Yu. M., Tsekanovskii E. R.* On the theory of nonnegative self-adjoint extensions of a nonnegative symmetric operator // Repts Nat. Acad. Sci. Ukraine. – 2002. – № 11. – P. 30 – 37.
15. *Albeverio S., Nizhnik L.* A Schrödinger operator with a δ'-interaction on a Cantor set and Krein – Feller operator. – Bonn, 2003. – 16 p. – (Preprint / Univ. Bonn, № 99).

Received 07.02.2005