UDC 517.9+ 531.19

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CORRELATED BROWNIAN MOTIONS AS AN APPROXIMATION TO DETERMINISTIC MEAN-FIELD DYNAMICS

КОРЕЛЬОВАНИЙ БРОУНІВСЬКИЙ РУХ ЯК ГРАНИЦЯ НАБЛИЖЕННЯ ДЕТЕРМІНІСТСЬКОЇ ДИНАМІКИ СЕРЕДНЬОГО ПОЛЯ

We analyze the transition from deterministic mean-field dynamics of several large particles and infinitely many small particles to a stochastic motion of the large particles. In this transition the small particles become the random medium for the large particles and the motion of the large particles becomes stochastic. Assuming that the empirical velocity distribution of the small particles is governed by a probability density ψ, the mean-field force can be represented as the negative gradient of a scaled version of ψ. The stochastic motion is described by a system of stochastic ordinary differential equations driven by Gaussian space-time white noise and the mean-field force as a shift-invariant integral kernel. The scaling preserves a small parameter in the transition, the so-called correlation length. In this set-up the separate motion of each particle is a classical Brownian motion (Wiener process), but the joint motion is correlated through the mean-field force and the noise. Therefore, it is not Gaussian. The motion of 2 particles is analyzed in detail and a diffusion equation is derived for the difference in the positions of the 2 particles. The diffusion coefficient in the latter equation is spatially dependent, which allows us to determine regions of attraction and repulsion of the two particles by computing the probability fluxes. The result is consistent with observations in the applied sciences, namely that Brownian particles get attracted to each other if the distance between them is smaller than a critical small parameter. In our case, this parameter is shown to be proportional to the aforementioned correlation length.

Проаналізовано перехід від детерміністської динаміки середнього поля декількох великих частинок та нескінченної кількості малих частинок до стохастичного руху великих частинок. Під час цього переходу маленькі частинки перетворюються у випадкове середовище для великих частинок, а рух великих частинок стає стохастичним. Якщо припустити, що розподіл емпіричної швидкості малих частинок визначається щільністю розподілу ψ, то силу середнього поля можна подати як від'ємний градієнт масштабного перетворення ψ. Стохастичний рух описано системою стохастичних звичайних диференціальних рівнянь, керованих гауссовим просторово-часовим білим шумом та силою середнього поля як інтегральним ядром, інваріантним відносно зсуву. Масштабування зберігає малий параметр при переході (так звану довжину кореляції). У даній постановці окремий рух кожної частинки є класичним броунівським рухом (вінерівським процесом), але спільний рух корелюється силою середнього поля і шумом. Тому він не є гауссівським. Детально проаналізовано рух двох частинок і виведено рівняння дифузії для різниці положень двох частинок. Коефіцієнт дифузії в останньому рівнянні є просторово залежним, що дозволяє визначити області притягання та відштовхування двох частинок шляхом розрахунку течій імовірностей. Результат узгоджується із спостереженнями у прикладних науках, а саме, з фактом, що броунівські частинки притягуються одна до одної, якщо відстань між ними менша за критичний малий параметр. У випадку, що вивчається, показано, що цей параметр пропорціональний згаданій вище довжині кореляції.

1. Introduction: microscopic model and space-time scales. Kotelenez [1] analyzes a model in which one can observe the transition from a completely deterministic motion of a system of large and small particles with random initial conditions into a stochastic motion. The small particles have very large velocities. The large particles' momenta are completely determined by the momenta of the small particles through a mean field type interaction.

In a scaling limit Kotelenez (loc.cit.) shows that, under certain assumptions, the positions of the large particles become correlated Brownian motions, where the spatial correlations can be computed through a transformation of the interaction force. A similar result holds for the velocities in a Hamiltonian framework. To explain the transition from a deterministic to a stochastic motion the following assumptions and observations are made. First, the state space of the particles is $\mathbf{R}^d$ with $d \geq 2$. This allows small particles to escape to infinity and to be replaced by other small particles. Secondly, assuming that

the initial mean velocity of the small particles tends to infinity in the scaling limit, the fast moving small particles interact with the large particles only for a short time before escaping to infinity. Consequently, in different (small) time intervals the displacements of the large particles are caused essentially by different small particles which started (by assumption) independently. Therefore, these displacements become independent in the limit.

The interactions between large particles can be neglected in this limit as their effects can be included in the macroscale (after deriving approximate formulas for the action of the small particles upon the large ones). The fractional step method (cf. [2]) provides a rigorous framework to include these effects.

Microscopic and mesoscopic units are defined as fractions of macroscopic units. Let $\mathbf{R}^d$ be partitioned into small cubes which are parallel to the axes. The cubes are denoted by $(\bar{R}^\lambda]$, where $\bar{R}^\lambda$ is the center of the cube and $\lambda \in \mathbf{N}$. These cubes are open on the left and closed on the right (in the sense of d-dimensional intervals) and have side length $\delta R \approx \dfrac{1}{n}$, and the origin 0 is the center of a cell. The cells and their centers will be used to "coarse-grain" the motion of the particles, placing the particles within a cell at the midpoint (cf. [3]). Further, the small particles within a cell are grouped as clusters, where particles in a cluster have similar initial velocities. Thus, we average locally over the initial data of particles while maintaining globally a spatially distributed structure of the particle systems. We may call δR a mesoscopic length unit (cf. [4, 5] for the use of this terminology). Similarly, we choose a mesoscopic time unit $\delta\sigma$, during which the interaction between small and large particles forms a pattern, and which can be used to simplify the calculations (s. the following Remark 1.1). Here we choose $\delta\sigma \approx \dfrac{1}{n^d}$, which follows from the need to control the variance of sums of independent random variables and its generalization in Doob's inequality, where the second moments are $\approx n^d$, i.e., the number of cells in a unit cube. $\delta\sigma$ then becomes a normalizing factor at the forces acting on the large particle motion. Assuming, without loss of generality, that the proportionality factors in the relations for δR and $\delta\sigma$ equal 1, we now define the mesoscopic and macroscopic time and spatial scales:

$$\delta\sigma = \frac{1}{n^d} \ll 1, \qquad \delta R = \frac{1}{n} \ll 1. \tag{1.1}$$

***Remark* 1.1.** Stochastic approximations to elastic collisions have been obtained by numerous authors for cases involving only one large particle and assuming no interaction (collisions) between the small particles. Dürr, Goldstein and Lebowitz [6, 7] obtain an Ornstein – Uhlenbeck approximation to the collision dynamics, generalizing a result of Holley [8] from dimension $d = 1$ to dimension $d = 3$. Sinai and Soloveichik [9] obtain an Einstein – Smoluchowski approximation in dimension $d = 1$ and prove that almost all small particles collide with the large particle only finitely often. A similiar result was obtained by Szász and Tóth [10, 11]. Szász and Tóth [11] obtain both Einstein – Smoluchowski and Ornstein – Uhlenbeck approximations for the time evolution of one large particle in dimension $d = 1$ (cf. also [12]).

Note that in the case of just one large particle, the fluid around that particle may look homogeneous and isotropic, which leads to a relatively simple statistical description of the displacement of that particle as a result of the "bombardment" of this large particle by small particles. Further, whether or not the "medium" of small particles is spatially

correlated cannot influence the motion of only one large particle, as long as in the scaling limit the time correlation length δs tends to 0.

In contrast, if there are at least two large particles and they move closely together, the fluid around each of them will no longer be homogeneous and isotropic. In fact, the fluid will get depleted (Asakura, and Oosawa [13], Goetzelmann, Evans, and Dietrich [14] and the references therein as well as Kotelenez, Leitman, and Mann (Jr.) [15]). The simplest argument to explain depletion is that if the large particles get closer together than the diameter of a typical small particle, the space between the large particles must get depleted (Goetzelmann, Evans, and Dietrich (loc.cit.)). Therefore, the forces, generated by the collisions and acting on two different large particles, become statistically correlated if the large particles move together closer than a critical length $\sqrt{\varepsilon} > 0$. To capture this effect in the scaling limit, we choose the following $\delta s \ll \delta\sigma$ as the microscopic time unit as follows:

$$\delta s := \frac{\sqrt{\varepsilon}}{\bar{\sigma}_n}, \tag{1.2}$$

where $\bar{\sigma}_n$ is the mean velocity of the small particles.

Suppose that the time evolution of the particles is stationary (starting with some Gibbs distribution).

Denote by $B_{\hat{\varepsilon}_n}(q)$ the $\hat{\varepsilon}_n$-neighborhood of q, i.e., the ball of radius $\hat{\varepsilon}_n$ and center q. Let $\tilde{m}$ be the mass of a small particle. We first replace $q^{\lambda,l}$, the position of the particle starting in $(\bar{R}^\lambda]$ with velocity $\dfrac{p^{\lambda,l}}{\tilde{m}}$ by the midpoint of the cell $(\bar{R}^\lambda]$ (coarse graining).

Suppose there are approximately $\tilde{M}_n$ small (noninteracting) particles starting in the same $(\bar{R}^\lambda]$ and $n^d\tilde{M}_n$ small particles in a unit cube. Let us make the following simplifying assumptions:

(i) We assume that the small particles hit the large particle at its center, averaging over all possible angles of contact. One might call this procedure "coarse graining" in momentum space.

(ii) We consider collisions over a time interval of length $\delta\sigma$. The size of the "observed" collisions is $L_n \le n^d\tilde{M}_n\hat{\epsilon}_n^{d-1}\dfrac{\delta\sigma}{\delta s}$, the average number of collisions during a time of length $\delta\sigma$. If $L_n \ll n^d\tilde{M}_n\hat{\epsilon}_n^{d-1}\dfrac{\delta\sigma}{\delta s}$, we may consider L_n to be the size of a sample. $\delta\sigma$ is large enough so that most small particles will be able to escape to infinity after colliding with the give large particle (cf. Sinai and Soloveichik (loc.cit.)) for the one-dimensional case. Hence, we may assume that the "samples" in disjoint time intervals $[(l-1)\delta\sigma, l\delta\sigma)$ are roughly independent. We take the rate of change in the velocity of the large particle, $V_R(t)$, as a result of one collision to be proportional to δs. Adding up those changes over a time interval of length $\delta\sigma$, we assume that the law of large numbers (LLN) is applicable.

As a result of these assumptions, we first obtain that the (random) force acting on the large particle R during the time interval $\delta\sigma$ should be proportional

$$\tilde{m}\frac{V_R(t+\delta\sigma)-V_R(t)}{\delta\sigma} \approx \tilde{m}\frac{c_{\varepsilon,L_n}}{\delta\sigma}\sum_\lambda(\bar{R}-\bar{R}^\lambda)\sum_{l=1}^{L_n}1_{\{|\bar{R}-\bar{R}^\lambda-\frac{p^{\lambda,l}\delta s}{\tilde{m}}|\le\hat{\varepsilon}_n\}}. \tag{1.3}$$

Here, c_{ε,L_n} is some constant, depending on ε and L_n. Next, suppose that as a result of the law of large numbers (LLN)

$$\frac{1}{L_n}\sum_{l=1}^{L_n} 1_{\{|\bar{R}-\bar{R}^\lambda-\frac{p^{\lambda,l}\delta s}{\hat{m}}|\le\hat{\varepsilon}_n\}} \approx$$

$$\approx E\frac{1}{L_n}\sum_{l=1}^{L_n} 1_{\{|\bar{R}-\bar{R}^\lambda-\frac{p^{\lambda,l}\delta s}{\hat{m}}|\le\hat{\varepsilon}_n\}} = g_\varepsilon(\bar{R}-\bar{R}^\lambda)\hat{\varepsilon}_n^d. \tag{1.4}$$

Here, $g_\varepsilon(\bar{R}-\bar{R}^\lambda)$ is an even density of the velocities of a cluster of small particles with start in the cell $(\bar{R}^\lambda]$ and moving with approximately the same velocity. Set

$$G_\varepsilon(\bar{R}-\bar{R}^\lambda) := \frac{c_{\varepsilon,L_n}L_n\hat{\varepsilon}_n^d}{\delta\sigma}(\bar{R}-\bar{R}^\lambda)g_\varepsilon(\bar{R}-\bar{R}^\lambda).$$

The procedure of replacing the collision effects by averages over small time intervals may be called "coarse graining" in time.

***Example* 1.1.** Let $D>0$ be a diffusion coefficient and consider the rescaled Maxwellian density

$$g_\varepsilon(q) := \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}}\exp\left(-\frac{|\,q\,|^2}{2\varepsilon}\right). \tag{1.5}$$

Suppose

$$\frac{c_{\varepsilon,L_n}L_n\hat{\varepsilon}_n^d}{\delta\sigma} \approx \sqrt{D}(4\pi\varepsilon)^{\frac{d}{4}}\left(\frac{2}{d\varepsilon}\right)^{\frac{1}{2}}.$$

Then, by (1.3)

$$\frac{c_{\varepsilon,L_n}}{\delta\sigma}(\bar{R}-\bar{R}^\lambda)\sum_{l=1}^{L_n} 1_{\{|\bar{R}-\bar{R}^\lambda-\frac{p^{\lambda,l}\delta s}{\hat{m}}|\le\hat{\varepsilon}_n\}} \approx$$

$$\approx \sqrt{D}(4\pi\varepsilon)^{\frac{d}{4}}\left(\frac{2}{d\varepsilon}\right)^{\frac{1}{2}}(\bar{R}-\bar{R}^\lambda)\frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}}\exp\left(-\frac{|\,\bar{R}-\bar{R}^\lambda\,|^2}{2\varepsilon}\right). \tag{1.6}$$

We obtain

$$G_\varepsilon(\bar{R}-\bar{R}^\lambda) =$$

$$= \sqrt{D}(4\pi\varepsilon)^{\frac{d}{4}}\left(\frac{2}{d\varepsilon}\right)^{\frac{1}{2}}(\bar{R}-\bar{R}^\lambda)\,\frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}}\exp\left(-\frac{|\,\bar{R}-\bar{R}^\lambda\,|^2}{2\varepsilon}\right). \tag{1.7}$$

So, in the case of a Maxwellian velocity field, the displacement of the large particles due to the collisions with (fast) moving small particles, analyzed in the mesoscale, is approximately governed by the smooth interaction potential

$$U_\varepsilon(R-q) := \hat{m}D\left(\frac{2\varepsilon}{d}\right)^{\frac{1}{2}}\frac{1}{(\pi\varepsilon)^{\frac{d}{4}}}\exp\left(\frac{-|R-q|^2}{2\varepsilon}\right). \tag{1.8}$$

In addition to the previous heuristic "derivation" of a mean-field force from short range collisions, a mean-field dynamics, on a microscopic level, can result from long range potentials, like a Coulomb potential or a smoothed Lenard – Jones potential, and an appropriate mixture of both effects can lead to more complicated mean-field forces.

2. A deterministic mean-field model and its stochastic limit. The interaction between large and small particles is governed by a twice continuously differentiable odd

$\mathbf{R}^d$-valued function G_ε. We assume that all partial derivatives up to order 2 are square integrable and that $|G_\varepsilon|^m$ is integrable for $1 \le m \le 4$, where "integrable" refers to the Lebesgue measure on $\mathbf{R}^d$. The function G_ε will be approximated by odd $\mathbf{R}^d$-valued functions $G_{\varepsilon,n}$ with bounded supports. Further, we use the idea of coarse graining, as described in the introduction. $\hat{m}$ will be the mass of a large particle, and m denotes the mass of a cluster of small particles within a cell. The velocities of the clusters in a cell can be characterized as follows:

Hypothesis 2.1. *There is a partitioning of the velocity space*

$$\mathbf{R}^d = \cup_{\iota \in \mathbf{N}} B_\iota,$$

and the velocities of each cluster take values in exactly one B_ι, where for the sake of simplicity we assume that all B_ι are small cubic d-dimensional intervals (left open, right closed), all with the same volume $\le \dfrac{1}{n^d}$.

Set

$$Y(dQ,t) := m \sum_{\lambda,\iota} \delta_{\bar{Q}_n(t,\lambda,\iota)}(dQ), \qquad X_N(dR,t) := \hat{m} \sum_{j=1}^{N} \delta_{\bar{R}_n^j(t)}(dR),$$

where $\bar{R}_n^j(t)$ and $\bar{Q}_n(t,\lambda,\iota)$ are the positions at time t of the large and small particles, respectively. " $\bar{}$ " means that the midpoints of those cells are taken, where the particles are at time t, and the labels λ, ι mean that the small particle started at $t = 0$ in $(\bar{R}^\lambda]$ with velocity from B_ι. Y and X_N are called the empirical distributions of the small and large particles, respectively. We include a friction coefficient β_n for the large particles (cf. Uhlenbeck and Ornstein [16] for the rationale) [1]. Let "$\vee$" denote "$\max$". The assumptions on the most important parameters are put together in the following assumption:

Hypothesis 2.2.

$$\begin{gathered} \beta_n = n^{\tilde{p}}, \qquad d > \tilde{p} > 0, \\ m = n^{-\eta}, \quad \eta \ge 0, \\ \bar{\sigma}_n = n^p, \quad p > (4d+2) \vee (2\tilde{p} + 2\eta + 2d + 2). \end{gathered}$$

The velocity of the ith large particle at time s, will be denoted $V_n^i(s)$, where $V_n^i(0) = 0 \ \forall i$. $w_{0,n}^{\lambda,\iota} \in B_\iota$ will be the initial velocity of the small particle starting at time 0 in the cell $(\bar{R}^\lambda]$, $\iota \in \mathbf{N}$. Note that, for the infinitely many small particles, the resulting friction due to the collision with the finitely many large particles should be negligible. Further, in a dynamical Ornstein – Uhlenbeck type model with friction β_n the "fluctuation force" has to be governed by a function $\tilde{G}_n$. The relation to an Einstein – Smoluchowski diffusion is given by

$$\tilde{G}_n(R) \approx \beta_n G_n(R),$$

as $\beta_n \longrightarrow \infty$ (cf., e.g., [17, 18] or [19]). This factor will disappear, as we move from a second order differential equation to a first order equation.

[1] In the original draft of the derivation of Brownian motions friction was missing. I want to thank J. A. Mann (Jr.) from the Chemical Engineering Department at CWRU for pointing out that the source of Brownian motion is also the source of friction and had to be included.

Next, we identify the clusters in the small cells with velocity from B_ι with random variables. The empirical distributions of particles and velocities in cells define, in a canonical way, probability distributions. The absence of interaction of the small material particles among themselves leads to the assumption that their initial positions and velocities are independent if modelled as random variables.

Let $\alpha \in (0,1)$ be the expected average volume (in a unit cube) occupied by small particles (for large n) and assume that the initial "density" function $\varphi_n(q)$ for the small particles satisfies:

$$0 \le \varphi_n(q) \le \alpha n^d.$$

We need a state to describe the outcome of finding no particle in the cell (the "empty state"). Let $\diamond$ denote this empty state and set $\hat{\mathbf{R}}^d := \mathbf{R}^d \cup \{\diamond\}$. The usual metric on $\mathbf{R}^d$ is extended to $\hat{\mathbf{R}}^d$ by defining the distance between $\diamond$ and an arbitrary element $r \in \mathbf{R}^d$ to be 1. The Borel sets in $\mathbf{R}^d$ and in $\hat{\mathbf{R}}^d$ will be denoted by $\mathcal{B}^d$ and $\hat{\mathcal{B}}^d$, respectively. Set

$$\Omega := \{\hat{\mathbf{R}}^d \times \mathbf{R}^d\}^{\mathbf{N}}.$$

The velocity field of the small particles will be governed by a strictly positive probability density $\psi(w)$ on $\mathbf{R}^d$, which we rescale as follows:

$$\psi_n(w) := \frac{1}{n^{pd}} \psi\left(\frac{w}{n^p}\right).$$

Define the initial velocities of the small particles starting in the cell $(\bar{R}^\lambda]$ and from a cluster, characterized by B_ι, as random variables

$$W_{0,n}^{\lambda,\iota} \sim \psi_n(w) 1_{B_\iota}(w) \frac{1}{\int\limits_{B_\iota} \psi_n(w)dw}. \tag{2.1}$$

Further, let

$$\eta_{\lambda,\iota,n} := \int\limits_{(\bar{R}^\lambda]} \varphi_n(q)dq \int\limits_{B_\iota} \psi_n(w)dw$$

be the probability of finding a small particle at $t = 0$ in $(\bar{R}^\lambda]$ and with velocity from the cluster B_ι. Define random variables $\zeta_n^{\lambda,\iota}$ for the initial positions of the small particles as follows:

$$\zeta_n^{\lambda,\iota} := \begin{cases} \bar{R}^\lambda, & \text{with probability (w.p.)} \quad \eta_{\lambda,\iota,n}, \\ \diamond, & \text{w.p.} \quad 1 - \eta_{\lambda,\iota,n}. \end{cases} \tag{2.2}$$

Denote by $\mu_{n,\lambda,\iota}$ and $\nu_{n,\lambda,\iota}$ the distributions of $\zeta_n^{\lambda,\iota}$ and $W_{0,n}^{\lambda,\iota}$, respectively. Set

$$\omega_{\lambda,\iota} := \hat{q}^{\lambda,\iota} \times w^{\lambda,\iota} \in \hat{\mathbf{R}}^d \times \mathbf{R}^d$$

and

$$P_{n,\lambda,\iota} := \mu_{n,\lambda,\iota} \otimes \nu_{n,\lambda,\iota}.$$

We assume the initial positions and velocities to be independent, i.e., we define the initial joint probability distribution of positions and velocities on Ω to be the product measure:

$$P_n := \otimes_{\lambda\in\mathbf{N}} \otimes_{\iota\in\mathbf{N}} P_{n,\lambda,\iota}. \tag{2.3}$$

Formally, the coarse-grained particle evolution in the mesoscale can be described by the following Euler scheme:

$$\begin{gathered} R_n^i(t) = R_n^i(0) + \sum_{s\le t} V_n^i(s)\delta\sigma, \\ V_n^i(s) = \sum_{0<u\le s} \exp[-\beta_n(s-u)]\frac{1}{\hat{m}m}\int \beta_n G_n(\bar{R}_n^i(u-) - q)Y(dq,u)\delta\sigma, \\ Q_n(s,\lambda,\iota) = \\ = \bar{R}^\lambda + w_{0,n}^{\lambda,\iota}s + \frac{1}{\hat{m}m}\sum_{u<s}\sum_{v\le u}\int \beta_n G_n(\bar{Q}_n(v,\lambda,\iota) - R)X_N(dR,v)(\delta\sigma)^2, \end{gathered} \tag{2.4}$$

if $\zeta_n^{\lambda,\iota} = \bar{R}^\lambda$.

Note that the empirical distribution of the small particles is not a priori finite on bounded sets. In particular, we do not know whether or not infinitely many small particles interact with a given large particle at a given time u. Hence, we have to show existence of the coarse-grained particle model:

Proposition 2.1. *The Euler scheme* (2.4) *is defined for all* $s \ge 0$ *a.s.*

Proof. Use induction (for details, cf. [1]).

Let $w(dq, ds)$ standard Gaussian white noise on $\mathbf{R}^d \times \mathbf{R}_+$ (which can be interpreted as the increments of a space-time Brownian sheet), defined on the same probability space as $(R^1(0), \dots, R^N(0))$ such that $(R^1(0), \dots, R^N(0))$ and $w(dq, ds)$ are independent. Consider the stochastic integral equations:

$$R^i(t) = R^i(0) + \sqrt{\alpha}\int_0^t \int G_\varepsilon(R^i(u) - q)w(dq, du), \quad i = 1, \dots, N. \tag{2.5}$$

The equations (2.5) are coupled only through the Brownian sheet $w(dq, du)$. The assumptions imply that the integrals define continuous square integrable martingales for any adapted processes $r^i(\cdot)$. Let us verify that (2.5) satisfies a uniform Lipschitz assumption. Let $r^i, r^j \in \mathbf{R}^d$ two d-dimensional components of an (Nd)-vector $\hat{r}$ and let $G_{\varepsilon,k}, G_{\varepsilon,\ell}$ denote the k-th and ℓ-th one-dimensional components of G_ε, respectively. In the following equation the process in brackets denotes the mutual quadratic variation of the stochastic integrals. Set

$$D_{k\ell,ij}(\hat{r}) := \int G_{\varepsilon,k}(r^i - r^j - q)G_{\varepsilon,\ell}(-q)dq. \tag{2.6}$$

The properties of the Brownian sheet imply:

$$\begin{gathered} = \left[\int G_{\varepsilon,k}(r^i - q)w(dq, du), \int G_{\varepsilon,\ell}(r^j - q)w(dq, du)\right] = \\ = \int G_{\varepsilon,k}(r^i - q)G_{\varepsilon,\ell}(r^j - q)dq = D_{k\ell,ij}(\hat{r}) \end{gathered}$$

by the shift invariance of the Lebesgue measure. Let " $\wedge$ " denote "minimum". It follows

$$\left[\int G_{\varepsilon,k}(r^i-q)-G_{\varepsilon,\ell}(r^j-q)w(dq,du)\right]=$$

$$=2\int(G_{\varepsilon,k}(-q)-G_{\varepsilon,k}(r^i-r^j-q))G_{\varepsilon,\ell}(-q)dq\le c_G(|r^i-r^j|\wedge 1) \tag{2.7}$$

by the smoothness and integrability assumptions of G_ε, where $c_G<\infty$. Therefore, (2.5) has a unique solution and is a Markov process in $\mathbf{R}^{Nd}$ (cf. [20]). The following theorem establishes the Einstein – Smoluchowski model as the approximation to the evolution of the positions of the large particles.

Hypothesis 2.3. $\{R_n^1(0),\dots,R_n^N(0)\}$ *and* $\left\{\zeta_n^{\lambda,\iota},W_{n,0}^{\lambda,\iota}:\lambda,\iota\in\mathbf{N}\right\}$ *are independent;*

$$(R_n^1(0),\dots,R_n^N(0))\Longrightarrow(R^1(0),\dots,R^N(0)),\ \textit{as}\ n\longrightarrow\infty,$$

where $(R_n^1(0),\dots,R_n^N(0))$ *are the initial positions of the large particles,* $(R^1(0),\dots$ $\dots,R^N(0))$ *the initial positions of* (2.5) *and "* $\Longrightarrow$ *" denotes weak convergence.*

Theorem 2.1. *Under Hypotheses* 2.1 – 2.3

$$(R_n^1(\cdot),\dots,R_n^N(\cdot))\Longrightarrow(R^1(\cdot),\dots,R^N(\cdot))$$
$$\textit{in}\quad D_{\mathbf{R}^{dN}}[0,\hat{t}],\quad \textit{as}\quad n\to\infty.$$

Here $(R^1(\cdot),\dots,R^N(\cdot))$ are the unique solutions of (2.5), $(R_n^1(\cdot),\dots,R_n^N(\cdot))$ are the solutions of the Euler scheme (2.4) and $D_{\mathbf{R}^{dN}}[0,\hat{t}]$ is the Skorokhod space of $\mathbf{R}^{dN}$-valued cadlag functions.

The proof is given in [1].

3. Properties of the stochastic limit. We now consider the n-particle motion described by (2.5), where we can incorporate the coefficient α into the definition of G_ε. Set

$$M^i(t):=\int_0^t\int G_\varepsilon(R^i(u)-q)w(dq,du)$$

and

$$c_{k\ell}:=\int G_{\varepsilon,k}(q)G_{\varepsilon,\ell}(q)dq.$$

Proposition 3.1. *For each* $i=1,\dots,N$ $M^i(\cdot)$ *is a* d*-dimensional Brownian motion with incremental covariance*

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1d}\\ c_{21} & c_{22} & \dots & c_{2d}\\ \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ c_{d1} & c_{d2} & \dots & c_{dd}\end{pmatrix}. \tag{3.1}$$

Proof. The kernel G_ε is spatially homogeneous. Therefore, the statement follows from a d-dimensional generalization of Paul Levy's theorem (cf. [21], Chapter 7, Theorem 1.1).

The motion of the each large particle is Brownian, if we consider the appropriate d-dimensional "marginal" process distribution of the motion described by (2.5) (with deterministic initial conditions). However, the joint motion is not Brownian. First of all,

let us state the fact that two large particles never hit each other if they start in different positions. The proof is due to Dawson [22].

Proposition 3.2. *Suppose that* $i \neq j$ *and that*

$$E|R^i(0) - R^j(0)|^{-2} < \infty. \tag{3.2}$$

Then

$$\mathrm{P}\Big\{\omega : \exists\, t \in [0,T] \quad \textit{such that} \quad R^i(t,\omega) = R^j(t,\omega)\Big\} = 0. \tag{3.3}$$

By (2.6) the mutual quadratic variation of the one-dimensional components of M^i and M^j for $i \neq j$:

$$[M^i_k, M^j_\ell] = \int G_{\varepsilon,k}(R^i(s) - q) G_{\varepsilon,\ell}(R^j(s) - q) ds = \\ = \int G_{\varepsilon,k}(R^i(s) - R^j(s) - q) G_{\varepsilon,\ell}(-q) ds. \tag{3.4}$$

This implies that the joint quadratic variation depends on the distance of the particles. In particular, assuming (3.2), the mutual quadratic variation does not become deterministic. Hence, the joint motion cannot be Gaussian and, a fortiori, cannot be Brownian.

Next, we are going to derive a partial shift invariance of the solutions of (2.5). To this end we define an infinite series expansion for $w(dp, dt)$.

Let $\mathbf{H}_0$ be the space of measurable functions on $\mathbf{R}^d$ which are square integrable with respect to the Lebesgue measure and let $|\cdot|_0$ be the usual L_2-norm, which is induced by the scalar product

$$\langle f, g\rangle_0 := \int f(q) g(q) dq$$

for $f, g \in \mathbf{H}_0$. Let $\{\tilde{\phi}_n\}_{n\in N}$ be a complete orthonormal system (CONS) in $\mathbf{H}_0$ and define an $\mathcal{M}_{d\times d}$-valued function ϕ_n whose entries on the main diagonal are all $\tilde{\phi}_n$ and whose other entries are all 0. Let $\mathcal{R}_{d,[0,\infty)}$ denote the space of $\mathbf{R}^d$-valued adapted continuous processes. Then for $r(\cdot) \in \mathcal{R}_{d,[0,\infty)}$

$$\int G_\varepsilon(r(t) - q) w(dq, dt) = \sum_{n=1}^{\infty} \int G_\varepsilon(r(t) - q) \phi_n(q) dq d\beta^n(t), \tag{3.5}$$

where $\beta^n(t)$ are $\mathbf{R}^d$-valued i.i.d. standard Wiener processes. The right-hand side of (3.5) defines the increment of an $\mathbf{R}^d$-valued square integrable continuous martingale M.

If f and g are random variables with values in some measurable space, we will write

$$f \sim g,$$

if f and g have the same distribution. Let $h \in \mathbf{R}^d$ and define the shifted Brownian sheet by:

$$w_{-h}(r,t) := w(r - h, t).$$

The representation (3.5) implies that $w_{-h}(dr,t) \sim w(dr,t)$, considered as distribution valued Wiener processes, have the same distribution and that $w_{-h}(r,t)$ is itself a Brownian sheet. The distribution space can be the Schwarz space of tempered distributions over $\mathbf{R}^d$ or a suitably chosen Hilbert subspace of the Schwarz space (cf. [23]).

In what follows we will focus on two particles, $R^1(\cdot),\ R^2(\cdot),$ described by (2.5). Denote the solutions of (2.5) with initial conditions R_0^i and driving noise $w(dr,dt)$ by $(R^1(\cdot,R_0^1,w),\ R^2(\cdot,R_0^2,w))$.

For $h\in\mathbf{R}^d$ and $i=1,2$ we see that

$$R^i(t)+h=R^i(0)+h+\int\limits_0^t\int G_\varepsilon(R^i(u)+h-(q+h)w(dq,du)=$$

$$=R^i(0)+h+\int\limits_0^t\int G_\varepsilon(R^i(u)+h-(q)w_{-h}(dq,du). \tag{3.6}$$

Hence, the left-hand side of (3.6) is the solution $\Big(R^1(\cdot,R_0^1+h,w_{-h}),\ R^2(\cdot,R_0^2+$ $+h,w_{-h})\Big)$. Further, the assumptions on G_ε imply that the solutions of (2.5) have a version which is measurable in all parameters, i.e., in the initial conditions and in the noise process, considered as a distribution-valued Wiener process (cf. [24]). This implies that $\Big(R^1(\cdot,R_0^1+h,w_{-h}),\ R^2(\cdot,R_0^2+h,w_{-h})\Big)\sim\Big(R^1(\cdot,R_0^1+h,w),R^2(\cdot,R_0^2+h,w)\Big)$. Thus, we obtain:

Proposition 3.3.

$$\Big(R^1(\cdot,R_0^1,w)+h,R^2(\cdot,R_0^2,w)+h\Big)\sim\Big(R^1(\cdot,R_0^1+h,w),R^2(\cdot,R_0^2+h,w)\Big), \tag{3.7}$$

where the pair processes are considered as $C([0,\infty);\mathbf{R}^{2d})$-valued random variables.

Note that (3.7) is only correct if we shift both R^1 and R^2 by the same d-dimensional vector h. Abbreviate $\hat r:=(r^1,r^2)\in\mathbf{R}^{2d}$ with d-dimensional coordinates r^1 and r^2, respectively. Recalling (2.6), the generator, $\hat A_\varepsilon$, of the Markov pair process $\hat R(\cdot):=$ $:=(R^1(\cdot),R^2(\cdot))$ is given by

$$\hat A_\varepsilon:=\frac12\sum_{k,\ell=1}^d\sum_{i,j=1}^2 D_{k\ell,ij}(\hat r)\frac{\partial^2}{\partial r_k^i\partial r_\ell^j}. \tag{3.8}$$

A core for this generator is, e.g., $C_0^2(\mathbf{R}^{2d},\mathbf{R})$, the space of twice continuously differentiable real valued functions on $\mathbf{R}^{2d}$ which vanish at infinity (cf. [25] (loc.cit.), Ch. 8.2, Theorem 2.5). We are interested in local effects of the diffusion coefficient on the distance $|R^2-R^1|$. Let I_d be the identity matrix in $\mathbf{R}^d$ and define the following orthogonal transformation of coordinates:

$$\begin{pmatrix}R^1\\R^2\end{pmatrix}=\frac{1}{\sqrt2}\begin{pmatrix}I_d&-I_d\\I_d&I_d\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}. \tag{3.9}$$

Let B be a Borel set in $\mathbf{R}^d$ and set $\Gamma_B:=\mathcal{A}(\mathbf{R}^d\times B)$, where $\mathcal{A}$ is the unitary matrix in the right-hand side of (3.9). Consider the probability for the two-particle motion to be at time t in Γ_B, having started in $\hat r$:

$$P(t,\hat r,\Gamma_B)=$$

$$=\left(P\left(\frac{1}{\sqrt2}\Big(R^1(t,r^1)+R^2(t,r^2)\Big),\frac{1}{\sqrt2}\left(R^2(t,r^1)-R^1(t,r^2)\right)\right)\in\mathbf{R}^d\times B\right)=$$

$$=P\left(\frac{1}{\sqrt2}(R^2(t,r^1)-R^1(t,r^2))\in B\right). \tag{3.10}$$

Recalling (3.7), we easily see that (3.10) only depends on the difference $a := \frac{1}{\sqrt{2}}(r^2 - -r^1)$ and we may define the "marginal" probability distributions

$$\tilde{P}(t,a,B) := P(t,\hat{r},\Gamma_B)\,|_{\frac{1}{\sqrt{2}}(r^2-r^1)=a}. \tag{3.11}$$

Next, let $f \in C_0^2(\mathbf{R}^{2d},\mathbf{R})$. Set $\hat{f}(u,v) := (f\circ\mathcal{A})(u,v)$. We denote those $\hat{f}$ which do not depend on u by $\tilde{f}$. Then $\tilde{f}$ can be considered as an element of $C_0^2(\mathbf{R}^d,\mathbf{R})$. Define function with values in the nonnegative definite matrices by

$$\bar{D}_{k\ell}(\sqrt{2}a) := [D_{k\ell,11}(0) - D_{k\ell,12}(\sqrt{2}a)], \quad k,\ell = 1,\dots,d, \tag{3.12}$$

and let $(\sigma_{kj}(a))$ be the unique nonnegative definite square root of $(\bar{D}_{k\ell}(\sqrt{2}a))$.

It can be seen, that the "marginal" probability distributions, defined by (3.11) generate a Feller – Markov process on $\mathbf{R^d}$ with generator

$$(A_\varepsilon\tilde{f})(a) := \frac{1}{2}\sum_{k,\ell=1}^{d}\bar{D}_{k\ell}(\sqrt{2}a)\left(\frac{\partial^2}{\partial a_k\partial a_\ell}\tilde{f}\right)(a). \tag{3.13}$$

Let β_j be i.i.d. one-dimensional standard Brownian motions, $j = 1,\dots,d$. Then, this Markov process can be represented as the unique solution to the following stochastic Itô differential equation:

$$db(t) = \sum_j \sigma_{\cdot j}(b)\beta_j(dt), \quad b(0) := a. \tag{3.14}$$

Really, note that our assumptions on G_ε imply that $\bar{D}_{k\ell}(\sqrt{2}a)$ are twice continuously differentiable with bounded partial derivatives. Therefore, (i) follows from Ethier and Kurtz (loc.cit.) [21], Section 8.2, Theorem 2.5). Moreover, the diffusion matrix $(\sigma_{kj}(a))$ is uniformly Lipschitz.

It is tempting to use the definition of the probability flux (cf. [5]) to compute regions where there is a bias towards attraction between the two particles or a bias in favor of repulsion. The problem with that definition is that the attractive and repulsive domains, determined by the flux, depend on time t and the initial distribution of a density. However, if the density is near a point b approximately constant the sign of the flux at b is completely determined by the diffusion coefficients. Set

$$\begin{gathered} -\bar{J}_u(b) := \sum_{k,\ell=1}^{d}\frac{\partial}{\partial b_\ell}[(D_{k\ell,11}(0) - D_{k\ell,12}(\sqrt{2}b))], \\ E_+ := \left\{b\in\mathbf{R}^d := \sum_{\ell=1}^{d} b_\ell \ge 0\right\}, \qquad E_- := \left\{b\in\mathbf{R}^d := \sum_{\ell=1}^{d} b_\ell < 0\right\}. \end{gathered} \tag{3.15}$$

Since we can replace the derivatives by difference quotients, we can determine which sign of $-\bar{J}_u(b)$ implies a bias towards attraction and which one would indicate a bias towards repulsion. Hence, we suggest the following:

Definition 3.1. *A Borel set B is called "attraction-biased" if either $B \subset E_+$ and $-\bar{J}_u(b) > 0 \ \ \forall b \in B$ or $B \subset E_-$ and $-\bar{J}_u(b) < 0 \ \ \forall b \in B$.*

The set B is called a"repulsion-biased" if either $B \subset E_+$ and $-\bar{J}_u(b) < 0 \ \forall b \in B$ or $B \subset E_-$ and $-\bar{J}_u(b) > 0 \ \forall b \in B$.

In the Maxwellian case from (1.7) there is a positive constant c_ε such that

$$-\bar{J}_u(b) = \left[\sum_{k=1}^{d} b_k\right] c_\varepsilon \exp\left(-\frac{|b|^2}{2\varepsilon}\right)\left[d+2-\frac{|b|^2}{\varepsilon}\right]. \tag{3.16}$$

Hence, in the Maxwellian case a Borel set B is attraction-biased if and only if for all $b \in B$ we have

$$|b| < \sqrt{\varepsilon(d+2)}.$$

Further, B is repulsion-biased if and only if

$$|b| > \sqrt{\varepsilon(d+2)}.$$

***Remarks* 3.1.** (i) We expect that for short times the distance between the two particles in attraction-biased regions would decrease and in repulsion-biased regions this distance would increase. In particular, for distances close to 0 the attraction bias would be consistent with the depletion phenomena, which has been observed in fluids (cf. references in the introduction). However, we do not claim that distances in an repulsion-biased region will gradually get greater and perhaps converge to ∞, as $t \longrightarrow \infty$. A similar comment holds for the attraction-biased case. Indeed, the proof of the following result can be found in [25], Ch. 1.3: Assume $d = 1$ and let $b(t,a)$ be the unique solution of (3.14) with $b(0) = a$, where $a \neq 0$. Then,

$$P\left\{\lim_{t\to\infty} b(t,a) = 0\right\} = 1. \tag{3.17}$$

(3.17) means that the distance between both particles will eventually tend to 0. In other words, the whole positive and negative real lines are attractive regions and the repulsion-biased regions will have no long-term effect on the distances of the two particles. Recall that Brownian motion in $\mathbf{R}$ is recurrent. Now if the distance between the two particles is much greater than $\sqrt{3\varepsilon}$, the process $b(\cdot)$ is very similar to a one-dimensional Brownian motion which follows from the asymptotics of $\bar{D}(\sqrt{2}b)$, as $b \gg \sqrt{3\varepsilon}$ in the Maxwellian case. The recurrence, however, lets the distance become less than $\sqrt{3\varepsilon}$. In this region the motion of the distance becomes skewed, favoring a decrease of the distance.

(ii) In dimension $d \geq 3$ Brownian motion is transient and, therefore, attraction-biased and repulsion-biased domains may play a different role in the long-time behavior of the distance between the two particles than in the case $d = 1$. This will be investigated in a future research.

Acknowledgement. The author wants to thank W. I. Skrypnik from the Institute of Mathematics of the Ukrainian Academy of Sciences for informing him about the existence of the papers [6, 7].

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Received 16.11.2004