UDC 517.5

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NORM OF A COMPOSITION OPERATOR FROM THE SPACE OF CAUCHY TRANSFORMS INTO ZYGMUND-TYPE SPACES

НОРМА ОПЕРАТОРА КОМПОЗИЦІЇ З ПРОСТОРУ ПЕРЕТВОРЕНЬ КОШІ У ПРОСТОРИ ТИПУ ЗИГМУНДА

The problem of evaluation of the norm of a composition operator acting between the spaces of holomorphic functions is quite difficult. In the Hardy and weighted Bergman spaces, the norm of the composition operator is unknown even for the choice of a fairly simple symbol. We compute the operator norm of composition operator acting between the space of Cauchy transforms and Zygmund-type spaces. We also characterize bounded and compact composition operators acting between these spaces.

Проблема визначення норми оператора композиції, що діє між просторами голоморфних функцій, є дуже складною. В просторах Гарді та зважених просторах Бергмана норма оператора композиції є невідомою навіть у випадку, коли вибрано досить простий символ. Ми знаходимо операторну норму оператора композиції, що діє між просторами перетворень Коші та просторами типу Зигмунда. Крім того, наведено характеристику обмежених та компактних операторів композиції, що діють між цими просторами.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, H^∞ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, $dA(w)$ the normalized area measure on $\mathbb{D}$ (i.e., $A(\mathbb{D}) = 1$) and $\mathcal{M}$, the space of all complex Borel measures on $\mathbb{T}$. Let

$$\eta_z(w) = \frac{z-w}{1-\bar{z}w}, \qquad z, w \in \mathbb{D},$$

that is, η_z is the involutive automorphism of $\mathbb{D}$ interchanging points z and 0. The space of Cauchy transforms $\mathcal{F}$ is the collection of functions $f \in H(\mathbb{D})$ which admit a representation of the form

$$f(w) = \int\limits_{\mathbb{T}} \frac{d\mu(x)}{1-\overline{x}w}.$$

The space $\mathcal{F}$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{F}} = \inf\left\{\|\mu\| : f(w) = \int\limits_{\mathbb{T}} \frac{d\mu(x)}{1-\overline{x}w}\right\},$$

where $\|\mu\|$ denotes the total variation of the measure μ. It is well known that

$$|f(w)| \le \frac{\|f\|_{\mathcal{F}}}{1-|w|} \tag{1.1}$$

for every $w \in \mathbb{D}$ and $f \in \mathcal{F}$. For more about these spaces see [2–5, 8–12].

A strictly positive continuous function ν on $\mathbb{D}$ is called weight. A weight ν is called radial if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. A weight ν is normal if there exist positive numbers η and τ, $0 < \eta < \tau$, and $\delta \in [0,1)$ such that

$$\frac{\nu(r)}{(1-r)^{\eta}} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^{\eta}} = 0,$$

$$\frac{\nu(r)}{(1-r)^{\tau}} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^{\tau}} = \infty.$$

If we say that a function $\nu : \mathbb{D} \to [0,\infty)$ is a normal weight function, then we also assume that it is radial. It is well known that classical weights $\sigma_\alpha(w) = (1-|w|^2)^\alpha$, $\alpha > -1$ are normal weights. For a normal weight ν, the Zygmund-type class $\mathcal{Z}_\nu = \mathcal{Z}_\nu(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$b_\nu(f) := \sup_{w\in\mathbb{D}} \nu(w)|f''(w)| < \infty$$

with the norm

$$\|f\|_{\mathcal{Z}_\nu} = |f(0)| + |f'(0)| + b_\nu(f)$$

the Zygmund-type class becomes a Banach space, called the Zygmund-type space.

The little Zygmund-type space, denoted by $\mathcal{Z}_{\nu,0} = \mathcal{Z}_{\nu,0}(\mathbb{D})$ is the closed subspace of $\mathcal{Z}_\nu$ consisting of all functions f such that

$$\lim_{|w|\to 1} \nu(w)|f''(w)| = 0.$$

If $\nu(w) = 1-|w|^2$, then we get the Zygmund space and the little Zygmund space. To characterize composition operators, we need an equivalent norm for $\mathcal{Z}_\nu$. The following lemma is possibly a known result, but we have not managed to find it in the literature, so we give a proof.

Lemma 1. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function and $d\lambda(w) = dA(w)/(1-|w|^2)^2$. Then $f \in \mathcal{Z}_\nu$ if and only if*

$$\|f\|^2_{\mathcal{Z}_\nu} \asymp |f(0)|^2 + |f'(0)|^2 + \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f''(w)|^2 \nu^2(w)(1-|\eta_z(w)|^2)^2 d\lambda(w) < \infty,$$

where the notation $A \asymp B$ means that $B \lesssim A \lesssim B$ and $A \lesssim B$ means that there is some positive constant C such that $A \leq CB$.

Proof. Let $D(z,1/2)=\big\{w\in\mathbb{D}\colon |\eta_z(w)|<1/2\big\}$. Then

$$|f''(z)|^2=|f''(\eta_z(0))|^2\le 4\int\limits_{|z|<1/2}|f''(\eta_z(w))|^2dA(w)=$$

$$=4\int\limits_{D(z,1/2)}|f''(w)|^2|\eta_z'(w)|^2dA(w). \tag{1.2}$$

Using the identity

$$(1-|w|^2)|\eta_z'(w)|=1-|\eta_z(w)|^2=\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}$$

and the fact that $\nu(z)\asymp\nu(w)$ for $w\in D(z,1/2)$ in (1.2), we have

$$|f''(z)|^2\lesssim\frac{1}{\nu^2(z)}\int\limits_{D(z,1/2)}|f''(w)|^2\nu^2(w)(1-|\eta_z(w)|^2)^2d\lambda(w).$$

Thus we obtain

$$b_\nu^2(f)\lesssim\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|f''(w)|^2\nu^2(|w|)(1-|\eta_z(w)|^2)^2d\lambda(w).$$

Therefore,

$$\|f\|^2_{\mathcal{Z}_\nu}\lesssim|f(0)|^2+|f'(0)|^2+\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|f''(w)|^2\nu^2(w)(1-|\eta_z(w)|^2)^2d\lambda(w). \tag{1.3}$$

Again

$$\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|f''(w)|^2\nu^2(w)(1-|\eta_z(w)|^2)^2d\lambda(w)\le$$

$$\le b_\nu^2(f)\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}dA(z)\lesssim b_\nu^2(f).$$

Thus,

$$|f(0)|^2+|f'(0)|^2+\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|f''(w)|^2\nu^2(w)(1-|\eta_z(w)|^2)^2d\lambda(w)\lesssim\|f\|^2_{\mathcal{Z}_\nu}. \tag{1.4}$$

Combining (1.3) and (1.4), we get the desired result.

The compactness of a closed subset $L\subset\mathcal{Z}_{\nu,0}$ can be characterized as follows.

Lemma 2. *A closed set L in $\mathcal{Z}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{Z}_\nu}$ and satisfies*

$$\lim_{|w|\to 1} \sup_{f\in L} \nu(w)|f''(w)| = 0.$$

This result for the for little Bloch space with $\nu(w) = (1-|w|^2)$ was proved by Madigan and Matheson [13]. The above lemma can be proved by a slight modification in their proof (see also Lemma 4.4 in [14]).

The composition operator C_φ induced by φ is defined by $C_\varphi f = f\circ\varphi$ for $f \in H(\mathbb{D})$. Recently, there has been some interest in computing the exact norm of a composition operator acting between two holomorphic function spaces. Of course the problem of computing the operator norm of a composition operator is quite difficult. On Hardy and weighted Bergman spaces, the norm of a composition operator even for a choice of fairly simple symbol is unknown, see [1, 7]. In this paper, we compute the operator norm of composition operator acting between Cauchy transforms and Zygmund spaces. We also characterize bounded and compact composition operators acting between these spaces, thereby, continuing the line of research in the papers [3, 15]. For more about composition operators on the space of Cauchy integral transforms, we refer [3, 8, 9, 15].

2. Boundedness and compactness of $C_\varphi\colon \mathcal{F}\to\mathcal{Z}_\nu$. In this section, we give the operator norm of composition operator acting from the space of Cauchy transforms $\mathcal{F}$ to Zygmund spaces $\mathcal{Z}_\nu$.

Theorem 1. *Let ν be a normal weight and φ be a holomorphic self-map of $\mathbb{D}$. Then the following conditions are equivalent:*

(a) *$C_\varphi\colon \mathcal{F}\to\mathcal{Z}_\nu$ is bounded,*

(b) $L := \sup_{x\in\mathbb{T}}\sup_{w\in\mathbb{D}} \nu(w)\left|\dfrac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \dfrac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| < \infty,$

(c) $M := \sup_{x\in\mathbb{T}}\sup_{w\in\mathbb{D}} \displaystyle\int_{\mathbb{D}} \left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2 \nu^2(w)(1-|\eta_z(w)|^2)^2 d\lambda(w) < \infty.$

Moreover, if $C_\varphi\colon \mathcal{F}\to\mathcal{Z}_\nu$ is bounded, then

$$\|C_\varphi\|_{\mathcal{F}\to\mathcal{Z}_\nu} = \sup_{x\in\mathbb{T}}\left|\frac{1}{1-\overline{x}\varphi(0)}\right| + \sup_{x\in\mathbb{T}}\left|\frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2}\right| + L \asymp \tag{2.1}$$

$$\asymp \sup_{x\in\mathbb{T}}\left|\frac{1}{1-\overline{x}\varphi(0)}\right| + \sup_{x\in\mathbb{T}}\left|\frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2}\right| + M^{1/2}. \tag{2.2}$$

Proof. (a) $\Leftrightarrow$ (b). First suppose that (a) holds. Consider the family of functions

$$f_x(w) = \frac{1}{1-\overline{x}w}, \qquad x\in\mathbb{T}. \tag{2.3}$$

Then $\|f_x\|_{\mathcal{F}} = 1$, for each $x\in\mathbb{T}$ (see, e.g., [2, p. 468]). Thus, by the boundedness of $C_\varphi\colon \mathcal{F}\to\mathcal{Z}_\nu$ we have

$$|f_x(\varphi(0))| + |f_x'(\varphi(0))| + \sup_{w\in\mathbb{D}} \nu(w)|(f_x\circ\varphi)''(w)| = \|C_\varphi f_x\|_{\mathcal{Z}_\nu} \le \|C_\varphi\|_{\mathcal{F}\to\mathcal{Z}_\nu}$$

for every $x \in \mathbb{T}$. Therefore,

$$\left|\frac{1}{1-\overline{x}\varphi(0)}\right| + \left|\frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2}\right| +$$

$$+\sup_{w\in\mathbb{D}} \nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| \le \|C_\varphi - C_\psi\|_{\mathcal{F}\to\mathcal{Z}_\nu}.$$

Taking supremum over $x \in \mathbb{T}$, we see that (b) holds and

$$\sup_{x\in\mathbb{T}}\left|\frac{1}{1-\overline{x}\varphi(0)}\right| + \sup_{x\in\mathbb{T}}\left|\frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2}\right| + L \le \|C_\varphi - C_\psi\|_{\mathcal{F}\to\mathcal{Z}_\nu}. \tag{2.4}$$

Conversely, suppose that (b) holds. Let $f \in \mathcal{F}$. Then there is a $\mu \in \mathcal{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}}$ and

$$f(w) = \int_{\mathbb{T}} \frac{d\mu(x)}{1-\overline{x}w}. \tag{2.5}$$

Differentiating (2.5) with respect to w, we get

$$f'(w) = \int_{\mathbb{T}} \frac{\overline{x}}{(1-\overline{x}w)^2} d\mu(x). \tag{2.6}$$

Again differentiating (2.6) with respect to w, we obtain

$$f'(w) = \int_{\mathbb{T}} \frac{2(\overline{x})^2}{(1-\overline{x}w)^3} d\mu(x). \tag{2.7}$$

Replacing w in (2.6) and in (2.7) by $\varphi(w)$, multiplying such obtained inequalities, respectively, by $\varphi''(w)$ and $2(\varphi'(w))^2$ and adding the obtained equations, we have

$$f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2 =$$

$$= \int_{\mathbb{T}} \left(\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right) d\mu(x). \tag{2.8}$$

By using (2.8) and an elementary inequality, we obtain

$$\nu(w)|(f\circ\varphi)''(w)| = \nu(w)|f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2| \le$$

$$\le \nu(w)\int_{\mathbb{T}} \left|\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| d|\mu|(x) \le$$

$$\le \sup_{x\in\mathbb{T}}\sup_{w\in\mathbb{D}} \nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| \int_{\mathbb{T}} d|\mu|(x) \le$$

$$\leq \sup_{x\in\mathbb{T}} \sup_{w\in\mathbb{D}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right| \|f\|_{\mathcal{F}}. \tag{2.9}$$

Thus, it follows that

$$b_\nu(C_\varphi f) \leq \sup_{x\in\mathbb{T}} \sup_{w\in\mathbb{D}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right| \|f\|_{\mathcal{F}}. \tag{2.10}$$

Replacing w in (2.5) by $\varphi(0)$ and using an elementary inequality, we get

$$|(C_\varphi f)(0)| \leq \int_{\mathbb{T}} \left| \frac{1}{1-\overline{x}\varphi(0)} \right| d|\mu|(x) \leq$$

$$\leq \sup_{x\in\mathbb{T}} \left| \frac{1}{1-\overline{x}\varphi(0)} \right| \int_{\mathbb{T}} d|\mu|(x) \leq$$

$$\leq \sup_{x\in\mathbb{T}} \left| \frac{1}{1-\overline{x}\varphi(0)} \right| \|f\|_{\mathcal{F}}. \tag{2.11}$$

Again, replacing w in (2.6) by $\varphi(0)$ and using an elementary inequality, we obtain

$$|(C_\varphi f)'(0)| \leq \int_{\mathbb{T}} \left| \frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2} \right| d|\mu|(x) \leq \sup_{x\in\mathbb{T}} \left| \frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2} \right| \|f\|_{\mathcal{F}}. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we see that

$$\|C_\varphi f\|_{\mathcal{Z}_\nu} \leq \left\{ \sup_{x\in\mathbb{T}} \left| \frac{1}{1-\overline{x}\varphi(0)} \right| + \sup_{x\in\mathbb{T}} \left| \frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2} \right| + L \right\} \|f\|_{\mathcal{F}}.$$

Thus, $C_\varphi : \mathcal{F} \to \mathcal{Z}_\nu$ is bounded and

$$\|C_\varphi\|_{\mathcal{F}\to\mathcal{Z}_\nu} \leq \sup_{x\in\mathbb{T}} \left| \frac{1}{1-\overline{x}\varphi(0)} \right| + \sup_{x\in\mathbb{T}} \left| \frac{\varphi'(0)}{(1-\overline{x}\varphi(0))^2} \right| + L. \tag{2.13}$$

Also from (2.6) and (2.13), (2.2) follows.

(b) $\Leftrightarrow$ (c). Assume that (b) holds. Since ν is normal, $\nu(z) \asymp \nu(w)$ when $w \in D(z, (1 - |z|)/2) = \{|w - z| < (1 - |z|)/2\}$. Also it is known that $|1 - \bar{z}w| \asymp 1 - |z|^2$, for $w \in \Omega(z, (1 - |z|)/2)$. Using these facts and the subharmonicity of the function

$$\left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right|^2,$$

we have

$$M = \sup_{x\in\mathbb{T}} \sup_{z\in\mathbb{D}} \int_{\Omega(z,(1-|z|)/2)} \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right|^2 \nu^2(w) d\lambda_z(w) \geq$$

$$\geq \sup_{x\in\mathbb{T}}\sup_{z\in\mathbb{D}}\nu(z)\left|\frac{\varphi''(z)}{(1-\overline{x}\varphi(z))^2}+\frac{2\overline{x}(\varphi'(z))^2}{(1-\overline{x}\varphi(z))^3}\right|^2\times$$

$$\times\sup_{z\in\mathbb{D}}\int\limits_{\Omega(z,(1-|z|)/2)}\frac{(1-|z|^2)^4}{|1-\bar{z}w|^4}dA(w)\geq$$

$$\geq C\sup_{x\in\mathbb{T}}\sup_{z\in\mathbb{D}}\nu(z)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2=CL^2. \tag{2.14}$$

Next assume that (c) holds. Then

$$M\leq L^2\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}dA(w)=L^2. \tag{2.15}$$

The asymptotic relation $M\asymp L^2$ follows from (2.14) and (2.15). Moreover, the equivalence of (2.1) and (2.2) follows.

***Corollary* 1.** *Let ν be a normal weight, $\mathcal{A}_\nu$ is the space of all $f\in H(\mathbb{D})$ such that $\|f\|_{\mathcal{A}_\nu}:=$ $:=\sup_{w\in\mathbb{D}}\nu(w)|f(w)|<\infty$ and φ be a holomorphic self-map of $\mathbb{D}$. For $x\in\mathbb{T}$, let*

$$f_x(w)=\left\{\varphi''(w)/(1-\overline{x}\varphi(w))^2\right\}+\left\{2\overline{x}(\varphi'(w))^2/(1-\overline{x}\varphi(w))^3\right\}.$$

Then $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ is bounded if and only if the family $\{f_x\colon x\in\mathbb{T}\}$ is norm bounded in $\mathcal{A}_\nu$.

By (1.1) it is easy to see that the unit ball of $\mathcal{F}$ is a normal family of holomorphic functions. A standard normal family argument then yields the proof of the following lemma (see, e.g., Proposition 3.11 of [6]).

Lemma 3. *Let ν be a normal weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ is compact if and only if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$ with $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{F}}\leq N$ which converges to zero on compact subsets of $\mathbb{D}$, we have $\lim_{n\to\infty}\|C_\varphi f_n\|_{\mathcal{Z}_\nu}=0$.*

Theorem 2. *Let ν be a normal weight, φ be a holomorphic self-map of $\mathbb{D}$ and $d\lambda_z(w)=\{(1-$ $-|\eta_z(w)|^2)^2/(1-|w|^2)^2\}dA(w)$ and $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ are bounded. Then $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ is compact if and only if*

$$\lim_{r\to1}\sup_{x\in\mathbb{T}}\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(w)|>r}\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\nu^2(w)d\lambda_z(w)=0. \tag{2.16}$$

Proof. First suppose that $\{f_j\}_{j\in\mathbb{N}}$ is a norm bounded sequence in $\mathcal{F}$ and $\sup_j\|f_j\|_{\mathcal{F}}\leq N$ and f_j converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Then by the Weierstrass theorem, f_j' and f_j'' also converges to 0 uniformly on compact subsets of $\mathbb{D}$ for each $j\in\mathbb{N}$. By Lemma 3, we need to show that $\|C_\varphi f_j\|_{\mathcal{Z}_\nu}\to0$ as $j\to\infty$. For each $j\in\mathbb{N}$, we can find a $\nu_j\in\mathcal{M}$ with $\|\nu_j\|=\|f_j\|_{\mathcal{F}}$ such that

$$f_j(w)=\int\limits_{\mathbb{T}}\frac{d\mu_j(x)}{1-\overline{x}w}.$$

Then proceeding as in (2.8), we have

$$f_j'(\varphi(w))\varphi''(w)+f_j''(\varphi(w))(\varphi'(w))^2=\int\limits_{\mathbb{T}}\left(\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right)d\mu_j(x).$$

Applying Jensen's inequality, as well as the norm boundedness of the sequence $\{f_j\}_{j\in\mathbb{N}}$, we obtain

$$|f_j'(\varphi(w))\varphi''(w)+f_j''(\varphi(w))(\varphi'(w))^2|^2\leq$$

$$\leq\|\mu_j\|^2\int\limits_{\mathbb{T}}\left|\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\frac{d|\mu_j|(x)}{\|\mu_j\|}\leq$$

$$\leq N\int\limits_{\mathbb{T}}\left|\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2d|\mu_j|(x). \tag{2.17}$$

By (2.16) we have that, for every $\varepsilon>0$, there is an $r_1\in(0,1)$ such that, for $r\in(r_1,1)$,

$$\sup_{x\in\mathbb{T}}\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|>r}\left|\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\nu^2(w)d\lambda_z(w)<\varepsilon. \tag{2.18}$$

Now

$$\|C_\varphi f_j\|^2_{\mathcal{Z}_\nu}\lesssim|f_j(\varphi(0))|^2+|f_j'(\psi(0))|^2+$$

$$+\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|\leq r}|f_j'(\varphi(w))\varphi''(w)+f_j''(\varphi(w))(\varphi'(w))^2|^2\nu^2(w)d\lambda_z(w)+$$

$$+\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|>r}|f_j'(\varphi(w))\varphi''(w)+f_j''(\varphi(w))(\varphi'(w))^2|^2\nu^2(w)d\lambda_z(w). \tag{2.19}$$

By taking $f(z)=z$ in $\mathcal{F}$ and using the fact that $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ is bounded, we have

$$\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi''(w)|^2\nu^2(w)d\lambda_z(w)<\infty. \tag{2.20}$$

Again, by taking $f(z)=z^2/2$ in $\mathcal{F}$ and using the fact that $C_\varphi\colon\mathcal{F}\to\mathcal{Z}_\nu$ is bounded, we obtain

$$\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi(w)\varphi''(w)+(\varphi'(w))^2|^2\nu^2(w)d\lambda_z(w)<\infty. \tag{2.21}$$

Thus using an elementary inequality, (2.20), (2.21) and the fact that $|\varphi(w)|<1$, we get

$$\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi'(w)|^4\nu^2(w)d\lambda_z(w)\leq\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi(w)\varphi''(w)+(\varphi'(w))^2|^2\nu^2(w)d\lambda_z(w)+$$

$$+\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi''(w)|^2\nu^2(w)d\lambda_z(w)<\infty. \tag{2.22}$$

Using (2.17), (2.18), (2.19), Fubini's theorem and the fact that $|f_j(\varphi(0))|^2<\varepsilon,\ |f_j'(\psi(0))|^2<\varepsilon,$ $\sup_{|w|\le r}|f_j''(w)|^2<\varepsilon$ and $\sup_{|w|\le r}|f_j''(w)|^2<\varepsilon,$ for sufficiently large j, say $j\ge j_0$, we have

$$\|C_\varphi f_j\|^2_{\mathcal{Z}_\nu}\lesssim|f_j(\varphi(0))|^2+f_j'(\psi(0))|^2+\sup_{|\varphi(z)|\le r}|f_j'(\varphi(w))|^2\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|\varphi''(w)|^2\nu^2(w)d\lambda_z(w)+$$

$$+\sup_{|\varphi(z)|\le r}|f_j''(\varphi(w))|^2\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|\le r}|\varphi'(w)|^4\nu^2(w)d\lambda_z(w)+$$

$$+\int\limits_{\mathbb{T}}\int\limits_{|\varphi(z)|>r}\left|\frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2d\lambda_z(w)d|\mu_j|(x)<$$

$$<C\left(1+\int\limits_{\mathbb{T}}d|\mu_j|(x)\right)\varepsilon<C\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, so by Lemma 3, it follows that $C_\varphi:\mathcal{F}\to\mathcal{Z}_\nu$ is compact.

Conversely, suppose that $C_\varphi:\mathcal{F}\to\mathcal{Z}_\nu$ is compact. For each $\epsilon>0$, using (2.20) and (2.22), we can choose $r\in(0,1)$ such that

$$\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|>r}|\varphi''(w)|^2\nu^2(w)d\lambda_z(w)<\varepsilon \tag{2.23}$$

and

$$\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(z)|>r}|\varphi'(w)|^4\nu^2(w)d\lambda_z(w)<\varepsilon. \tag{2.24}$$

Let $f\in B_{\mathcal{F}}$ and $f_t(w)=f(tw)$, $0<t<1$. Then for each $t\in(0,1)$, $f_t\in\mathcal{F}$ and $\sup_{0<t<1}\|f_t\|_{\mathcal{F}}\le$ $\le\|f\|_{\mathcal{F}}$. Moreover, $f_t\to f$ uniformly on compact subsets of $\mathbb{D}$ as $t\to1$. By the compactness of $C_\varphi:\mathcal{F}\to\mathcal{Z}_\nu$, we obtain

$$\lim_{t\to1}\|C_\varphi f_t-C_\varphi f\|_{\mathcal{Z}_\nu}=0.$$

Hence, for every $\varepsilon>0$, there is a $t\in(0,1)$ such that

$$\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}|(f_t'(\varphi(w))\varphi''(w)+f_t''(\varphi(w))(\varphi'(w))^2)-$$

$$-(f'(\varphi(w))\varphi''(w)+f''(\varphi(w))(\varphi'(w))^2)|^2\nu^2(w)d\lambda_z(w)<\varepsilon. \tag{2.25}$$

From the inequalities (2.23), (2.24), and (2.25), we have

$$\sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} |f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2|^2 \nu^2(w) d\lambda_z(w) \leq$$

$$\leq \sup_{z\in\mathbb{D}} \int\limits_{\mathbb{D}} |(f_t'(\varphi(w))\varphi''(w) + f_t''(\varphi(w))(\varphi'(w))^2) -$$

$$-(f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2)|^2 \nu^2(w) d\lambda_z(w) +$$

$$+\sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} |f_t'(\varphi(w))\varphi''(w) + f_t''(\varphi(w))(\varphi'(w))^2|^2 \nu^2(w) d\lambda_z(w) \leq$$

$$\leq \varepsilon C(1 + \|f_t'\|_\infty^2) \quad \text{for some constant} \quad C > 0.$$

Hence, for every $f \in B_{\mathcal{F}}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f, \varepsilon)$, such that for $r \in (\delta_0, 1)$

$$\sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} |f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2|^2 \nu^2(w) d\lambda_z(w) < \varepsilon. \tag{2.26}$$

From the compactness of $C_\varphi \colon \mathcal{F} \to \mathcal{Z}_\nu$, we have that for every $\varepsilon > 0$ there is a finite collection of functions $f_1, f_2, \ldots, f_m \in B_{\mathcal{F}}$ such that for each $f \in B_{\mathcal{F}}$, there is a $k \in \{1, 2, \ldots, m\}$ such that

$$\sup_{z\in\mathbb{D}} \int\limits_{\mathbb{D}} |(f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2) -$$

$$-(f_k'(\varphi(w))\varphi''(w) + f_k''(\varphi(w))(\varphi'(w))^2)|^2 \nu^2(w) d\lambda_z(w) < \varepsilon. \tag{2.27}$$

On the other hand, from (2.26) it follows that if $\delta := \max_{1\leq j\leq m} \delta_j(f_k, \varepsilon)$, then, for $r \in (\delta, 1)$ and all $k \in \{1, 2, \ldots, m\}$, we have

$$\sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} |f_k'(\varphi(w))\varphi''(w) + f_k''(\varphi(w))(\varphi'(w))^2|^2 \nu^2(w) d\lambda_z(w) < \varepsilon. \tag{2.28}$$

From (2.27) and (2.28), we have that for $r \in (\delta, 1)$ and every $f \in B_{\mathcal{F}}$, there is a constant $C > 0$ such that

$$\sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} |f'(\varphi(w))\varphi''(w) + f''(\varphi(w))(\varphi'(w))^2|^2 \nu^2(w) d\lambda_z(w) < \varepsilon C. \tag{2.29}$$

Applying (2.29) to the family of functions $f_x(w) = 1/(1 - \overline{x}w)$, $x \in \mathbb{T}$, we obtain

$$\sup_{x\in\mathbb{T}} \sup_{z\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} \left| \frac{\varphi''(w)}{(1 - \overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1 - \overline{x}\varphi(w))^3} \right|^2 \nu^2(w) d\lambda_z(w) < \varepsilon C.$$

Since $\varepsilon > 0$ is arbitrary, (2.16) follows.

***Corollary* 2.** *Let ν be a normal weight, φ be a holomorphic self-map of $\mathbb{D}$ and $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ are bounded. If*

$$\lim_{r\to 1}\sup_{x\in\mathbb{T}}\sup_{|\varphi(w)|>r}\nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| = 0, \tag{2.30}$$

then $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ is compact.

Proof. Suppose that (2.30) holds. Then

$$\sup_{x\in\mathbb{T}}\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(w)|>r}\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\nu^2(w)d\lambda_z(w) \le$$

$$\le \sup_{x\in\mathbb{T}}\sup_{|\varphi(w)|>r}\nu^2(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\sup_{z\in\mathbb{D}}\int\limits_{|\varphi(w)|>r}d\lambda_z(w) \le$$

$$\le \sup_{x\in\mathbb{T}}\sup_{|\varphi(w)|>r}\nu^2(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|^2\sup_{z\in\mathbb{D}}\int\limits_{\mathbb{D}}\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}dA(w) \le$$

$$\le \left(\sup_{x\in\mathbb{T}}\sup_{|\varphi(w)|>r}\nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right|\right)^2.$$

Taking limit as $r \to 1$, by Theorem 2, we see that $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ is compact.

3. Boundedness and compactness of $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$. In this section, we characterize the boundedness and compactness of $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$.

Theorem 3. *Let ν be a normal weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ is bounded if and only if $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ is bounded and*

$$\lim_{|w|\to 1}\nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| = 0 \tag{3.1}$$

for every $x \in \mathbb{T}$.

Proof. First suppose that $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ is bounded. Then it is obvious that $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ is bounded. Once again consider the family of test functions in (2.3). Then $\|f_x\|_{\mathcal{F}} = 1$ and

$$f_x^k(\varphi(w))\varphi''(w) + f_x''(\varphi(w))(\varphi'(w))^2 = \frac{\overline{x}\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2(\overline{x})^2(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}.$$

Thus, by the boundedness of $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$, we have $C_\varphi f_x \in \mathcal{Z}_{\nu,0}$ for every $x \in \mathbb{T}$ and so

$$\lim_{|z|\to 1}\nu(w)\left|\frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2}+\frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3}\right| = 0$$

for every $x \in \mathbb{T}$.

Conversely, suppose that $C_\varphi\colon \mathcal{F} \to \mathcal{Z}_\nu$ is bounded and (3.1) hold. By (3.1), the integrand in (2.9) tends to zero for every $x \in \mathbb{T}$, as $|z| \to 1$, and is dominated by M, where M is as in (b) of Theorem 1. Thus by the Lebesgue convergence theorem, the integral in (2.9) tends to zero as $|w| \to 1$, implying

$$\lim_{|w|\to 1} \nu(w)|(C_\varphi f)''(z)| = 0.$$

Hence, for every $f \in \mathcal{F}$, we have that $C_\varphi f \in \mathcal{Z}_{\nu,0}$, from which the boundedness of $C_\varphi \colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ follows.

Theorem 4. *Let ν be a normal weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi \colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ is compact if and only if*

$$\lim_{|w|\to 1} \sup_{x\in\mathbb{T}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right| = 0. \tag{3.2}$$

Proof. By Lemma 2, a closed set E in $\mathcal{Z}_{\nu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|w|\to 1} \sup_{f\in E} \nu(w)|f''(w)| = 0.$$

Thus, the set $\{C_\varphi f : f \in \mathcal{F}, \|f\|_{\mathcal{F}} \le 1\}$ has compact closure in $\mathcal{Z}_{\nu,0}$ if and only if

$$\lim_{|w|\to 1} \sup \left\{ \nu(w)|(C_\varphi f)''(w)| : f \in \mathcal{F}, \|f\|_{\mathcal{F}} \le 1 \right\} = 0. \tag{3.3}$$

Let $f \in B_{\mathcal{F}}$, then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}}$ and

$$f(z) = \int\limits_{\mathbb{T}} \frac{d\mu(x)}{1-\overline{x}z}.$$

Thus, we easily get that for each $f \in B_{\mathcal{F}}$

$$\nu(w)|(C_\varphi f)''(z)| \le \int\limits_{\mathbb{T}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right| d|\mu|(x) \le$$

$$\le \|\mu\| \sup_{x\in\mathbb{T}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right| \le$$

$$\le \sup_{x\in\mathbb{T}} \nu(w) \left| \frac{\varphi''(w)}{(1-\overline{x}\varphi(w))^2} + \frac{2\overline{x}(\varphi'(w))^2}{(1-\overline{x}\varphi(w))^3} \right|. \tag{3.4}$$

Using (3.2) in (3.4), we get (3.3). Hence $C_\varphi \colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ is compact. Conversely, suppose that $C_\varphi \colon \mathcal{F} \to \mathcal{Z}_{\nu,0}$ is compact. Taking the test functions in (2.3), we can easily obtain that (3.2) follows from (3.3).

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Received 28.11.16