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TOPOLOGICAL SPACES
WITH THE SKOROKHOD REPRESENTATION PROPERTY[*]

ТОПОЛОГІЧНІ ПРОСТОРИ
ІЗ ВЛАСТИВІСТЮ ЗОБРАЖЕННЯ СКОРОХОДА

We give a survey of recent results that generalize and develop a classical theorem of Skorokhod on representation of weakly convergent sequences of probability measures by almost everywhere convergent sequences of mappings.

Наведено огляд отриманих останнім часом результатів, що узагальнюють класичну теорему Скорохода про зображення слабко збіжної послідовності ймовірнісних мір майже напевно збіжними послідовностями відображень.

1. Introduction. A classical result of A. V. Skorokhod [1, 2], obtained half a century ago and now-a-days included in textbooks on probability and measure theory, states that for every sequence of Borel probability measures μ_n on a complete separable metric space X that converges weakly to a Borel probability measure μ_0, one can find Borel mappings $\xi_n \colon [0,1] \to X$, $n = 0, 1, \ldots$, such that the image of Lebesgue measure λ under ξ_n is μ_n for every $n \geq 0$ and $\lim_{n \to \infty} \xi_n(t) = \xi_0(t)$ for almost all $t \in [0,1]$. Numerous extensions of this useful result have been found since then, see, e.g., Blackwell and Dubbins [3], Dudley [4], Fernique [5], Jakubowski [6], Letta and Pratelli [7], Schief [8], Szczotka [9], Wichura [10], and a series of our papers [11 – 15]; a section on this topic is presented in volume 2 of the book [16]; for related problems and applications see also Choban [17], Cuesta-Albertos and Matrán-Bea [18], Tuero [19]. The purpose of this work is to survey recent achievements in this direction.

One of the most important extensions of Skorokhod's theorem was discovered independently by Blackwell and Dubbins [3] and Fernique [5], who showed that all Borel probability measures on a Polish space X can be parameterized simultaneously by mappings from $[0,1]$ with the preservation of the above correspondence. More precisely, to every Borel probability measure μ on X one can associate a Borel mapping $\xi_\mu \colon [0,1] \to X$ such that the image of Lebesgue measure under ξ_μ equals μ and whenever measures μ_n converge weakly to μ, one has $\lim_{n \to \infty} \xi_{\mu_n}(t) = \xi_\mu(t)$ for almost all $t \in [0,1]$. It has been recently shown in [11] that this result can be derived from its simple one dimensional case and general topological and functional-analytic results. In addition, it has been shown in [11] that there are interesting links between the Skorokhod parameterization of probability measures on topological spaces and topological properties of those spaces. A study of nonmetrizable topological spaces to which the Skorokhod and Blackwell – Dubbins – Fernique theorems can be extended was initiated in the works [6] and [11, 12], respectively. Most of the results obtained fall into the following two groups: given a set $\mathcal{M}$ of Borel probability measures on a topological space X, on parameterizes the measures in $\mathcal{M}$ by Borel mappings from $[0,1]$ to X in the above sense such

that this correspondence is continuous either (a) for all convergent sequences in $\mathcal{M}$ or (b) only for certain subsequences. Certainly, by varying the initial set $\mathcal{M}$ one obtains a lot of particular cases. For example, one can take for $\mathcal{M}$ a convergent sequence, as Skorokhod did, or the space of all measures, as in the Blackwell–Dubbins–Fernique theorem, or a uniformly tight sequence, as in Jakubowski's work, or some compact set.

The principal concept in this area is a space with the strong Skorokhod property for Radon measures defined as a space on which all Radon probability measures admit a simultaneous parameterization $\mu \mapsto \xi_\mu$ by Borel mappings from $[0,1]$ endowed with Lebesgue measure such that one obtains the aforementioned correspondence between weak convergence of measures and almost everywhere convergence of mappings. There are very interesting links between the Skorokhod representation property (and its versions) and other probabilistic properties of topological nature such as the Prokhorov property, open mappings and others. These matters are discussed in Sections 2–4. A very important for applications version of Skorokhod's result is due to Jakubowski. Section 5 is devoted to developments of his idea. Jakubowski's theorem applies to a large class of spaces, but does not provide a simultaneous parameterization of all Radon probability measures. Section 6 contains a list of open problems. A common feature of many positive results on Skorokhod parameterization (strong or weak) for the most diverse topological spaces is that such spaces are obtained by applying certain topological operations on some "building blocks" that are constructed from metric spaces by means of special surjections or injections.

2. Notation and terminology. We assume throughout that X is a Tykhonoff (i.e., completely regular) topological space. Let $C_b(X)$ be the space of all bounded continuous functions on X and let $\mathcal{B}(X)$ be the Borel σ-field of X. The symbol $\mathcal{P}(X)$ denotes the space of all Borel probability measures on X. Let $\mathcal{P}_0(X)$ and $\mathcal{P}_r(X)$ denote, respectively, the spaces of all Baire and Radon (i.e., inner compact regular) probability measures on X. A probability measure μ on a space X is called *discrete* if $\mu(X \setminus C) = 0$ for some countable subset $C \subset X$. Dirac's measure at x is denoted by δ_x.

The weak topology on $\mathcal{P}(X)$, $\mathcal{P}_r(X)$ or $\mathcal{P}_0(X)$ is the restriction of the weak topology on the linear space of all bounded Borel (or Baire) measures that is generated by the seminorms

$$p_f(\mu) = \left| \int\limits_X f(x)\,\mu(dx) \right|, \quad f \in C_b(X).$$

Thus, a sequence of measures μ_n converges weakly to a measure μ precisely when

$$\lim_{n\to\infty} \int\limits_X f(x)\,\mu_n(dx) = \int\limits_X f(x)\,\mu(dx) \quad \forall f \in C_b(X).$$

It is well known that the weak topology is generated by the base of sets

$$W(\mu, U, a) := \{\nu \in \mathcal{P}_r(X) \colon \nu(U) > \mu(U) - a\},$$

where U is open in X and $a > 0$. Weak convergence is denoted by $\mu_n \Rightarrow \mu$. Recall that the weak topology is Hausdorff on $\mathcal{P}_0(X)$ and $\mathcal{P}_r(X)$ and that $\mathcal{P}(X) = \mathcal{P}_0(X) = \mathcal{P}_r(X)$ for any completely regular Souslin space X. See [16] for additional information about weak convergence of probability measures.

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $f \colon X \to Y$ is a measurable mapping, then the image of a measure μ on X under the mapping f is denoted by $\mu \circ f^{-1}$ and defined by the formula

$$\mu \circ f^{-1}(B) = \mu\big(f^{-1}(B)\big), \quad B \in \mathcal{B}.$$

We recall that a family $\mathcal{M}$ of nonnegative Borel measures on a topological space X is called uniformly tight if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset X$ such that $\mu(X \backslash K_\varepsilon) < \varepsilon$ for all $\mu \in \mathcal{M}$.

We shall call a topological space X *sequentially Prokhorov* if every sequence of Radon probability measures on X that converges weakly to a Radon measure is uniformly tight.

Definition 2.1. (i) *We shall say that a family $\mathcal{M}$ of Borel probability measures on a topological space X has the strong Skorokhod property if, to every measure $\mu \in \mathcal{M}$, one can associate a Borel mapping $\xi_\mu \colon [0,1] \to X$ with $\lambda \circ \xi_\mu^{-1} = \mu$, where λ is Lebesgue measure, such that if a sequence of measures $\mu_n \in \mathcal{M}$ converges weakly to a measure $\mu \in \mathcal{M}$, then*

$$\lim_{n \to \infty} \xi_{\mu_n}(t) = \xi_\mu(t) \quad \textit{for almost all} \quad t \in [0,1]. \tag{2.1}$$

If (2.1) holds under the additional assumption that $\{\mu_n\}$ is uniformly tight, then $\mathcal{M}$ is said to have the uniformly tight strong Skorokhod property.

(ii) *We shall say that a topological space X has the strong Skorokhod property for Radon measures if the family $\mathcal{P}_r(X)$ of all Radon measures has that property. If the family of all discrete probability measures on X has the strong Skorokhod property, then X is said to have that property for discrete measures.*

We shall abbreviate the strong Skorokhod property as SSP. The strong Skorokhod property for Radon measures will be abbreviated as SSP_r. Obviously, the SSP and SSP_r coincide for any Radon space, i.e., a space on which every Borel measure is Radon (we recall that every Souslin space is Radon).

The *uniformly tight Skorokhod property* for X is defined analogously. In a similar manner we define also the strong and uniformly tight strong Skorokhod properties for probability measures with finite supports and two-point supports.

It may be useful to consider a more general property when in Definition 2.1 one can take any probability space $(\Omega, \mathcal{F}, P)$ in place of the unit interval $[0,1]$ with Lebesgue measure. Such a property can be called the extended SSP. However, in this survey we consider the SSP with Lebesgue measure.

We shall use the terms *Skorokhod parameterization* and *Skorokhod representation* for mappings $\mu \mapsto \xi_\mu$ of the type described in this definition.

With this definition, the two principle results mentioned above can be restated as follows.

Theorem 2.1. (i) (Skorokhod). *Any weakly convergent sequence of Borel probability measures on a Polish space has the strong Skorokhod property.*

(ii) (Blackwell – Dubbins – Fernique). *Every Polish space has the strong Skorokhod property.*

It is clear that a sequentially Prokhorov space has the strong Skorokhod property for Radon measures if and only if it has the uniformly tight strong Skorokhod property for Radon measures.

An advantage of dealing with Radon measures is that the strong Skorokhod property for them is inherited by arbitrary subspaces (a simple verification is found in [11]).

Lemma 2.1. *Let X be a space with the strong Skorokhod property for Radon measures. Then every subset Y of X has this property as well. If Y is universally measurable, then the analogous assertion is valid for the strong Skorokhod property for Borel measures on the spaces X and Y.*

The following trivial lemma from [11] enables one to reduce the case of a general universally measurable set in a Polish space to the that of unit interval $[0, 1]$. The question whether such a reduction is possible has been posed by A. N. Shiryaev [20].

Given a continuous mapping $f\colon X \to Y$ between completely regular spaces, we say that the corresponding mapping

$$\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y), \quad \mu \mapsto \mu \circ f^{-1},$$

has a continuous right inverse in the weak topology if there exists a mapping $\Psi\colon \mathcal{P}_r(Y) \to \mathcal{P}_r(X)$ continuous in the weak topology such that $\widehat{f}(\Psi(\nu)) = \nu$ for all $\nu \in \mathcal{P}_r(Y)$.

The idea is this: suppose a space Y is obtained as the image of a subspace $X \subset [0, 1]$ under a continuous mapping f. This mapping induces the mapping $\widehat{f}$ between $\mathcal{P}_r(X)$ and $\mathcal{P}_r(Y)$. If $\widehat{f}$ admits a continuous right inverse Ψ, then one can use a parameterization $\mu \mapsto \xi_\mu$ of $\mathcal{P}_r(X)$ to obtain a parameterization of $\mathcal{P}_r(Y)$ in the form $\nu \mapsto \xi_{\Psi(\nu)}$.

Lemma 2.2. *Let f be a continuous mapping from a topological space X with the strong Skorokhod property for Radon measures onto a topological space Y such that the corresponding mapping $\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$ is surjective and has a continuous right inverse in the weak topology. Then Y has the strong Skorokhod property for Radon measures.*

In order to employ this lemma, we need, of course, to know when a continuous right inverse exists. This question has been investigated in [11]. Here we need only the following lemma, which is deduced in [11] (see also [16], § 8.5) from a well-known functional-analytic result due to Milyutin. For the reader's convenience we recall this result. Let $f\colon X \to Y$ be a continuous surjection of compact spaces X and Y. A linear operator $U\colon C(X) \to C(Y)$ is called a regular operator of averaging for f if $U\psi \geq 0$ whenever $\psi \geq 0$ and $U(\varphi \circ f) = \varphi$ for all $\varphi \in C(Y)$. Such an operator is automatically continuous and has unit norm. It is readily seen that the operator $V = U^*\colon \mathcal{M}_r(Y) = C(Y)^* \to \mathcal{M}_r(X) = C(X)^*$ maps $\mathcal{P}_r(Y)$ to $\mathcal{P}_r(X)$ and $\widehat{f} \circ V$ is the identity mapping on $\mathcal{M}(Y)$, i.e., V is a continuous right inverse for $\widehat{F}$. Indeed, for all $\nu \in \mathcal{M}_r(Y)$ we have

$$\int\limits_Y \varphi(y) \left[\widehat{f} \circ V(\nu)\right](dy) = \int\limits_X \varphi\big(f(x)\big) V(\nu)(dx) =$$

$$= \int\limits_Y U(\varphi \circ f)(y)\, \nu(dy) = \int\limits_Y \varphi(y)\, \nu(dy) \quad \forall\, \varphi \in C(Y).$$

A compact space S is called a Miljutin space if, for some cardinality τ, there exists a continuous surjection $f\colon \{0,1\}^\tau \to S$, where $\{0,1\}$ is the two point space, having a regular operator of averaging. The Miljutin lemma (see [21], Theorem 5.6) states that a closed interval is a Miljutin space. It follows that every metrizable compact space S is a Miljutin space, and the set $\mathbb{N}$ can be taken for τ in the above definition. Since the space $\{0,1\}^\mathbb{N}$ is homeomorphic to the classical Cantor set $C \subset [0, 1]$ that consists of all

points in the closed interval $[0, 1]$, whose triadic decomposition does not contain 1, we conclude that for every nonempty metrizable compact space S, there exists a continuous surjection $f\colon C \to S$ that has a regular operator of averaging. Therefore, the following assertion is true.

Lemma 2.3. *Let S be a nonempty metrizable compact space. Then there exists a continuous surjection $f\colon C \to S$ such that the mapping $\widehat{f}$ has a linear continuous right inverse.*

It is readily verified by using the distribution functions that $[0, 1]$ has the SSP (see [16], § 8.5). Therefore, we arrive at the following assertion.

Theorem 2.2. *Let X be a set in a Polish space. Then X has the* SSP *for Radon measures.*

Let us mention a number of assertions related to Lemma 2.3, although we do not employ them below. See [11] and [22] for proofs and related comments and references.

Theorem 2.3. (i) *Let X and Y be completely regular spaces and let $f\colon X \to Y$ be an open surjective mapping that satisfies the following local conservativity condition: for every open set $V \subset X$ and every Radon probability measure ν on $f(V)$, there exists a Radon probability measure μ on V with $\mu \circ f^{-1} = \nu$. Then*

$$\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$$

is an open surjection.

(ii) *Let X and Y be completely regular Souslin spaces and let $f\colon X \to Y$ be an open surjective mapping. Then the mapping $\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$ is an open surjection in the weak topology.*

(iii) *Let $f\colon X \to Y$ be an open surjective mapping of complete metric spaces. Then the mapping $\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$ is an open surjection and has a continuous right inverse in the weak topology.*

(iv) *For every set Y in a Polish space, there exist a subset X of the space $\mathcal{R}$ of the irrational numbers in $[0, 1]$ and a continuous surjective mapping $f\colon X \mapsto Y$ such that the mapping $\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$ has a continuous right inverse in the weak topology. If Y is universally measurable, then X is universally measurable as well.*

(v) *For every set Y in a Polish space, there exist a subset X of the Cantor set C and a continuous surjective mapping $f\colon X \mapsto Y$ such that the mapping*

$$\widehat{f}\colon \mathcal{P}_r(X) \to \mathcal{P}_r(Y)$$

has a continuous right inverse in the weak topology. In the case of compact Y, the set X can be chosen compact. If Y is universally measurable, then X is universally measurable as well.

***Remark* 2.1.** (i) It is to be noted that in Blackwell, Dubins [3], there is a very short sketch of the proof of their theorem (moreover, as noted in [23], there is a flaw in this sketch), but a detailed and correct proof on this way with the verification of all details is not that short as one can see in Fernique [5] and Lebedev [24] (Ch. 5). Unlike the topological approach explained above, the method of proof in the works cited develops the original method of Skorokhod.

There is an interesting approach to parameterization of measures by mappings connected with the Monge – Kantorovich problem and other extremal problems for measures

with given marginals. Its main idea is to obtain mappings parameterizing measures as solutions of certain variational problems; for example, if we consider Borel probability measures on the cube $V = [0,1]^d$ in $\mathbb{R}^d$ and denote by λ Lebesgue measure on V, then, for any Borel probability measure μ on V, there is a Borel mapping ξ_μ at which the expression

$$\int\limits_V |\xi(x) - x|^2 \, \lambda(dx)$$

attains its minimum over the class of Borel mappings ξ with $\lambda \circ \xi^{-1} = \mu$. This approach is discussed by Cuesta – Albertos and Matrán-Bea [18], Krylov [25], Tuero [19]. In particular, Krylov [25] considered measures on $\mathbb{R}^n$ and obtained a parameterization with certain differentiability properties. However, there are other natural generalizations of increasing functions that parameterize Borel probability measures, but cannot be taken as Skorokhod parameterizations. For example, the so called increasing triangular mappings (see [26]), i.e., Borel mappings T on V such that $T = (T_1, \ldots, T_d)$, where every component T_k depends only on the variables $x_1, \ldots, x_k$ and the functions $x_k \mapsto T_k(x_1, \ldots, x_k)$ is increasing, have the property that every Borel probability measure μ on V is the image of λ under a unique (up to a modification) increasing triangular Borel mapping T_μ. However, these mappings lack the desired continuity property.

(ii) Theorem 2.3 shows that it would be also enough to prove that the Hilbert cube $Q = [0,1]^\infty$ has the strong Skorokhod property, since by the Uryson theorem, every separable metric space is homeomorphic to a subset of Q. Hence it suffices to have at least one compact metric space Z with the strong Skorokhod property such that Z can be mapped onto Q by the aid of an open mapping. Since one can deduce from [25] that the space $\mathbb{R}^d$ has the strong Skorokhod property, it remains to find a compact set in $\mathbb{R}^d$ that can be mapped onto Q by an open mapping. Such compacta exist indeed [27]; moreover, one can find a compact set $Z \subset \mathbb{R}^3$ with the desired property (see [28] (Ch. 4, § 4) or [27]).

(iii) Although the proof of Theorem 2.2 in the case $X = [0,1]$ is straightforward, the statement is yet surprising in view of the fact that the space $\mathbb{B}$ of all Borel functions from $[0,1]$ to $[0,1]$ has neither metric nor topology in which the convergent sequences are exactly those that converge almost everywhere (the same is true even for the subspace of continuous functions, see [16], Exercise 2.12.62). Theorem 2.2 says that the factor-space of $\mathbb{B}$ by the equivalence relation determined by the equality of the images of Lebesgue measure under the corresponding functions is metrizable (by a metric that metrizes the weak topology). If (X, d) is a complete separable metric space, then the subset Λ_X in the space $\mathbb{B}([0,1], X)$ of Borel mappings from $[0,1]$ to X formed by the parameterezing mappings ξ_μ, $\mu \in \mathcal{P}_r(X)$, can be equipped with the metric of convergence in measure, i.e.,

$$d_0(f,g) = \int\limits_0^1 \frac{d(f(t), g(t))}{d(f(t), g(t)) + 1} \, dt.$$

Then a sequence in Λ_X converges in this metric if and only if it converges almost everywhere (in order to see this, it suffices to observe that convergence in measure implies weak convergence of induced measures). Such a metric can be also used for metrization of the weak topology on $\mathcal{P}_r(X)$.

The case of a general metric space follows by the fact that for any complete metric space Y, there is a cardinality τ such that Y is the image of the space $B(\tau)$ that is the countable power of a discrete space of cardinality τ under an open mapping. It is verified in [11] that $B(\tau)$ has the SSP_r. This yields the following assertion.

Theorem 2.4. *Every metric space has the* SSP_r.

A powerful tool in the study of open mappings is the following classical result, called Michael's selection theorem (see [29] or [30, p. 190]). Let M be a metrizable space, let P be a complete metrizable closed subset of locally convex space L, and let $\Phi\colon M \to 2^P$ be a lower semicontinuous mapping with values in the set of nonempty convex closed subsets of P. Then, there exists a continuous mapping $f\colon M \to P$ such that $f(x) \in \Phi(x)$ for all x. For our purposes, it is enough to deal with the case where L is a normed space; a short proof for this case can be found in [30] (A§ 1) (note that Filippov [31] constructed an example showing that one cannot omit the requirement that P is closed even if P is a G_δ-set in a Hilbert space). Namely, we deal with the situation where $M = \mathcal{P}_r(Y)$ and $P = \mathcal{P}_r(X)$ for some Polish spaces X and Y; the weak topology on these sets is generated by the Kantorovich–Rubinstein norm on $\mathcal{M}_r(X)$ and $\mathcal{M}_r(Y)$. A typical application of this theorem is this: let $T\colon P \to M$ be a continuous open affine mapping of a complete metrizable convex closed set P in a locally convex space L to a metrizable set M in a locally convex space. Then $\Phi(x) = T^{-1}(x)$ satisfies the hypotheses of Michael's theorem, therefore, T has a continuous right inverse. In our situation, an open mapping from X to Y generates an open affine mapping between $P := \mathcal{P}_r(X)$ and $M := \mathcal{P}_r(Y)$. This demonstrates yet another interesting link between Skorokhod's representation and general topology.

3. Nonmetrizable spaces with the strong Skorokhod property. Now we discuss what happens if we leave the area of metric spaces. As shown in [11], the space $\mathbb{R}_0^\infty$ of all finite real sequences with its natural topology of the inductive limit of the spaces $\mathbb{R}^n$ fails to have even the uniformly tight strong Skorokhod property; moreover, one can find a weakly convergent uniformly tight sequence of probability measures on $\mathbb{R}_0^\infty$ that does not admit a Skorokhod parameterization by mappings. It is worth noting that $\mathbb{R}_0^\infty$ has the weak Skorokhod property considered in Section 5. However, we shall see that the class of spaces with the SSP is wider than the class of metric spaces and includes all almost metrizable sequentially Prokhorov spaces. In particular, we construct a class of nonmetrizable topological spaces with the strong Skorokhod property for Radon measures. Our simplest example is a countable set which is the set of natural numbers with an extra point from its Stone–Čech compactification. This example will be justified by more general facts.

First we note that the uniformly tight strong Skorokhod property for Radon measures is preserved by bijective continuous proper mappings. In particular, the strong Skorokhod property for Radon measures is preserved by bijective continuous proper mappings onto sequentially Prokhorov spaces. We recall that a mapping $f\colon X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact for every compact subspace $K \subset Y$.

Theorem 3.1. *Let X and Y be two topological spaces such that there exists a bijective continuous proper mapping $F\colon X \to Y$. Assume that X has the uniformly tight strong Skorokhod property for Radon measures. Then Y possesses this property as well. In particular, if Y is sequentially Prokhorov, then Y has the strong Skorokhod property for Radon measures.*

Remark **3.1.** For a bijective continuous proper mapping $f\colon X \to Y$ and a uniformly tight weakly convergent sequence $\mu_n \Rightarrow \mu$ of probability Radon measures on Y, the sequence of measures $\mu_n \circ f^{-1}$ on X is weakly convergent to $\mu \circ f^{-1}$.

We define a topological space X to be *almost metrizable* if there exists a bijective continuous proper mapping $f\colon M \to X$ from a metrizable space M. If M is discrete, then X is called *almost discrete*.

One can easily show by examples that an almost metrizable space may not be metrizable (such examples are given below). On the other hand, each almost metrizable k-space is metrizable. We recall that a topological space X is a *k-space* if a subset $Z \subset X$ is closed in X precisely when $Z \cap K$ is closed for every compact subset $K \subset X$ (see [32]).

One can readily show that almost metrizable spaces and almost discrete spaces have the following properties.

Proposition 3.1. (i) *Any subspace of an almost metrizable space is almost metrizable.*

(ii) *A topological space is metrizable if and only if it is an almost metrizable k-space.*

(iii) *A topological space X is almost metrizable if and only if the strongest topology inducing the original topology on each compact subset of X is metrizable.*

(iv) *A topological space is almost discrete if and only if it contains no infinite compact subspaces.*

(v) *A countable product of almost metrizable spaces is almost metrizable.*

(vi) *The classes of almost metrizable and almost discrete spaces are stable under formation of arbitrary topological sums.*

(vii) *The images of almost metrizable and almost discrete spaces under continuous bijective proper mappings belong to the respective classes.*

The following theorem characterizes almost metrizable spaces with the strong Skorokhod property for Radon measures.

Theorem 3.2. *Any almost metrizable space has the uniformly tight strong Skorokhod property for Radon measures. Moreover, an almost metrizable space has the strong Skorokhod property for Radon measures if and only if it is sequentially Prokhorov.*

Since the countable product of sequentially Prokhorov spaces is also sequentially Prokhorov (see, e.g., [33], § 8.3), we arrive at the following statement.

Corollary **3.1.** *The countable product of almost metrizable spaces with the SSP_r has this property as well.*

For almost discrete spaces we have even stronger results. We recall that a topological space X is called *sequentially compact* if each sequence in X contains a convergent subsequence, see [32] (§ 3.10).

Theorem 3.3. *For a topological space X the following conditions are equivalent*:

(i) *X is an almost discrete space.*

(ii) *Each compact subset of X is sequentially compact and each uniformly tight sequence of Radon probability measures μ_n on X that converges weakly to a Radon measure is convergent in the variation norm (equivalently, one has convergence $\mu_n(x) \to \mu(x)$ for each $x \in X$).*

Corollary **3.2.** *Let X be an almost discrete sequentially Prokhorov space, let E be a completely regular space, and let a sequence of Radon probability measures μ_n on $X \times E$ converge weakly to a Radon probability measure μ. Then, for each $x \in X$, the*

restrictions of the measures μ_n to the set $x \times E$ converge weakly to the restriction of μ, i.e., one has $\mu_n|_{x \times E} \Rightarrow \mu|_{x \times E}$.

Almost metrizable spaces need not be metrizable (we shall encounter below even countable almost metrizable nonmetrizable spaces). A somewhat unexpected example is the Banach space l^1 endowed with the weak topology. The space l^1 is known to have the Shur property. We recall that a Banach space X has the *Shur property* if each weakly convergent sequence in X is norm convergent.

Theorem 3.4. *Let X be a Banach space with the Shur property and let τ be an intermediate topology between the norm and weak topologies on X. Then the space (X, τ) is almost metrizable and has the uniformly tight strong Skorokhod for Radon measures.*

It should be noted that, as shown in [34, p. 127], the space l^1 with the weak topology (as well as any infinite dimensional Banach space with the weak topology) is not sequentially Prokhorov.

We shall now give an example of an almost discrete space without the strong Skorokhod property.

***Example* 3.1.** Let X_n, $n \in \mathbb{N}$, be pairwise disjoint finite sets in $\mathbb{N}$ with $\mathrm{Card}\,(X_n) < \mathrm{Card}\,(X_{n+1})$ for each n. Fix any point $\infty \notin \bigcup_{n \in \mathbb{N}} X_n$ and define a topology on the union $X = \{\infty\} \cup \bigcup_{n=1}^{\infty} X_n$ as follows. All points except for ∞ are isolated and the neighborhood base of a unique nonisolated point ∞ is formed by the sets $X \setminus F$, where $F \subset \bigcup_{n \in \mathbb{N}} X_n$ is a subset for which there is $m \in \mathbb{N}$ such that $\mathrm{Card}\,(F \cap X_n) \le m$ for every n. It can be shown that the space X is almost discrete and fails to have the SSP_r. To this end, it suffices to note that X has no nontrivial convergent sequences. On the other hand, the sequence of measures μ_n, where each μ_n is concentrated on X_n and assigns equal values $[\mathrm{Card}\,(X_n)]^{-1}$ to the points of X_n, converges weakly to Dirac's measure at ∞. The existence of a Skorokhod parameterization of a subsequence of μ_n would give a nontrivial convergent sequence. For the same reason, no subsequence in $\{\mu_n\}$ is uniformly tight (otherwise such a subsequence could be Skorokhod parameterized by Remark 3.1).

Finally, we shall see that nonmetrizable almost metrizable spaces with the strong Skorokhod property for Radon measures exist indeed. Such spaces will be constructed as subspaces of extremally disconnected spaces. A topological space X is called *extremally disconnected* if the closure $\overline{U}$ of any open subset U of X is open, see, e.g. [32]. A standard example of an extremally disconnected space is $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$. More generally, the Stone–Čech compactification βX of a Tykhonoff space X is extremally disconnected if and only if the space X is extremally disconnected [32] ($\S\,6.2$).

Theorem 3.5. *Any countable subspace X of an extremally disconnected completely regular space K is almost discrete and has the strong Skorokhod property for Radon measures.*

***Corollary* 3.3.** *For every $p \in \beta N \setminus \mathbb{N}$, the space $X = \{p\} \cup \mathbb{N}$ with the induced topology is a nonmetrizable almost discrete space with the strong Skorokhod property for Radon measures.*

It turns out that Theorem 3.5 holds true for a wider class of spaces. We say that a completely regular space X is a *Grothendieck space* if the space $C_b(X)$ with the sup-norm is a Grothendieck Banach space. We recall that a Banach space E is said to be a Grothendieck Banach space if $*$-weak convergence of countable sequences in E^* is equivalent to weak convergence (i.e., convergence in the topology $\sigma(E^*, E^{**})$). As

show by Grothendieck, each extremally disconnected Tykhonoff space is Grothendieck (see also [35] for a discussion and further generalizations of the Grothendieck theorem).

Theorem 3.6. *Any countable subspace X of a Grothendieck space K is almost discrete and has the strong Skorokhod property.*

Corollary 3.4. *A subspace X of a Grothendieck space K is almost discrete and has the SSP_r if and only if all compact subsets of X are metrizable (equivalently, finite). In particular, this is true if K is an extremally disconnected completely regular space.*

The space $\{p\}\cup\mathbb{N}$, $p\in\beta\mathbb{N}\backslash\mathbb{N}$, is probably the simplest example of a nonmetrizable space with the strong Skorokhod property. The fact that it is not metrizable is seen from the property that p belongs to the closure of $\mathbb{N}$, but there are no infinite convergent sequences with elements from $\mathbb{N}$ (if such a sequence $\{n_i\}$ converges, then the function $f(n_{2i}) = 0$, $f(n_{2i+1}) = 1$ has no continuous extensions to $\beta\mathbb{N}$).

It should be noted that although weak convergence of countable sequences of probability measures on the space X in Corollary 3.3 is the same one that corresponds to the discrete metric on X, the two weak topologies on the space of probability measures are different (otherwise X would be metrizable in the topology from $\beta\mathbb{N}$).

Thus, in the class of countable spaces with a unique nonisolated point, there are almost metrizable nonmetrizable spaces which have (or have not) the strong Skorokhod property.

On the other hand, all countable spaces with a unique nonisolated point have the strong Skorokhod property for uniformly tight families of Radon measures.

Proposition 3.2. *Any uniformly tight family of probability measures on a countable space with a unique nonisolated point has the strong Skorokhod property.*

Note that the nonmetrizable spaces with the strong Skorokhod property which have been constructed so far are not topologically homogeneous. There exist also nonmetrizable topologically homogeneous spaces with the strong Skorokhod property for Radon measures. A topological space X is called *topologically homogeneous* if for every pair of points $x, y \in X$, there is a homeomorphism $h \colon X \to X$ such that $h(x) = y$. As a rule, pathological examples of topologically homogeneous spaces are constructed by using the technique of left topological groups. We recall that a *left topological group* is a group $(G, *)$ endowed with a left invariant topology, i.e., a topology τ such that for each $g \in G$, the left shift $l_g \colon x \mapsto g * x$ is continuous on (G, τ). A rich theory of left topological groups has been developed by I. V. Protasov [36, 37]. According to [37] (Theorem 4.1), each infinite group G admits a nondiscrete regular extremally disconnected left invariant topology. The theorem cited together with Theorem 3.6 implies the following assertion.

Corollary 3.5. *Each countable group G admits a left invariant topology τ such that (G, τ) is a nonmetrizable countable almost discrete topologically homogeneous extremally disconnected space with the strong Skorokhod property for Radon measures.*

Since compact subsets of almost metrizable spaces are metrizable, we conclude that each Radon measure μ on an almost metrizable space X is concentrated on a σ-compact space $C \subset X$ with a countable network in the sense that $\mu(C) = 1$. We recall that a space X has a *countable network* if there is a countable family $\mathcal{N}$ of subsets of X such that, for every point $x \in X$ and every neighborhood $U \subset X$ of x, there is an element $N \in \mathcal{N}$ with $x \in N \subset U$.

4. The strong Skorokhod property in linearly ordered and compact spaces. Any linear order $\le$ on a set X generates two natural topologies on X. The usual *interval topology* is generated by the pre-basis consisting of the rays

$$(\leftarrow, a) = \{x \in X \colon\, x < a\}$$

and

$$(a, \rightarrow) = \{x \in X \colon\, x > a\}, \quad \text{where} \quad a \in X.$$

The *Sorgenfrey topology* on X is generated by the pre-basis consisting of the rays $(a, \rightarrow)$ and $(\leftarrow, a] = \{x \in X \colon\, x \le a\}$, where $a \in X$. The space X endowed with the interval topology will be denoted by $\big(X, (\le)\big)$. The space X endowed with the Sorgenfrey topology will be denoted by $\big(X, (\le])$. According to [32] (2.7.5) the space $\big(X, (\le)\big)$ is hereditarily normal, and the space $\big(X, (\le])$ is Tykhonoff and zero-dimensional.

We recall that a *Souslin line* is a linearly ordered nonseparable compact space with countable cellularity, see [32] (2.7.9). It is known that the existence of a Souslin line is independent of the ZFC axioms.

It is worth noting that the Sorgenfrey line can be topologically embedded into the product $A = [0, 1] \times \{0, 1\}$ endowed with the interval topology generated by the lexicographic order $\le \colon (x_1, t_1) \le (x_2, t_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2$ and $t_1 \le t_2$. The space A (called *the Alexandroff two arrows space*) is a nonmetrizable separable first countable compact space.

A topological space X is called *linearly ordered* if it carries the interval topology generated by some liner order on X. Standard examples of linearly ordered spaces are the real line and intervals of ordinals.

Theorem 4.1. *If* $\le$ *is a linear order on a set* X, *then the spaces* $\big(X, (\le)\big)$ *and* $\big(X, (\le])$ *have the strong Skorokhod property for discrete probability measures.*

***Corollary* 4.1.** *If each Radon probability measure on a linearly ordered topological space* X *is discrete, then the space* X *has the strong Skorokhod property for Radon measures.*

A topological space X is called *scattered* if each subspace E of X has an isolated point. It is well known that each Radon measure on a scattered space is discrete, see [38] (Lemma 294).

***Corollary* 4.2.** (i) *Each scattered linearly ordered space has the* SSP_r.

(ii) *For every ordinal* α, *the interval* $[0, \alpha]$ *endowed with the usual order topology has the* SSP_r.

(iii) *For any linear order* $\le$ *on a set* X, *the space* $\big(X, (\le])$ *has the* SSP_r.

(iv) *The Sorgenfrey line* $\big(\mathbb{R}, (\le])$ *has the* SSP_r.

(v) *The Alexandroff two arrows space* A *has the strong Skorokhod property for discrete probability measures.*

Since the space A admits a continuous surjection onto the interval $[0, 1]$, it carries nondiscrete probability measures, e.g., the measure whose projection is Lebesgue measure on $[0, 1]$ (there is only one such measure on A).

Now we briefly discuss the strong Skorokhod property in nonmetrizable compact topological spaces. We already know that the ordinal interval $[0, \alpha]$ has the SSP_r. In particular, there exist nonmetrizable compact spaces with this property. However, it seems that such examples are not numerous. One of the simplest examples of nonmetrizable compacta is *the Alexandroff supersequence*, which is the one point compactification $\alpha \aleph_1$ of a discrete space of the smallest uncountable size.

***Example* 4.1.** The Alexandroff supersequence $\alpha \aleph_1$ fails to have the strong Skorokhod property even for probability measures with two-point supports. The same is true for any topological space containing a copy of the Alexandroff supersequence $\alpha \aleph_1$.

We recall that a space is *dyadic* if it is the image of certain power $\{0,1\}^\tau$ under a continuous mapping. In the class of dyadic compacta only metrizable ones enjoy the strong Skorokhod property.

Theorem 4.2. *Each dyadic compact with the strong Skorokhod property for probability measures with two-point supports is metrizable. In particular, if a dyadic compact has the strong Skorokhod property for probability measures with two-point supports, then it has the strong Skorokhod property for Radon measures.*

Besides the class of dyadic compact spaces, there are many interesting classes of compact spaces for which their relation to the class of spaces with the strong Skorokhod property has not yet been clarified. In this respect, it would be interesting to investigate the classes of Eberlein, Corson, and Rosenthal compacta (see, e.g., [39, 40]). We recall that a compact space K is defined to be

an *Eberlein compact* if K is homeomorphic to a compact subset of the Σ^*-product $\Sigma^*(\tau) = \big\{(x_i)_{i\in\tau} \in \mathbb{R}^\tau \colon \forall \varepsilon > 0 \text{ the set } \{i \in \tau \colon |x_i| > \varepsilon\} \text{ is finite }\big\}$ for some cardinal τ;

a *Corson compact* if K is homeomorphic to a compact subset of the Σ-product $\Sigma(\tau) = \big\{(x_i)_{i\in\tau} \in \mathbb{R}^\tau \colon \text{ the set } \{i \in \tau \colon x_i \neq 0\} \text{ is countable }\big\}$ for some cardinal τ;

a *Rosenthal compact* if K is homeomorphic to a compact subset of the space $B_1(P) \subset \mathbb{R}^P$ of all functions of the first Baire class on a Polish space P.

It is clear that any Eberlein compact is Corson. It is known that each separable Corson compact as well as each Eberlein compact with countable cellularity is metrizable. The Alexandroff two arrows space is a standard example of a Rosenthal compact which is not a Corson compact. The Alexandroff supersequence is both an Eberlein and Rosenthal compact. Thus, there exist Eberlein and Rosenthal compacta without the strong Skorokhod property. The space $[0, \omega_1]$ is neither Corson nor Rosenthal, see [41, p. 256, 259].

5. The weak Skorokhod property.

Definition 5.1. (i) *We shall say that a topological space X has the weak Skorokhod property if to every Radon probability measure μ on X, one can associate a μ-measurable mapping $\xi_\mu \colon [0,1] \to X$ on $([0,1], \mathcal{B}([0,1]), \lambda)$ with $\lambda \circ \xi_\mu^{-1} = \mu$ in such a way that for every uniformly tight sequence of Radon probability measures μ_n on X, there is a subsequence in $\{\xi_{\mu_n}\}$ that converges almost everywhere.*

(ii) *We shall say that a topological space X has the weak Skorokhod property for uniformly tight sequences if, for every uniformly tight sequence $\{\mu_n\}$ of Radon probability measures on X, there exist a subsequence $\{\mu_{n_k}\} \subset \{\mu_n\}$, and a sequence of λ-measurable mappings $\xi_k \colon [0,1] \to X$ with $\lambda \circ \xi_{n_k}^{-1} = \mu_{n_k}$ that converges almost everywhere.*

In both cases one can consider the extended WSP when $[0,1]$ with Lebesgue measure is replaced by an arbitrary probability space $(\Omega, \mathcal{F}, P)$. We shall abbreviate the weak Skorokhod property as WSP. The reader is warned that for simplicity of terminology we alter the terminology from [11] where the WSP has been definied as the extended WSP above.

Lemma 5.1. *Suppose that there is a sequence of continuous functions ψ_j separating the points in X. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\xi_n \colon \Omega \to X$ be $(\mathcal{F}, \mathcal{B}(X))$-measurable mappings convergent almost everywhere to a mapping ξ such that the sequence of measures $P \circ \xi_n^{-1}$ is uniformly tight. Then the mapping ξ is $(\mathcal{F}_P, \mathcal{B}(X))$-measurable, where $\mathcal{F}_P$ is the completion of $\mathcal{F}$ with respect to P.*

Proof. Let us take increasing compact sets $K_i \subset X$ such that $P \circ \xi_n^{-1}(K_i) \geq 1 - 2^{-i}$ for all n and i. Set $\psi(x) := (\psi_1(x), \psi_2(x), \ldots)$. The sets K_i are metrizable, since ψ separates their points. Hence ψ is a Borel isomorphism between $E := \bigcup_{i=1}^{\infty} K_i$ and $\psi(E) \subset \mathbb{R}^{\infty}$. The mapping $\psi \circ \xi$ with values in $\mathbb{R}^{\infty}$ is defined P-a.e. and is P-measurable. We have $\psi(\xi(\omega)) \in \psi(E)$ for P-a.e. ω, since $P \circ (\psi \circ \xi_n)^{-1}(\psi(K_i)) \geq$ $\geq 1 - 2^{-i}$ for all n and i, which yields $P \circ (\psi \circ \xi)^{-1}(\psi(K_i)) \geq 1 - 2^{-i}$ for all i. Therefore, $\xi(\omega) \in E$ for P-a.e. ω. Let us fix any $x_0 \in X$. The mapping $h \colon \mathbb{R}^{\infty} \to X$ defined by $h = g^{-1}$ on $\psi(E)$ and $h(y) = x_0$ if $y \notin \psi(E)$ is Borel measurable, because $\psi(E)$ is a countable union of compact sets. It remains to note that $\xi(\omega) =$ $= h(\psi(\xi(\omega)))$ for P-a.e. ω.

We recall that in general a pointwise limit of Borel mappings with values in a topological space may fail to be Borel measurable.

Theorem 5.1. (*Jakubowki* [6]). *Every space that possesses a countable family of continuous functions separating the points has the weak Skorokhod property for uniformly tight sequences.*

This is a very useful general result. In particular, it applies to all completely regular Souslin spaces.

The following result is proved in [11].

Proposition 5.1. (i) *Let X be a topological space that has a continuous injection into $\mathbb{R}^{\infty}$ and let $\Phi \colon X \to Y$ be a Borel mapping with values in a metrizable Souslin space Y such that the sets $\Phi(A)$ and $\Phi^{-1}(B)$ have compact closure for all compact sets A and B. Then X has the weak Skorokhod property.*

(ii) *Let f be a continuous surjection of a space X with the weak Skorokhod property onto a space Y such that the preimages of compact sets are compact. Then Y has the weak Skorokhod property.*

Let us consider examples of spaces with the weak Skorokhod property. More examples can be produced by using the fact that every subset of a space with the weak Skorokhod property has this property as well.

Example 5.1. (i) Let $X = \prod_{m=1}^{\infty} X^m$, where each space X_m admits a continuous injection into $\mathbb{R}^{\infty}$ and has a fundamental sequence $\{K_n^m\}$ of compacts sets (i.e., every compact subset of X_m is contained in one of the sets K_n^m). Then X has the weak Skorokhod property.

(ii) Let a locally convex space X be the strict inductive limit of an increasing sequence of its closed metrizable Souslin linear subspaces X_j. Then X has the weak Skorokhod property. The same is true for any countable product of such inductive limits.

Note that the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ has a fundamental sequence of compact sets; the same is true for the duals to separable Banach spaces with the weak-$*$ topology. The above example covers also the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ (which is homeomorphic to a closed subspace of a countable product of spaces having fundamental sequences of compact sets). We observe that part (ii) of the example covers the space $\mathcal{D}(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$ (with its natural topology of the inductive limit of the spaces $C_0^{\infty}(U_n)$, where U_n is the centered open ball of radius n).

The next theorem is proved in [15]. A mapping $f \colon X \to Y$ between topological spaces is called compact-covering if for every compact set $K \subset Y$ there is a compact set $l(K) \subset X$ such that $K = f(l(K))$. If one can choose the sets $l(K)$ in such a way that $l(K) \subset l(K')$ whenever $K \subset K'$, then f is called monotone compact-covering.

Theorem 5.2. *A completely regular space Y has the WSP if and only if there is a monotone compact-covering mapping $f\colon X \to Y$ from a space X possessing the WSP.*

In particular, a completely regular space that is the image of a metrizable space under a monotone compact-covering mapping has the WSP.

The last result justifies the following definition.

Definition 5.2. *A Hausdorff topological space X is called a monotone $\aleph$-space if X is the image of a metrizable space M under a monotone compact-covering mapping $f\colon M \to X$. If M is separable, then X is called a monotone $\aleph_0$-space.*

Thus, any completely regular monotone $\aleph$-space possesses the WSP.

Monotone $\aleph$-spaces are thoroughly investigated in Sections 3 and 4 in [15]. Here we mention only the following result. A topological space is called *submetrizable* if it admits a continuous bijective mapping onto a metrizable space.

Theorem 5.3. (i) *Let X be a submetrizable sequentially Prokhorov completely regular monotone $\aleph$-space. Then the space $P_r(X)$ with the weak topology is a monotone $\aleph$-space, hence has the WSP.*

(ii) *Let X be a sequentially Prokhorov completely regular monotone $\aleph_0$-space. Then the space $P_r(X)$ is a monotone $\aleph_0$-space.*

Note also that if X is a separable Banach space, then X^* with the topology $\sigma(X^*, X)$ is a monotone $\aleph_0$-space. If also X^* is separable, then the same is true for X with the weak topology.

6. Open problems. Here we mention a number of open questions that are closely related to the above discussion and have simple formulations.

Question 1. Is there an infinite extremally disconnected compact space with the strong Skorokhod property for Radon measures? In particular, does $\beta\mathbb{N}$ possess the strong Skorokhod property for Radon measures?

Let us give the definitions of two interesting countable spaces. The *Fréchet – Urysohn fan V* is defined as follows (cf. [32], 1.6.18). Let

$$V := \{k + (n + 1)^{-1}\colon k, n \in \mathbb{N}\} \cup \{0\}$$

be endowed with the following topology: every point $k + (n + 1)^{-1}$ has its usual neighborhoods from the space $V\setminus\{0\}$ and the point 0 has an open base formed by the sets

$$U_{n_1,\ldots,n_j,\ldots} := \{k + (n + 1)^{-1}\colon k \in \mathbb{N}, n \geq n_k\} \cup \{0\},$$

where $\{n_j\}$ is a sequence of natural numbers. The *Arens fan A_2* is the space

$$A_2 = \{(0,0), (1/i, 0), (1/i, 1/j)\colon 1 \leq i \leq j < \infty\}$$

endowed with the strongest topology inducing the original topology on each compact $K_n = \{(0,0), (1/k, 0), (1/i, 1/j)\colon k \in \mathbb{N}, 1 \leq i \leq n, i \leq j < \infty\}$.

Question 2. Do the Fréchet – Urysohn and Arens fans have the strong Skorokhod property?

Let us note that according to Proposition 3.2 the Fréchet – Urysohn fan has the strong Skorokhod property for uniformly tight families of probability measures.

Question 3. Does the product $[0, 1] \times [0, \omega_1]$ have the strong Skorokhod property for Radon measures or for discrete probability measures?

Question 4. Does every linearly ordered compact space have the strong Skorokhod property for Radon measures? In particular, does the Souslin line have the strong Skorokhod property for Radon measures?

Question 5. Is it true that the Alexandroff two arrows space A possess the strong Skorokhod property for Radon measures?

Question 6. Is every Eberlein (Corson, Rosenthal) compact with the strong Skorokhod property metrizable?

It is worth noting that each countable family $\mathcal{M}$ of probability measures on an Eberlein compact E admits a Skorokhod representation. Indeed, one can show that there is a metrizable compact subset $K \subset E$ with $\mu(K) = 1$ for all $\mu \in \mathcal{M}$.

Question 7. Suppose a compact K has the strong Skorokhod property and admits a continuous finite-to-one map onto a metrizable compact. Is K metrizable?

Question 8. Stability of the class of spaces with the strong Skorokhod property (or with the other related properties mentioned above) with respect to formation of finite and countable products.

In particular, it would be interesting to study whether the product $X \times Y$, where X has the strong property and Y is separable metric (say, $Y = \mathbb{N}^{\infty}$), retains the same Skorokhod property. We do not know an answer even for finite or countable Y (in particular, we do not know whether the topological sum of two spaces with the strong Skorokhod property has that property).

Question 9. Suppose that a completely regular space X admits a continuous injection into a Polish space. Does X have the weak Skorokhod property?

It would be also interesting to study a version of Skorokhod's representation in which convergence of mappings almost everywhere is replaced by convergence in measure, which makes sense for subspaces of locally convex spaces (then one has to consider convergence in measure with respect to every fixed seminorm defining the topology of a given space) or for uniform spaces.

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