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ARBITRARY BINARY RELATIONS, CONTRACTION MAPPINGS, AND b-METRIC SPACES*

ДОВІЛЬНІ БІНАРНІ СПІВВІДНОШЕННЯ, СТИСКАЮЧІ ВІДОБРАЖЕННЯ ТА b-МЕТРИЧНІ ПРОСТОРИ

We prove some results on the existence and uniqueness of fixed points defined on a b-metric space endowed with an arbitrary binary relation. As applications, we obtain some statements on coincidence points involving a pair of mappings. Our results generalize, extend, modify and unify several well-known results especially those obtained by Alam and Imdad [J. Fixed Point Theory and Appl., **17**, 693 – 702 (2015); Fixed Point Theory, **18**, 415 – 432 (2017); Filomat, **31**, 4421 – 4439 (2017)] and Berzig [J. Fixed Point Theory and Appl., **12**, 221 – 238 (2012)]. Also, we provide an example to illustrate the suitability of results obtained.

Доведено деякі результати про існування та єдиність нерухомих точок на b-метричних просторах, що наділені довільним бінарним відношенням. В якості застосувань отримано деякі твердження про точки збігу для пар відображень. Ці результати узагальнюють, розширюють, модифікують та уніфікують деякі відомі результати Alam і Imdad [J. Fixed Point Theory and Appl., **17**, 693 – 702 (2015)]; Fixed Point Theory, **18**, 415 – 432 (2017); Filomat, **31**, 4421 – 4439 (2017)] та Berzig [J. Fixed Point Theory and Appl., **12**, 221 – 238 (2012)]. Також наведено приклад для ілюстрації застосовності отриманих результатів.

1. Relation theoretic notions and preliminary results. We begin with some preliminary definitions and notations which will be required in the sequel.

Definition 1.1 [7, 11]. *Let X be a (nonempty) set and $k \geq 1$ be a given real number. A function $d \colon X \times X \to [0, +\infty)$ is a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:*

(b_1) $d(x,y) = 0$ *if and only if* $x = y$,

(b_2) $d(x,y) = d(y,x)$,

(b_3) $d(x,z) \leq k\left(d(x,y) + d(y,z)\right)$.

The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively large than that of metric spaces, since a b-metric is a metric when $k = 1$. Following example show that in general a b-metric need not necessarily be a metric (see also [1, 14, 17]).

***Example* 1.1.** Let (X, d) be a metric space, and $\rho(x,y) = (d(x,y))^p$, $p > 1$ is a real number. Then ρ is a b-metric with $k = 2^{p-1}$, but ρ is not metric on X.

Otherwise, for more concepts such as b-convergence, b-completeness, b-Cauchy sequence and b-closed set in b-metric spaces, we refer the reader to [1, 13, 14, 17] and the references mentioned therein. Also, for the concepts such as partial order, comparable, well ordered, nondecreasing, increasing, dominated, dominating and other, we refer the reader to [1, 9, 12, 14, 17, 18, 21].

In the sequel, let $\mathbb{N}$ denote the set of all nonnegative integers, $\mathbb{R}$ the set of all real numbers. Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation but for the sake of simplicity, we often write binary relation instead of nonempty binary relation.

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Definition 1.2. *Let $\mathcal{R}$ be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.*

Definition 1.3. *A binary relation $\mathcal{R}$ on a nonempty set X is called*:

(1) *reflexive if $(x, x) \in \mathcal{R}$ for every $x \in X$,*

(2) *symmetric if whenever $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$,*

(3) *antisymmetric if whenever $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$,*

(4) *transitive if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$,*

(5) *complete or connected or dichotomous if $[x, y] \in \mathcal{R}$ for all $x, y \in X$,*

(6) *weakly complete or weakly connected or trichotomous if $[x, y] \in \mathcal{R}$ or $x = y$ for all $x, y \in X$.*

Definition 1.4. *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.*

(1) *The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} := := \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.*

(2) *The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.*

Definition 1.5. *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subseteq X$ is called $\mathcal{R}$-preserving if (x_n, x_{n+1}) belongs to $\mathcal{R}$ for all $n \in \mathbb{N}_0$.*

Definition 1.6 [2]**.** *Assume that P, Q are self-mappings on a nonempty set X. A binary relation $\mathcal{R}$ on X is called (P, Q)-closed if for all $x, y \in X$, $(Qx, Qy) \in \mathcal{R}$, then (Px, Py) belongs to $\mathcal{R}$.*

If we take $Q =$identity mapping, we have $\mathcal{R}$ is P-closed.

If $\mathcal{R}$ is P-closed, then R^s is also P-closed.

Definition 1.7. *Let $(X, d, k \geq 1)$ be a b-metric space and $\mathcal{R}$ a binary relation on X. We say that (X, d) is R-complete if every R-preserving b-Cauchy sequence in X converges.*

Definition 1.8. *Let $(X, d, k \geq 1)$ be a b-metric space and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-closed if every $\mathcal{R}$-preserving b-convergent sequence in E converges to a point of E.*

In the following lines, we extend a weaker version of the notion of d-self-closeness of a partial order $\preceq$ (defined by Turinici [19]) to an arbitrary binary relation.

Definition 1.9. *Let $(X, d, k \geq 1)$ be a b-metric space and $Q : X \to X$. A binary relation $\mathcal{R}$ defined on X is called (Q, b_d)-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \to^d x$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $[Qx_{n_i}, Qx] \in \mathcal{R}$ for all $i \in \mathbb{N}$.*

If Q is identity mapping, we have the following definitions.

Definition 1.10. *Let $(X, d, k \geq 1)$ be a b-metric space. A binary relation $\mathcal{R}$ defined on X is called b_d-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \to^d x$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $(x_{n_i}, x) \in \mathcal{R}$ for all $i \in \mathbb{N}$.*

Definition 1.11 [20]**.** *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.*

Definition 1.12 [2]**.** *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. For $x, y \in \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions*:

(i) $z_0 = x$ *and* $z_k = y$,

(ii) $(z_i, z_{i+1}) \in R$ *for each* i $(0 \leq i \leq k-1)$.

Definition 1.13. *Let* $(X, d, k \geq 1)$ *be a b-metric space,* $\mathcal{R}$ *a binary relation on* X *and* P *and* Q *two self-mappings on* X. *We say that* P *and* Q *are* $\mathcal{R}$*-compatible if for any sequence* $\{x_n\} \subset X$ *such that* $\{Px_n\}$ *and* $\{Qx_n\}$ *are* $\mathcal{R}$*-preserving and* $\lim_{n\to\infty} Q(x_n) = \lim_{n\to\infty} P(x_n)$, *we have* $\lim_{n\to\infty} d(QPx_n, PQx_n) = 0$.

Following auxiliary results will be used in sequel.

Lemma 1.1 [15]**.** *Let* X *be a nonempty set and* T *a self-mapping on* X. *Then there exists a subset* $E \subseteq X$ *such that* $T(E) = T(X)$ *and* $T\colon E \to X$ *is one-to-one.*

Lemma 1.2 ([17], Lemma 3.1)**.** *Let* $\{y_n\}$ *be a sequence in a b-metric space* (X, d) *with* $k \geq 1$ *such that*

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \tag{1}$$

for some $\lambda \in \left[0, \frac{1}{k}\right)$ *and each* $n = 1, 2, \ldots$. *Then* $\{y_n\}$ *is a b-Cauchy sequence in b-metric space* (X, d).

Proposition 1.1 [2]**.** *For a binary relation* $\mathcal{R}$ *on a nonempty set* X, $(x, y) \in \mathcal{R}^s$ *if and only if* $[x, y] \in \mathcal{R}$.

Proposition 1.2. *Let* X *be a nonempty set,* $\mathcal{R}$ *a binary relation on* X *and* P, Q *are self-mappings on* X. *If* $\mathcal{R}$ *is* (P, Q)*-closed, then* $\mathcal{R}^s$ *is also* (P, Q)*-closed.*

Proposition 1.3. *If* $(X, d, k \geq 1)$ *be a b-metric space and* $T, S\colon X \to X$ *be self mappings, then the following contractive conditions are equivalent*:

$$d(Tx, Ty) \leq \lambda d(Sx, Sy) \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad (Sx, Sy) \in \mathcal{R},$$

$$d(Tx, Ty) \leq \lambda d(Sx, Sy) \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad [Sx, Sy] \in \mathcal{R}$$

for some $\lambda \in \left[0, \frac{1}{k}\right)$.

For $S =$identity mapping, we have the following result.

Proposition 1.4. *If* $(X, d, k \geq 1)$ *be a b-metric space and* $T\colon X \to X$ *be self mapping, then the following contractive conditions are equivalent*:

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad (x, y) \in \mathcal{R},$$

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \textit{for all} \quad x, y \in X \quad \textit{with} \quad [x, y] \in \mathcal{R}.$$

for some $\lambda \in \left[0, \frac{1}{k}\right)$.

In this paper, we use the following notations:

(i) $F(T)$ = the set of all fixed points of T,

(ii) $X(T; \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$,

(iii) $\gamma(x, y, \mathcal{R})$:= the class of all paths in $\mathcal{R}$ from x to y.

In this paper, we prove some results on the existence and uniqueness of fixed points defined on a b-metric space endowed with an arbitrary binary relation. As applications, some results on coincidence points involving a pair of mappings are also obtained. Our results generalize, extend, modify and unify several well-known results especially those obtained in [2, 3, 5, 20].

2. Main results. First we introduce the concept of C_T-relation theoretic contractive mappings in the setting of b-metric space.

Definition 2.1. *Let $(X, d, k \geq 1)$ be a b-metric space and $\mathcal{R}$ a binary relation on X. A mapping $T\colon X \to X$ is called a C_T-contraction if there exists $\lambda \in \left[0, \dfrac{1}{k}\right)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{2}$$

for every $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Now we are ready to state our first main result.

Theorem 2.1. *Let $(X, d, k \geq 1)$ be a b-complete b-metric space, $\mathcal{R}$ a binary relation on X and $T\colon X \to X$ a C_T-relation theoretic contraction satisfying the following conditions*:

(i) *$X(T; \mathcal{R})$ is nonempty,*

(ii) *$\mathcal{R}$ is T-closed,*

(iii) *either T is b-continuous or $\mathcal{R}$ is b_d-self-closed.*

Then T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Further, if

(iv) *$\gamma(x, y, \mathcal{R}^s)$ is nonempty, for each $x, y \in X$, then T has a unique fixed point.*

Proof. Let $x_0 \in X(T, \mathcal{R})$. Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \quad \text{for all} \quad n \geq 0.$$

Now, we shall show that the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Then, by definition, we have $(x_0, x_1) \in \mathcal{R}$. By T-closedness of $\mathcal{R}$, we obtain $(Tx_0, Tx_1) = (x_1, x_2) \in \mathcal{R}$. Repetition of this argument gives

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus, the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Since T is a C_T-contractive mapping and sequence $\{x_n\}$ is $\mathcal{R}$-preserving, we obtain, for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda d(x_{n-1}, x_n).$$

Therefore, by using Lemma 1.2, it follows that $\{x_n\}$ is a b-Cauchy sequence.

Since $(X, d, k \geq 1)$ is b-complete, there exists $x^* \in X$ such that

$$x_n \to x^* \text{ as } n \to \infty.$$

From the b-continuity of T, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. Due to the uniqueness of the limit, we derive that $Tx^* = x^*$, that is, x^* is a fixed point of T.

Alternately, we assume that $\mathcal{R}$ is b_d-self-closed. Since $\{x_n\}$ is $\mathcal{R}$-preserving and $x_n \to x$, by b_d-self-closedness of $\mathcal{R}$ there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $[x_{n_j}, x] \in \mathcal{R}$ for all $j \in \mathbb{N}$.

Since T is a C_T-contraction and using Proposition 1.4, we obtain

$$d(x^*, Tx^*) \leq k[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] =$$
$$= k[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \leq$$
$$\leq k[d(x^*, x_{n+1}) + \lambda d(x_n, x^*)] \to 0,$$

when $n \to \infty$. Hence $Tx^* = x^*$ and x^* is a fixed point of T.

Suppose that y^* is another fixed point of T.

By assumption (iv), there exists a path (say $\{z_0, z_1, z_2, \ldots, z_k\}$) of some finite length $k \in \mathcal{R}^s$ from x to y so that $z_0 = x,\ z_k = y,\ [z_i, z_{i+1}] \in \mathcal{R}$ for each $i,\ 0 \leq i \leq k-1$.

As $\mathcal{R}$ is T-closed, $\mathcal{R}^s$ is T-closed. Therefore, we have $[T^n z_i, T^n z_{i+1}] \in \mathcal{R}$ for each $i,\ 0 \leq i \leq \leq k-1$, and for each $n \in \mathbb{N}$.

Using the above arguments and inequality (2), we have

$$d(x^*, y^*) = d(T^n x^*, T^n y^*) \leq \sum_{i=0}^{k-1} d(T^n z_i, T^n z_{i+1}) \leq$$
$$\leq \lambda \sum_{i=0}^{k-1} d(T^{n-1} z_i, T^{n-1} z_{i+1}) \leq \ldots \leq \lambda^n \sum_{i=0}^{k-1} d(z_i, z_{i+1}) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $x^* = y^*$. Hence, T has a unique fixed point.

Theorem 2.1 is proved.

***Example* 2.1.** Let $X = \mathbb{N} \cup \{\infty\}$ and $d\colon X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if} \quad m = n, \\ \left|\dfrac{1}{m} - \dfrac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with $k = \dfrac{5}{2}$ (see [16]).

Define a binary relation $\mathcal{R} = \{(x, y) \in R^2 \colon x - y \geq 0\}$ on X. Consider a mapping $T\colon X \to X$ as

$$Tx = \begin{cases} 5x, & x \in \mathbb{N}, \\ \infty, & x = \infty. \end{cases}$$

Clearly, $\mathcal{R}$ is T-closed and T is continuous. Now, for $x, y \in X$ with $(x, y) \in \mathcal{R}$, we have the following cases.

Case I: x, y are even numbers. Then $Tx = 5x,\ Ty = 5y,\ d(x,y) = \left|\dfrac{1}{x} - \dfrac{1}{y}\right|,\ d(Tx, Ty) = = \dfrac{1}{5}\left|\dfrac{1}{x} - \dfrac{1}{y}\right|$, hence, $d(Tx, Ty) = \dfrac{1}{5}\left|\dfrac{1}{x} - \dfrac{1}{y}\right| \leq \dfrac{1}{k} d(x, y)$.

Case II: x, y are odd numbers (and $x \neq y$). Then $Tx = 5x,\ Ty = 5y,\ d(x, y) = 5,\ d(Tx, Ty) = = \dfrac{1}{5}\left|\dfrac{1}{x} - \dfrac{1}{y}\right|$, hence, $d(Tx, Ty) = \dfrac{1}{5}\left|\dfrac{1}{x} - \dfrac{1}{y}\right| \leq \dfrac{1}{k} d(x, y)$.

Case III: x, y are natural numbers of different parity. Then $Tx = 5x,\ Ty = 5y,\ d(x,y) = 2,$ $d(Tx,Ty) = \frac{1}{5}\left|\frac{1}{x} - \frac{1}{y}\right|$, hence, $d(Tx,Ty) = \frac{1}{5}\left|\frac{1}{x} - \frac{1}{y}\right| \leq \frac{1}{k}d(x,y).$

Case IV: x is even, $y = \infty$. Then $Tx = 5x,\ Ty = 5y,\ d(x,y) = \frac{1}{x},\ d(Tx,Ty) = \frac{1}{5x}$, hence, $d(Tx,Ty) = \frac{1}{5x} \leq \frac{1}{k}d(x,y).$

Case V: x is odd and $y = \infty$. Then $Tx = 5x,\ Ty = 5y,\ d(x,y) = 5,\ d(Tx,Ty) = \frac{1}{5x}$, hence, $d(Tx,Ty) = \frac{1}{5x} \leq \frac{1}{k}d(x,y).$

Hence, all the conditions of Theorem 2.1 are satisfied and T has a fixed point in X, that is ∞.

Theorem 2.2. *Let $(X, d, k \geq 1)$ be a b-complete b-metric space, $\mathcal{R}$ a binary relation on X. Suppose that $P, Q : X \to X$ are self mappings on X. Further, following conditions hold:*

(i) *$(X, P, \mathcal{R}), (X, Q, \mathcal{R})$ are nonempty and $P(X) \subseteq Q(X)$,*

(ii) *$\mathcal{R}$ is a (P, Q)-closed,*

(iii) *there exists $\lambda \in \left[0, \frac{1}{k}\right)$, such that $d(Px, Py) \leq \lambda d(Qx, Qy)$ for all $x, y \in X$ with $(Qx, Qy) \in \mathcal{R}$,*

(iv) *either P is $(Q, \mathcal{R})$-continuous or P and Q are continuous.*

Then P, Q have a point of coincidence.

Proof. In view of assumption (i), let x_0 be an arbitrary element of $X(P, Q, \mathcal{R})$, then $(Qx_0, Px_0) \in \mathcal{R}$. If $Q(x_0) = P(x_0)$, then x_0 is a coincidence point of P and Q and, hence, we are through.

Otherwise, if $Q(x_0) \neq P(x_0)$, then, from $P(X) \subseteq Q(X)$, we can choose $x_1 \in X$ such that $Q(x_1) = P(x_0)$. Again from $P(X) \subseteq Q(X)$, we can choose $x_2 \in X$ such that $Q(x_2) = P(x_1)$. Continuing this process, we construct a sequence $\{x_n\} \subset X$ (of joint iterates) such that

$$Q(x_{n+1}) = P(x_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{3}$$

Now, we claim that $\{Qx_n\}$ is $\mathcal{R}$-preserving sequence, i.e.,

$$(Qx_n, Qx_{n+1}) \in \mathcal{R} \quad \text{for all} \quad n \in \mathbb{N}. \tag{4}$$

We prove this fact by mathematical induction. By using equation (3) (with $n = 0$) and the fact that $x_0 \in X(P, Q, \mathcal{R})$, we have $(Qx_0, Qx_1) \in \mathcal{R}$, which shows that (4) holds for $n = 0$.

Suppose that (4) holds for $n = r > 0$, i.e., $(Qx_r, Qx_{r+1}) \in \mathcal{R}$. As $\mathcal{R}$ is (P, Q)-closed, we have $(Px_r, Px_{r+1}) \in \mathcal{R}$, which, by using (1), yields that $(Qx_{r+1}, Qx_{r+2}) \in \mathcal{R}$, i.e., (4) holds for $n = r + 1$. Hence, by induction, (4) holds for all $n \in \mathbb{N}$.

In view of (3) and (4), the sequence $\{Px_n\}$ is also an $\mathcal{R}$-preserving, i.e.,

$$(Px_n, Px_{n+1}) \in \mathcal{R} \quad \text{for all} \quad n \in \mathbb{N}$$

By using (3), (4) and assumption (iii), we obtain

$$d(Qx_n, Qx_{n+1}) = d(Px_{n-1}, Px_n) \leq \lambda d(Qx_{n-1}, Qx_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

Therefore, by using Lemma 1.2, it follows that $\{Qx_n\}$ is a b-Cauchy sequence.

Owing to (3), $\{Qx_n\} \subseteq P(X)$ so that $\{Qx_n\}$ is $\mathcal{R}$-preserving b-Cauchy sequence in X. As X is b-complete, there exists $u \in Q(X)$ such that

$$\lim_{n\to\infty} Q(x_n) = Q(u). \tag{5}$$

By using (3) and (5), we obtain

$$\lim_{n\to\infty} P(x_n) = Q(u). \tag{6}$$

Now, we show that u is a coincidence point of P and Q.

Firstly, suppose that P is $(Q, \mathcal{R})$-continuous, then, by using (4) and (5), we get

$$\lim_{n\to\infty} P(x_n) = P(u). \tag{7}$$

By using (6) and (7), we get $Q(u) = P(u)$. Hence, we are done.

Secondly, suppose that P and Q are continuous. Owing to Lemma 1.1, there exists a subset $E \subseteq X$ such that $Q(E) = Q(X)$ and $Q\colon E \to X$ is one-to-one. Now, define $T : Q(E) \to Q(X)$ by $T(Qa) = P(a)$ for all $Q(a) \in Q(E)$ where $a \in E$.

As $Q\colon E \to X$ is one-to-one and $P(X) \subseteq Q(X)$, T is well defined. Again since P and Q are continuous, it follows that T is continuous. By using the fact $Q(X) = Q(E)$, $P(X) \subseteq Q(X)$, we have $P(X) \subseteq Q(E)$, which follows that, without loss of generality, we are able to construct $\{x_n\} \subset E$ satisfying (3) and to choose $u \in E$. By using (5), (6) and continuity of T, we get $P(u) = T(Qu) = T(\lim_{n\to\infty} Qx_n) = \lim_{n\to\infty} T(Qx_n) = \lim_{n\to\infty} P(x_n) = Q(u)$. Thus, $u \in X$ is a point of coincidence of P and Q and, hence, we have the result.

Theorem 2.2 is proved.

Theorem 2.3. *In the hypotheses of Theorem* 2.2, *if instead of (iv), we take*

(v) *P and Q are $\mathcal{R}$-compatible, Q is $\mathcal{R}$-continuous, and either P is $\mathcal{R}$-continuous or $\mathcal{R}$ is (Q, b_d)-self-closed.*

Then P, Q have a point of coincidence.

Proof. On the lines of Theorem 2.2, we have $\{Qx_n\}$ is a b-Cauchy sequence. Owing to (3), $\{Qx_n\} \subseteq P(X)$ so that $\{Qx_n\}$ is $\mathcal{R}$-preserving b-Cauchy sequence in X. As X is b-complete, there exists $u \in Q(X)$ such that

$$\lim_{n\to\infty} Q(x_n) = Q(u). \tag{8}$$

By using (3) and (8), we obtain

$$\lim_{n\to\infty} P(x_n) = Q(u).$$

As Q is $\mathcal{R}$-continuous, we have

$$\lim_{n\to\infty} Q(Qx_n) = Q(\lim_{n\to\infty} Q(x_n)) = Q(Q(u)).$$

Also, we get

$$\lim_{n\to\infty} Q(Px_n) = Q(\lim_{n\to\infty} P(x_n)) = Q(Q(u)).$$

As $\{Px_n\}$ and $\{Qx_n\}$ are $\mathcal{R}$-preserving and $\lim_{n\to\infty} P(x_n) = Q(u) = \lim_{n\to\infty} Q(x_n)$ and P and Q are $\mathcal{R}$-compatible, we obtain $\lim_{n\to\infty} d(QP(x_n), PQ(x_n)) = 0$.

Now, we show that $Q(u)$ is a coincidence point of P and Q.

Firstly, we suppose that P is $\mathcal{R}$-continuous. By using (4), we have $\lim_{n\to\infty} P(Qx_n) = P(\lim_{n\to\infty} Q(x_n)) = P(Q(u))$. Assume that $Q(u) = z$. By triangle inequality, we have

$$d(Qz, Pz) \le k[d(Qz, Q(Px_n)) + d(Q(Px_n), Pz)] \le$$

$$\le kd(Qz, Q(Px_n)) + k^2[d(Q(Px_n), P(Qx_n)) + d(P(Qx_n), Pz)].$$

Taking the limit as $n \to \infty$, we obtain $d(Qz, Pz) \le 0$, which implies $Qz = Pz$, that is, $z = Q(u)$ is a coincidence point of P and Q.

Alternatively, suppose that $\mathcal{R}$ is (Q, b_d)-self closed. As $\{Qx_n\}$ is $\mathcal{R}$-preserving and $Qx_n \to Qu$, therefore, by using (Q, b_d)-self closedness of $\mathcal{R}$, there exists a subsequence $\{Qx_{n_i}\}$ of $\{Qx_n\}$ such that $[QQx_{n_i}, QQu]$ belongs to $\mathcal{R}$ for all $i \in \mathbb{N} \cup 0$. Since $Qx_{n_i} \to Qu$ and using Proposition 1.3, we obtain

$$d(PQx_{n_i}, PQu) \le \lambda d(QQx_{n_i}, QQu) \quad \text{for all} \quad i \in \mathbb{N} \cup \{0\}.$$

Take $Qu = z$. On using triangular inequality, we get

$$d(Qz, Pz) \le k[d(Qz, Q(Px_{n_i})) + d(Q(Px_{n_i}), Pz)] \le$$

$$\le kd(Qz, Q(Px_{n_i})) + k^2[d(Q(Px_{n_i}), P(Qx_{n_i})) + d(P(Qx_{n_i}), Pz)] \le$$

$$\le kd(Qz, Q(Px_{n_i})) + k^2[d(Q(Px_{n_i}), P(Qx_{n_i})) + \lambda d(Q(Qx_{n_i}), Qz)].$$

Taking the limit as $i \to \infty$, we obtain $d(Qz, Pz) \le 0$, which implies $Qz = Pz$, that is, $z = Q(u)$ is a coincidence point of P and Q.

Theorem 2.3 is proved.

3. Consequences.

Definition 3.1 [8]. *Let P and Q be two self-mappings on X. We say that P is Q-comparative if for any $x, y \in X$, $(Qx, Qy) \in \mathcal{R}^s$, then $(Px, Py) \in \mathcal{R}^s$.*

***Remark* 3.1.** It is clear that P is Q-comparative if and only if $\mathcal{R}^s$ is (P, Q)-closed.

Definition 3.2 [20]. *We say that $(X, d, \mathcal{R}^s)$ is regular if the following condition holds: if $\{x_n\}$ is nondecreasing sequence in X and the point $x \in X$ are such that $x_n \to x$, $(x_n, x) \in \mathcal{R}^s$ for all n.*

***Remark* 3.2.** Clearly, $(X, d, \mathcal{R}^s)$ is regular if and only if $\mathcal{R}^s$ is b_d-self-closed.

We extend the result of Berzig [20] in the framework of b-metric space.

***Corollary* 3.1.** *Let $(X, d, \le)$ be an b-complete b-metric space with $k \ge 1$ and $\mathcal{R}$ a binary relation on X. Assume that P and Q are two self-mappings on X. Suppose that the following conditions hold*:

(a) *$P(X) \subseteq Q(X)$, $Q(X)$ is closed,*

(b) *P is Q-comparative,*

(c) *there exists $x_0 \in X$ such that $(Q(x_0), P(x_0)) \in \mathcal{R}^s$,*

(d) *there exists $\lambda \in \left[0, \dfrac{1}{k}\right)$, such that $d(Px, Py) \le \lambda d(Qx, Qy)$ for all $x, y \in X$ with $(Q(x), Q(y)) \in \mathcal{R}^s$,*

(e) $(X, d, \mathcal{R}^s)$ *is regular.*

Then P, Q *have a point of coincidence.*

Definition 3.3 [10]. *Let* $(X, \leq)$ *be an ordered set and* P *and* Q *two self-mappings on* X. *We say that* P *is* Q*-increasing if for any* $x, y \in X$, $Q(x) \leq Q(y)$, *then* $P(x) \leq P(y)$.

Definition 3.4 [6]. *Given a mapping* $Q : XX$, *we say that an ordered b-metric space* $(X, d, k \geq \geq 1)$ *has* Q*-ICU (increasing-convergence-upper bound) property if* Q*-image of every increasing sequence* $\{x_n\}$ *in* X *such that* $x_n \to^d x$, *is bounded above by* Q*-image of its limit (as an upper bound), i.e.,* $Q(x_n) \leq Q(x)$, *for all* $n \in \mathbb{N}$.

Notice that under the restriction $Q = I$, the identity mapping on X, Definition 3.4 transforms to the notion of ICU property.

***Remark* 3.3.** It is clear that if ordered b-metric space $(X, d, k \geq 1)$ has ICU property (resp., Q-ICU property), then $\leq$ is b_d-self-closed (resp., (Q, b_d)-self-closed).

***Corollary* 3.2.** *Let* $(X, d, \leq)$ *be an b-complete ordered b-metric space with* $k \geq 1$. *Assume that* P *and* Q *are two self-mappings on* X. *Suppose that the following conditions hold*:

(a) $P(X) \subseteq Q(X)$,

(b) P *is* Q*-increasing,*

(c) *there exists* $x_0 \in X$ *such that* $Q(x_0) \leq P(x_0)$,

(d) *there exists* $\lambda \in \left[0, \dfrac{1}{k}\right)$, *such that* $d(Px, Py) \leq \lambda d(Qx, Qy)$ *for all* $x, y \in X$ *with* $Q(x) \leq Q(y)$,

(e) P *and* Q *are compatible,* Q *is continuous, and either* P *is continuous or* $(X, d, \leq)$ *has* Q*-ICU property.*

Then P, Q *have a point of coincidence.*

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