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STOCHASTIC AND DETERMINISTIC BUNDLES
СТОХАСТИЧНІ ТА ДЕТЕРМІНІСТИЧНІ РОЗШАРУВАННЯ

In honour of Professor A. Skorokhod for his 75 birthday

We consider a bundle determined by a classifying map with skelettum smooth in Chen – Souriau sense. We show that the stochastic classifying map is homotopic to a deterministic classifying map on the Hölder loop space.

Розглядаються розшарування, що визначаються класифікуючим відображенням із скелетом, гладким у сенсі Chen – Souriau. Показано, що стохастичне класифікуюче відображення гомотопне детерміністичному класифікуючому відображенню на просторах гьольдерових петель.

1. Introduction. Let $L_x(M)$ be the based loop space of a compact Riemannian manifold, endowed with the Brownian bridge measure.

There are two stochastic de Rham cohomology theories associated to it, for forms almost surely defined:

The first one is the de Rham cohomology in Nualart – Pardoux sense, which is a refinment of Malliavin Calculus [1 – 3]. If the manifold is simply connected, it is equal to the de Rham cohomology of the finite energy based loop space.

The second one is the de Rham cohomology in Chen – Souriau sense, which uses a wide variety of stochastic diffeologies on the based loop space [4 – 7]. The main theorem is that the stochastic cohomology in Chen – Souriau sense is equal to the deterministic de Rham cohomology of the Hölder loop space.

On a compact manifold, associated to a complex bundle, we can study characteristic classes. In particular, Chern – Weil isomorphism states that the complex K-theory tensorized by C of a compact manifold is equal to the complex de Rham cohomology associated to the manifold.

There are a big variety of definition of stochastic bundle (with fiber almost surly defined!) on the loop space:

Either, in Nualart – Pardoux sense, [8 – 10] consider stochastic K-theories, by considering finite dimensional random projectors. Let us recall that one of the main originality of Malliavin Calculus with respects of its preliminary versions (see works of Hida, Elworthy, Fomin, Albeverio, Berezanskii ...) is that the space of test functionals is an algebra. This allows to Léandre [8, 11] to define a stochastic Chern – Character in Nualart – Pardoux sense.

Or in the Chen – Souriau sense, various stochastic Z-valued Stiefel – Whitney classes were defined [8]. Léandre [8] uses a suitable classifying map in Chen – Souriau sense in the classifying space (see the book of Milnor – Stasheff [12] for the study of such objects in the deterministic context).

Moreover, for these two Calculi, in order to understand characteristic classes, various algebraic de Rham complexes were studied in [13] (see the book of Loday [14] for an extensive study of such objects associated to differential algebras).

On the other hand, if the based loop space is simply connected, a line bundle is determined by its curvature. This justifies the fact [7] that a stochastic line bundle (with fiber almost surely defined!) in Chen – Souriau sense is isomorphic to a true line bundle over the Hölder loop space, if this one is simply connected.

This motivates the question: for what theories is a stochastic bundle (with fiber almost-surely defined!) isomorphic to a deterministic bundle?

Let us recall that study of bundles on the loop spaces are motivated by the works of Witten [15, 16] relating the K-theory of the free loop space with elliptic cohomology. Moreover, Jaffe – Lesniewski – Osterwalder [17] have considered some K-theory on the loop space associated to the Wess – Zumino model on the loop space.

The goal of this work is to answer to this question.

Let us recall that we have defined in [18, 19] the notion of stochastic forms smooth in Chen – Souriau sense with skelettum.

We consider in this paper a bundle determined by a classifying map with skelettum smooth in Chen – Souriau sense. We show that the stochastic classifying map is homotopic to a deterministic classifying map on the Hölder loop space. We get:

Main theorem. *A stochastic bundle, smooth in Chen – Souriau sense* (*or in the Froelicher sense*), *with skelettum, over the based loop space is isomorphic to a deterministic bundle on the strong Hölder loop space.*

By strong Hölder loop space, we mean the set of loops $s \to \gamma(s)$ such that

$$\lim_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|^{1/2-\epsilon}} = 0$$

for some small ϵ where d is the Riemannian distance on the manifold.

We refer to the survey of Albeverio [20], and to the 3 surveys of Léandre for the relation between stochastic analysis and mathematical physics [11, 21, 22]. We thank D. Arnal for helpfull comments.

2. The model. We consider the injective limit $M_\infty(C)$ of the linear space $M_n(C)$ of linear applications from C^n into C^n. We consider the Hilbert norm $\|A\|^2 = \sum |Ae_i|^2$ where e_i is the canonical basis of C^n. This norm is compatible with the canonical injection from $M_n(C)$ into $M_{n+1}(C)$. We get a structure of Prehilbert space $M_\infty(C)$ (it is not complete). In order to take derivatives, we consider its natural completion $\overline{M}_\infty(C)$.

$B_\infty(U(n))$ is the set of orthogonal projectors of rank n belonging to $M_\infty(C)$. It is the classifying space [12] of n-dimensional complex Hermitian bundles endowed with an Hermitian structure. There is on $B_\infty(U(n))$ a canonical universal $U(n)$ bundle.

Let $M \subseteq R^n$ a Riemannian manifold of dimension d isometrically imbedded in R^n. We consider the based loop space $L_x(M)$ of strongs Hölder loops $s \to \gamma(s)$ of loops such that

$$\lim_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|^{1/2-\epsilon}} = 0.$$

It is a Banach manifold. It is endowed with the Brownian bridge measure.

Definition 2.1. *A stochastic plot of dimension* m, $\phi_{st} = (U, \phi_i, \Omega_i)_{i \in N}$ *is given by the following data:*

a open subset U *of* R^m;

a countable partition Ω_i *of* $L_x(M)$;

a family of smooth applications $(u, s, y) \to F_i(u, s, y)$ *from* $U \times S^1 \times M$ *bounded with bounded derivatives of all orders, with values in* R^n;

over Ω_i, $\phi_i(u) = \{s \to F_i(u, s, \gamma(s))\}$ *belongs to* $L_x(M)$.

Let $L_x(M)^N$ be the set of loops such that $\sup_{|s-t|<1/N} d(\gamma(s),\ \gamma(t)) < r$ where r is small enough. If γ belongs to $L_x(M)^N$, we can define its polygonal approximation γ^N.

We call $L_{x,\infty}(M)$ the finite energy based loop space.

Theorem 2.1. *A map Ψ from $L_{x,\infty}(M)$ into $B_\infty(U_n)$ is called smooth in the Frechet sense if it is Frechet smooth considered as map with values in $\overline{M}_\infty(C)$.*

Analoguous definition works for a Frechet smooth map from the strong Hölder loop space in the classifying space.

Definition 2.2. *A Frechet smooth map Ψ from $L_{x,\infty}(M)$ in the classifying space is called the skelettum of a map smooth in the Chen – Souriau sense (or in the Froelicher sense) Ψ_{st} with values in the classifying space if for all stochastic plot $u \to \Psi(\phi_{st}^N)$ tends almost surely smoothly in N (for a suitable compactification of R^+) for the smooth topology to a smooth random map $u \to \Psi_{st}(\phi_{st}(u))$ from U into the classifying space (almost surely defined!).*

The application Ψ_{st} defines a map smooth in the stochastic Chen – Souriau sense (or in the stochastic Froelicher sense) from $L_x(M)$ endowed with the Brownian bridge measure according to our diffeology and the definitions of [8] with values in the classifying space. In particular, the pullback by Ψ_{st} of the universal bundle $E_\infty(U(n))$ on the classifying space defines a stochastic bundle in the stochastic Chen – Souriau sense (or in the stochastic Froelicher sense) over $L_x(M)$. We denote $\Psi_{st}^* E_\infty(U_n)$ it. It is a stochastic bundle in Chen – Souriau sense with skelettum $\Psi^* E_\infty(U_n)$.

3. Proof of the main theorem. Let us recall what is a stochastic bundle in Chen – Souriau sense (or in Froelicher sense) over the strong Hölder loop space [8].

Definition 3.1. *Let us consider a map Ψ_{st} smooth in Chen – Souriau sense with values in the classifying space $B_\infty(U_n)$. This means, that for all stochastic plots ϕ_{st} $U \to L_x(M)$, we can define $\Psi_{st}(\phi_{st}(u))$ as a random smooth map from U into the classifying space. Moreover the set of these random smooth maps have to satisfied the two following requirements:*

If $j : U_1 \to U_2$ is a smooth deterministic map from U_1 into U_2, if $\phi_{st,2}$ is a stochastic plot from U_2 into $L_x(M)$, we can consider the composite plot $\phi_{st,1} = \phi_{st,2} \circ j$. Therefore, the following statement has to be satisfied: $\Psi_{st}(\phi_{st}(u_1)) = \Psi_{st}(\phi_{st,2}(j(u_1))$ has smooth random applications from U_1 into the classifying space.

If two stochastic plots $\phi_{st,1}$ and $\phi_{st,2}$ are deduced one from the other by a measurable transformation ψ of the strong Hölder based loop space on a set of probability not equal to 0, we have

$$\Psi_{st}(\phi_{st,1}) = \Psi_{st}(\phi_{st,2}) \circ \psi$$

almost surely.

All stochastic bundle in the Chen – Souriau sense (or in the Froelicher sense) can be seen as isomorphic to a pullback bundle $\Psi_{st}^* E_\infty(U_n)$ [8]. This generalizes this very well known fact in algebraic topology [12] in the deterministic context.

Let us recall the following theorem:

Theorem 3.1. *$\Psi_{st,1}^* E_\infty(U_n)$ and $\Psi_{st,2}^* E_\infty(U_n)$ are isomorphic as soon as there exists a smooth homotopy $\Psi_{st,t}$ between $\Psi_{st,1}$ and $\Psi_{st,2}$. This means that for all stochastic plot ϕ_{st}, $(u,t) \to \Psi_{st,t}(\phi(u))$ is almost surely smooth in t in u with values in the classifying space and satisfies consistency requirements analoguous to Definition* 3.1.

Lemma 3.1. *Let $L_0(R^n)$ be the strong $\frac{1}{2} - \epsilon$-Hölder based loop space of R^n. There exists θ and $p \in 2N$ such that the map which to γ in the strong Hölder based loop space of R^m associates*

$$F(\gamma) = \int\limits_{S^1} |\gamma(s)|^p ds + \int\limits_{S^1 \times S^1} \frac{|\gamma(s) - \gamma(t)|^p}{|s-t|^{1+\theta p}} ds\, dt$$

is Frechet smooth.

Proof. Let us perturb γ by h in the strong Hölder based loop space:

$$\big||\gamma(s) - \gamma(t)|^p - |\gamma(s) + h(s) - \gamma(t) - h(t)|^p\big| \leq$$

$$\leq |h(s) - h(t)|t \sum |\gamma(s) - \gamma(t)|^k |\gamma(s) + h(s) - \gamma(t) - h(t)|^{p-k-1}.$$

If θ is small enough,

$$\int\limits_{S^1 \times S^1} \frac{|\gamma(s) - \gamma(t)|^k |\gamma(s) + h(s) - \gamma(t) - h(t)|^{p-k-1}}{|s-t|^{1+\theta p - 1/2 + \epsilon}} ds\, dt$$

is finite. Therefore the result for the Frechet differentiability of F. Higher Frechet derivabilities are shown in the same way.

The lemma is proved.

$F^{1/p}$ defines a norm. Some Hölder loop space is continuously imbedded in the Banach space of loops such the norm $F^{1/p}$ is finite [23]. This shows us that if $F(\gamma) < C^k$ for some real number k, we can define $\gamma^{C'}$ for $C' \geq C$ if γ is a loop on the strong $\frac{1}{2} - \epsilon$-Hölder loop space of the manifold, imbedded in $L_0(R^n)$. Moreover, $\|\gamma^{C'} - \gamma\|_\infty < r$ for r small enough where $\|.\|_\infty$ is the supremum norm on the based loop space of R^n. Let π be the projection from a small tubular neighborhood from M in R^n into M. Let us consider $\tilde{\gamma}$ a loop in this tubular neighborhood. Let us introduce π_∞ — the Nemystki map:

$$\pi_\infty(\tilde{\gamma}) = \{s \to \pi(\tilde{\gamma})(s)\},$$

π_∞ is Frechet smooth for the strong Hölder topology (see [7] for a proof of this statement in a similar situation).

Proof of the main theorem. Let h be a smooth decreasing function from R^+ into $[0, 1]$ equal to 0 outside a small neighborhood of 0 and equal to 1 in a small neighborhood of 0. Let Ψ be the skelettum of Ψ_{st}. We put

$$\Psi_{st,t} = \Psi\left(\pi_\infty\left(\sum_{n\geq 0} \gamma^{nC\exp[C/t]}(-h(n^{-k}F(\gamma)) + h((n+1)^{-k}F(\gamma)\right)\right).$$

We remark that $\sum_{n\geq 0} -h(n^{-k}F(\gamma)) + h((n+1)^{-k}F(\gamma)) = 1$ and is a series of positives numbers almost all equal to 0. Moreover,

$$\left\|\sum \gamma^{nC\exp[C/t]}(h((n+1)^{-k}F(\gamma)) - h(n^{-k}F(\gamma)) - \gamma\right\|_\infty \leq r.$$

So we can define $\pi_\infty\left(\sum \gamma^{nC\exp[C/t]}\Big(h((n+1)^{-k}F(\gamma)) - h(n^{-k}F(\gamma)\Big)\Big)\right)$. Since F is Frechet smooth for the strong Hölder topology, and since Ψ is the skelettum of Ψ_{st}, we deduce that that $t \to \Psi_{st,t}$ is a smooth homotopy between $\Psi_{st,0} = \Psi_{st}$ and $\Psi_{st,1}$ with values in the classifying space. But $\Psi_{st,1}$ is a Frechet smooth map from the strong Hölder based loop space of M into the classifying space. Therefore the results.

1. *Léandre R.* Cohomologie de Bismut – Nualart – Pardoux et cohomologie de Hochschild entiere // Séminaire de Probabilités XXX in honour of P. A. Meyer and J. Neveu / Eds J. Azéma, M. Emery, M. Yor // Lect. Notes Math. – 1996. – **1626**. – P. 68 – 100.
2. *Léandre R.* Brownian cohomology of an homogeneous manifold // New Trends in Stochastic Analysis / Eds K. D. Elworthy, S. Kusuoka, I. Shigekawa. – World Sci., 1997. – P. 305 – 347.
3. *Léandre R.* Stochastic Adams theorem for a general compact manifold // Rev. Math. Phys. – 2001. – **13**. – P. 1095 – 1133.
4. *Léandre R.* Singular integral homology of the stochastic loop space // Inf. Dim. Anal. Quant. Probab. Rel. Top. – 1998. – **1**. – P. 17 – 31.
5. *Léandre R.* A sheaf theoretical approach to stochastic cohomology // XXXI Symp. Math. Phys. Torun / Ed R. Mrugala // Rep. Math. Phys. – 2000. – **46**. – P. 157 – 164.
6. *Léandre R.* Anticipative Chen – Souriau cohomology and Hochschild cohomology // Conf. Moshe Flato 1999 / Eds G. Dito, D. Sternheimer // Math. Phys. Stud. – 2000. – **22** – P. 185 – 199.
7. *Léandre R.* Stochastic cohomology of Chen – Souriau and line bundle over the Brownian bridge // Probab. Theory Relat. Fields. – 2001. – **120**. – P. 168 – 182.
8. *Léandre R.* Brownian motion and classifying space // Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads // Eds N. Obata, T. Matsui, A. Hora. – World Sci., 2002. – P. 329 – 345.
9. *Léandre R.* Stochastic complex K-theories: Chern character and Bott periodicity // Lett. Math. Phys. – 2003. – **64**. – P. 185 – 202.
10. *Léandre R.* Malliavin Calculus and real Bott periodicity // Stochastic analysis / Eds S. Albeverio, A. Boutet de Monvel, H. Ouerdiane. – Kluwer, 2004. – P. 37 – 51.
11. *Léandre R.* Analysis over loop space and topology // Math. Notes. – 2002. – **72**. – P. 212 – 229.
12. *Milnor J., Stasheff J.* Characteristic classes. – Princeton Univ. Press, 1974.
13. *Léandre R.* Stochastic algebraic de Rham complexes // Acta. Appl. Math. – 2003. – **79**. – P. 217 – 247.
14. *Loday J. L.* Cyclic homology. – Springer, 1998.
15. *Witten E.* The Index of Dirac operator in loop space // Elliptic Curves and Modular Forms in Algebraic Topology / Ed. P. S. Landweber // Lect. Notes. Math. – 1988. – **1326**. – P. 161 – 181.
16. *Kreck M.* Elliptic homology // Espaces de lacets / Eds R. Léandre, S. Paycha, T. Wuerzbacher. – Publ. Univ. Strasbourg, 1996. – P. 63 – 69.
17. *Jaffe A., Lesniewski A., Osterwalder K.* Quantum K-theory. The Chern character // Communs Math. Phys. – 1988. – **118**. – P. 1 – 19.
18. *Léandre R.* Hypoelliptic diffusions and cyclic cohomology // Stochastic Analysis. IV / Eds R. Dalang, M. Dozzi, F. Russo // Prog. Probab. – 2004. – **58**. – P. 165 – 185.
19. *Léandre R.* Stochastic equivariant cohomologies and cyclic cohomology // Ann. Probab. (to appear).
20. *Albeverio S.* Loop groups, random gauge fields, Chern – Simons models, strings: some recent mathematical developments // Espaces de lacets / Eds R. Léandre, S. Paycha, T. Wuerzbacher. – Publ. Univ. Strasbourg, 1996. – P. 5 – 34.
21. *Léandre R.* Cover of the Brownian bridge and stochastic symplectic action // Rev. Math. Phys. – 2000. – **12**. – P. 157 – 164.
22. *Léandre R.* The geometry of Brownian surfaces. – Preprint.
23. *Triebel H.* Interpolation theory, function spaces and differential operators // J. Ambrosius Barth. – 1995.

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