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ON THE EMBEDDING OF WATERMAN CLASS IN THE CLASS H_p^{ω}

ПРО ВКЛЮЧЕННЯ КЛАСУ ВАТЕРМАНА ДО КЛАСУ H_p^{ω}

In this paper the necessary and sufficient condition for the inclusion of class ΛBV in the class H_p^{ω} is found.

Знайдено необхідну і достатню умову для включення класу ΛBV до класу H_p^{ω}.

1. Introduction. The notion of function of bounded variation was introduced by Jordan [1]. Generalized this notion Wiener [2] has considered the class V_p of functions. Young [3] introduced the notion of functions of Φ-variation. In [4] Waterman has introduced the following concept of generalized bounded variation.

Definition 1. *Let* $\Lambda = \{\lambda_n : n \geq 1\}$ *be an increasing sequence of positive numbers such that* $\sum_{n=1}^{\infty}(1/\lambda_n) = \infty$. *A function* f *is said to be of* Λ*-bounded variation* $(f \in \Lambda BV)$, *if for every choice of nonoverlapping intervals* $\{I_n : n \geq 1\}$ *we have*

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty,$$

where $I_n = [a_n, b_n] \subset [0, 1]$ *and* $f(I_n) = f(b_n) - f(a_n)$.

If $f \in \Lambda BV$, then Λ-variation of f is defined to be the supremum of such sums, denoted by $V_{\Lambda}(f)$.

Properties of functions of the class ΛBV as well as the convergence and summability properties of their Fourier series were investigated in [4 – 10].

For everywhere bounded 1-periodic functions, Chanturia [11] introduced the concept of the modulus of variation.

If $\omega(\delta)$ is a modulus of continuity, then H_p^{ω}, $p \geq 1$, denotes the class of functions $f \in L^p([0, 1])$ for wich $\omega(\delta, f)_p = O(\omega(\delta))$ as $\delta \to 0+$, where

$$\omega(\delta, f)_p = \sup_{0<h\leq\delta} \left(\int_0^1 |f(x+h) - f(x)|^p \, dx \right)^{1/p}.$$

The relation between different classes of generalized bounded variation was taken into account in the works of Avdispahic[12], Kovacik [13], Belov [14], Chanturia [15], Akhobadze [16], Medvedeva [17], Kita, Yoneda [18], Goginava [19, 20].

2. Main result. The main result of this paper is presented in the following proposition:

Theorem 1. $\Lambda BV \subset H_p^{\omega}$ *for some* $p \in [1, \infty)$ *if and only if*

$$\overline{\lim_{n\to\infty}} \frac{1}{\omega(1/n)n^{1/p}} \max_{1\leq m\leq n} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} < +\infty. \tag{1}$$

3. *Proof.* The sufficiency of Theorem 1 follows immediately from the following theorem:

Theorem A [21]. *Let* $f \in \Lambda BV$ *and* $p \in [1, \infty)$. *Then*

$$\omega\left(\frac{1}{n}, f\right)_p \leq V_\Lambda(f) \left\{ \frac{1}{n} \max_{1 \leq m \leq n} \frac{m}{\left(\sum_{i=1}^{m} 1/\lambda_i\right)^p} \right\}^{1/p}.$$

Necessity. We suppose that condition (1) is not satisfied. As an example, we construct function from ΛBV which is not in H_p^ω.

Since condition (1) is not satisfied, there exists a sequence of integers $\{\gamma_k : k \geq 1\}$ such that

$$\lim_{k \to \infty} \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1 \leq m \leq \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} = \infty.$$

Let $\{\gamma'_k : k \geq 1\}$ be a sequence of integers for which $2^{\gamma'_k - 1} \leq \gamma_k < 2^{\gamma'_k}$. Then from the fact that $\omega(\delta)$ is nondecreasing we write

$$\frac{2^{1/p}}{\omega(2^{-\gamma'_k})2^{\gamma'_k/p}} \max_{1 \leq m \leq 2^{\gamma'_k}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} \geq$$

$$\geq \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1 \leq m \leq \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i},$$

consequently,

$$\overline{\lim_{k \to \infty}} \frac{1}{\omega(2^{-\gamma'_k})2^{\gamma'_k/p}} \max_{1 \leq m \leq 2^{\gamma'_k}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} < +\infty.$$

Then there exists a sequence of integers $\{n'_k : k \geq 1\} \subset \{\gamma'_k : k \geq 1\}$ such that

$$\lim_{k \to \infty} \frac{1}{\omega(2^{-n'_k})} \frac{1}{\sum_{i=1}^{m(n'_k)} 1/\lambda_i} \left(\frac{m(n'_k)}{2^{n'_k}}\right)^{1/p} < +\infty, \qquad (2)$$

where

$$\max_{1 \leq m \leq 2^{n'_k}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} = \frac{(m(n'_k))^{1/p}}{\sum_{i=1}^{m(n'_k)} 1/\lambda_i}.$$

The following three cases are possible:

a) there exists a sequence of integers $\{s'_k : k \geq 1\} \subset \{n'_k : k \geq 1\}$ such that

$$m(s'_k) < 2^{2s'_{k-1}},$$

b) there exists a sequence of integers $\{q'_k : k \geq 1\} \subset \{n'_k : k \geq 1\}$ such that

$$2^{2q'_{k-1}} \leq m(q'_k) < 2^{q'_k - q'_{k-1}},$$

c) $2^{n'_k - n'_{k-1}} \leq m(n'_k) < 2^{n'_k}$ for all $k \geq k_0$.

First, we consider the case a). We choose a sequence of integers $\{s_k : k \geq 1\} \subset \subset \{s'_k : k \geq 1\}$ such that

$$\sum_{i=1}^{m(s_k)} \frac{1}{\lambda_i} \geq 2^{2s_{k-1}/p}.$$

Then from (2) we get

$$\lim_{k\to\infty} \omega\left(\frac{1}{2^{s_k}}\right) 2^{s_k/p} = 0.$$

Let $\{r_k : k \geq 1\} \subset \{s_k : k \geq 1\}$ such that

$$\omega\left(\frac{1}{2^{r_k}}\right) 2^{r_k/p} \leq 4^{-k}. \tag{3}$$

Consider the function f defined by

$$f(x) = \begin{cases} 2c_j(2^{r_j}x - 1), & \text{if } x \in [2^{-r_j}, 3\cdot 2^{-r_j-1}), \\ -2c_j(2^{r_j}x - 2), & \text{if } x \in [3\cdot 2^{-r_j-1}, 2\cdot 2^{-r_j}) \text{ for } j = 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(x+l) = f(x), \qquad l = \pm 1, \pm 2, \ldots,$$

where

$$c_j = \sqrt{\omega\left(\frac{1}{2^{r_j}}\right) 2^{r_j/p}}.$$

From the construction of the function f and by (3), we get $f \in \Lambda BV$. Next, we shall prove that $f \notin H_p^\omega$. Since $f(x + 2^{-r_j-2}) - f(x) = c_j/2$, for $x \in [2^{-r_j}, 5\cdot 2^{-r_j-2}]$ we get

$$\int_0^1 \left| f(x + 2^{-r_j-2}) - f(x) \right|^p dx \geq$$

$$\geq \int_{2^{-r_j}}^{5\cdot 2^{-r_j-2}} \left| f(x + 2^{-r_j-2}) - f(x) \right|^p dx = \frac{1}{2^p} c_j^p 2^{-r_j-2}.$$

Consequently, by (3) we get

$$\frac{\omega(f, 2^{-r_j})_p}{\omega(2^{-r_j})} \geq \frac{1}{2}\frac{1}{4^{1/p}} \frac{c_j}{\omega(2^{-r_j}) 2^{r_j/p}} \geq \frac{2^{j-1}}{4^{1/p}} \to \infty \quad \text{as} \quad j \to \infty.$$

Now we consider the case b). Let $\{q_k : k \geq 1\} \subset \{q_k' : k \geq 1\}$ such that

$$\frac{1}{\omega(2^{-q_k})} \frac{1}{\sum_{i=1}^{m(q_k)} 1/\lambda_i} \left(\frac{m(q_k)}{2^{q_k}}\right)^{1/p} \geq 4^k. \tag{4}$$

Consider the function g_k defined by

$$g_k(x) = \begin{cases} h_k(2^{q_k}x - 2j + 1), & x \in [(2j-1)/2^{q_k}, 2j/2^{q_k}), \\ -h_k(2^{q_k}x - 2j - 1), & x \in [2j/2^{q_k}, (2j+1)/2^{q_k}) \\ & \text{for } j = m(q_{k-1}), \ldots m(q_k) - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$h_k = \frac{1}{2^k \sum_{j=1}^{m(q_k)} 1/\lambda_j}.$$

Let

$$g(x) = \sum_{k=2}^{\infty} g_k(x), \quad g(x+l) = g(x), \quad l = \pm 1, \pm 2, \ldots .$$

First, we prove that $g \in \Lambda BV$. For every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$, we get

$$\sum_{j=1}^{\infty} \frac{|g(I_j)|}{\lambda_j} \leq 2 \sum_{i=1}^{\infty} h_i \sum_{j=1}^{m(q_i)} \frac{1}{\lambda_j} = 2 \sum_{i=1}^{\infty} \frac{1}{2^i} = 2.$$

Hence, we have $g \in \Lambda BV$.

Next, we shall prove that $g \notin H_p^{\omega}$. Since $g_k(x + 2^{-q_k - 1}) = g_k(x) + h_k/2$, for $x \in$ $\in [(2j-1)2^{-q_k}, (4j-1)2^{-q_k-1})$ and $m(q_k) \geq 2m(q_{k-1})$ we obtain

$$\int_0^1 \left| g\left(x + \frac{1}{2^{q_k+1}}\right) - g(x) \right|^p dx \geq$$

$$\geq \sum_{j=m(q_{k-1})}^{m(q_k)-1} \int_{(2j-1)2^{-q_k}}^{(4j-1)2^{-q_k-1}} \left| g_k\left(x + \frac{1}{2^{q_k+1}}\right) - g_k(x) \right|^p dx =$$

$$= \frac{h_k^p}{2^p} \frac{1}{2^{q_k+1}} (m(q_k) - m(q_{k-1})) \geq \frac{h_k^p}{2^{p+2}} \frac{m(q_k)}{2^{q_k}},$$

consequently, by (4)

$$\frac{\omega(2^{-q_k}, f)_p}{\omega(2^{-q_k})} \geq \frac{1}{2^{1+2/p}} \frac{h_k}{\omega(2^{-q_k})} \left(\frac{m(q_k)}{2^{q_k}} \right)^{1/p} =$$

$$= \frac{1}{2^{1+2/p}\, 2^k} \frac{1}{\omega(2^{-q_k})} \frac{1}{\sum_{i=1}^{m(q_k)} 1/\lambda_i} \left(\frac{m(q_k)}{2^{q_k}} \right)^{1/p} \geq \frac{2^k}{2^{1+2/p}} \to \infty \quad \text{as} \quad k \to \infty.$$

Finally, we consider the case c). Let $\{n_k : k \geq 1\} \subset \{n'_k : k \geq k_0\}$ such that

$$n_k \geq 2n_{k-1} + 1, \tag{5}$$

$$\frac{1}{\omega(2^{-n_k})} \frac{1}{\sum_{i=1}^{m(n_k)} 1/\lambda_i} \left(\frac{m(n_k)}{2^{n_k}} \right)^{1/p} \geq 2^{2n_{k-1}/p + k}. \tag{6}$$

Consider the function φ_k defined by

$$\varphi_k(x) = \begin{cases} d_k(2^{n_k} x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}), \\ -d_k(2^{n_k} x - 2j - 1), & x \in [2j/2^{n_k}, (2j+1)/2^{n_k}) \\ & \text{for } j = 2^{n_{k-1} - n_{k-2}}, \ldots, 2^{n_k - n_{k-1} - 1} - 1, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$d_k = \frac{1}{2^k \sum_{j=1}^{m(n_k)} 1/\lambda_j}.$$

Let

$$\varphi(x) = \sum_{k=3}^{\infty} \varphi_k(x), \quad \varphi(x+l) = \varphi(x), \quad l = \pm 1, \pm 2, \ldots .$$

For every choice of nonoverlapping intervals $\{I_n : n \ge 1\}$, we get

$$\sum_{j=1}^{\infty} \frac{|\varphi(I_j)|}{\lambda_j} \le 2\sum_{i=2}^{\infty} d_i \sum_{j=1}^{2^{n_i - n_{i-1} - 1}} \frac{1}{\lambda_j} \le 2\sum_{i=2}^{\infty} d_i \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} \le 2\sum_{i=2}^{\infty} \frac{1}{2^i} = 1.$$

Hence, we have $\varphi \in \Lambda BV$.

Next, we shall prove that $\varphi \notin H_p^\omega$. From (5) we write

$$\int_0^1 \left|\varphi\left(x + \frac{1}{2^{n_k+1}}\right) - \varphi(x)\right|^p dx \ge$$

$$\ge \sum_{j=2^{n_{k-1}-n_{k-2}}}^{2^{n_k - n_{k-1}} - 1} \int_{(2j-1)2^{-n_k}}^{(4j-1)2^{-n_k - 1}} \left|\varphi_k\left(x + \frac{1}{2^{n_k+1}}\right) - \varphi_k(x)\right|^p dx \ge$$

$$\ge \frac{2^{n_k - n_{k-1}} - 1}{2^{p+2}} \frac{d_k^p}{2^{n_k}} \ge c\frac{d_k^p}{2^{n_{k-1}}} \frac{m(n_k)}{2^{n_k}}.$$

Consequently, by (6)

$$\frac{\omega(2^{-n_k}, f)_p}{\omega(2^{-n_k})} \ge c\frac{d_k}{2^{n_{k-1}/p}} \left(\frac{m(n_k)}{2^{n_k}}\right)^{1/p} \frac{1}{\omega(2^{-n_k})} =$$

$$= \frac{c}{2^{n_{k-1}/p+k}} \left(\sum_{j=1}^{m(n_k)} 1/\lambda_j\right)^{-1} \left(\frac{m(n_k)}{2^{n_k}}\right)^{1/p} \frac{1}{\omega(2^{-n_k})} \ge c\, 2^{n_{k-1}/p} \to \infty \quad \text{as} \quad k \to \infty.$$

Therefore we get $\varphi \notin H_p^\omega$ and the proof of Theorem 1 is complete.

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