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ULAM – HYERS STABILITY ANALYSIS OF A THREE-POINT BOUNDARY-VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

АНАЛІЗ СТАБІЛЬНОСТІ ЗА УЛАМОМ ТА ХАЙЄРСОМ ТРИТОЧКОВОЇ ГРАНИЧНОЇ ЗАДАЧІ ДЛЯ ДРОБОВИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We study the problem of existence and uniqueness of solution of a three-point boundary-value problem for a differential equation of fractional order. Further, we investigate various kinds of the Ulam stability, such as the Ulam – Hyers stability, the generalized Ulam – Hyers stability, the Ulam – Hyers – Rassias stability, and the generalized Ulam – Hyers – Rassias stability for the analyzed problem. We also provide examples to explain our results.

Вивчається проблема існування та єдиності розв'язку триточкової граничної задачі для дробового диференціального рівняння. Крім того, досліджено різні типи стабільності даної проблеми за Уламом, що включають стабільність за Уламом та Хайєрсом, узагальнену стабільність за Уламом та Хайєрсом, стабільність за Уламом, Хайєрсом та Рассіасом, а також узагальнену стабільність за Уламом, Хайєрсом та Рассіасом. Наведено приклади, що пояснюють отримані результати.

1. Introduction. Classical calculus has been generalized from integer order to arbitrary order. At the end of sixteenth century (1695), in a letter to Leibnitz, L. Hospital asked about the derivative of z with respect to t of order $\alpha = 1/2$. This was a question which moved minds towards generalization of integer order derivatives to fractional order. Lacroix was the first person who introduced fractional order derivative for first time [18]. Later on a great contribution in this field was made by researchers like Abel, Fourier, Riemann, Liouville, Grunwald, Letnikov and others, for detail see [11, 15, 20]. Now a days fractional calculus is the most developing and interesting area of research. There has been a lot of development in this field. This course has got great attention and importance for its many applications in various fields of science, engineering and technology like physics, chemistry, dynamics, control system, optimization theory, computer networking systems, mathematical biology, bioengineering, aerodynamics, electrodynamics, signal and image processing, mathematical modeling, etc. (see, for instance, [8, 9, 16, 19]). One of the most well-known area of research in fractional differential equations is concerning to the existence theory. For the last one hundred years this area was very well explored by many authors, for detail see [2, 4, 23, 28, 31]. Benchohra et al. [5], studied existence and uniqueness of solutions to the following antiperiodic boundary-value problem (BVP) provided by

$$ {}^cD^{\alpha}z(t) = \Theta(t, z(t), {}^cD^{\alpha-1}z(t)), \qquad 0 \le t \le 1, \quad 1 \le \alpha < 2, $$

$$ z(0) = -z(1), \qquad z'(0) = -z'(1). $$

In same line Shah et al. [27], studied the following BVP for multipoints:

$$-{}^cD^{\alpha}z(t) = \Theta(t, z(t), {}^cD^{\alpha-1}z(t)), \qquad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

$$z(0) = 0,\ z(1) = \sum_{i=1}^{m-2} \delta_i z(\vartheta_i), \qquad \text{where } \delta_i, \vartheta_i \in (0,1) \quad \text{with } \sum_{i=1}^{m-2} \delta_i(\vartheta_i) < 1.$$

To receive the existence and uniqueness results, the researchers used the classical fixed point theory of cone type. Besides from the aforesaid theory, they also applied pre estimate method known as topological degree method, Schauder's degree method and Brouwer's degree method, etc., for instance, we refer to [1, 3, 10, 26].

Another important area of research which has attracted more attention from researchers is devoted to the stability analysis of differential equations of both classical and fractional order. Historically, S. M. Ulam [29], did a fundamental question about the stability of functional equations which was answered in 1941 by Hyers [12] in Banach spaces. Obloza was the first to report Hyers – Ulam stability for linear differential equations. Later on this result was generalized and extended by Rassias, Jung and others, for instance, we refer to [13, 14, 25]. Recently Benchohra and his co-author [7], established Ulam – Hyers stability, generalized Ulam – Hyers stability, Ulam – Hyers – Rassias stability and generalized Ulam – Hyers – Rassias stability for the following initial value problem of implicit fractional order differential equation:

$${}^cD^{\alpha}z(t) = \Theta(t, z(t), {}^cD^{\alpha}z(t)), \ \ 0 \leq t \leq 1,\ 0 < \alpha \leq 1,$$

$$z(0) = z_0,$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative and $\Theta\colon J \times \Re \times \Re \to \Re$ is a given continuous function, $z_0 \in \Re,\ J = [0, T],\ T > 0$ and $\Re$ denotes the set of real numbers.

The aim of this paper is to investigate the existence and uniqueness results of solution and then to establish the above four types of Ulam stabilities for the following boundary-value implicit fractional order differential equation:

$$\begin{gathered} {}^cD^{\alpha}z(t) = \Theta(t, z(t), D^{\alpha}z(t)), \qquad 0 \leq t \leq 1, \quad 1 \leq \alpha < 2, \\ z(0) = 0, \quad z(1) = \delta z(\vartheta), \quad \delta, \vartheta \in (0,1), \end{gathered} \tag{1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative and $\Theta\colon J \times \Re \times \Re \to \Re$ is a given continuous function. Here we remark that over all in the subject of fractional calculus huge research is in progress in recent times which addresses existence theory, numerical analysis and stability theory, we present some recent work as [33 – 44].

2. Preliminaries. Now to receive the aforementioned goals, we remind some basic definitions and lemmas which will be used in our results.

Definition 1 [22]. *The arbitrary order integral of a function* $h \in L^1([0,T], \Re_+)$ *of order* $\alpha \in \in (0, \infty)$ *is defined by*

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided that integral on the right is pointwise defined on $(0,\infty)$, *where* Γ *is the Euler Gamma function defined as* $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha>0.$

Definition 2 [15]. *The Caputo fractional arbitrary order derivative of order* α *of function* h *is defined by*

$$^{c}D^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-s)^{n-\alpha-1}h^n h(s)ds,$$

provided that integral on the right is pointwise defined on $(0,\infty)$, *where n=*$[\alpha]$*+1 and* $[\alpha]$ *denotes the integer part of the real number* α.

Lemma 1 [17]. *For a fractional derivative and integral of order* α,*we have the following result:*

$$I^{\alpha c}D^{\alpha}h(t)=h(t)+b_0+b_1t+b_2t^2+\ldots+b_{n-1}t^{n-1},$$

where $b_i\in\Re,\ i=0,1,2,3,\ldots,n-1.$

Lemma 2 [5]. *The space* $\tilde{C}$ *defined by*

$$\tilde{C}(J,\Re)=\{z\in C(J,\Re):{}^{c}D^{\alpha}z\in C^2(J,\Re)\}$$

with the norm

$$||z||_\infty=\mathrm{Sup}\,\{|z(t)|:t\in[0,1]\}$$

is a Banach space under the defined norm.

Definition 3 [24]. *The equation* (1) *is said to be Ulam – Hyers stable if there exists a positive real number* $\aleph$ *such that for every* $\varepsilon>0$ *and for each solution* $w\in C^1(J,\Re)$ *of the inequality*

$$|^{c}D^{\alpha}w(t)-\Theta(t,w(t),^{c}D^{\alpha}w(t))|\leq\varepsilon,\qquad t\in J, \tag{2}$$

there exists a solution $z\in C^1(J,\Re)$ *of the equation* (1) *such that* $|w(t)-z(t)|\leq\aleph\varepsilon,\ t\in J.$

Definition 4 [24]. *The equation* (1) *is said to be generalized Ulam – Hyers stable if there exists* $\mu\in C(\Re^+,\Re^+),\ \mu(0)=0,$ *such that for each solution* $z\in C^1(J,\Re^+)$ *of the inequality* (2), *there exists a solution* $w\in C^1(J,\Re^+)$ *of the equation* (1) *such that*

$$|w(t)-z(t)|\leq\mu\varepsilon,\qquad t\in J.$$

Definition 5 [24]. *The equation* (1) *is said to be Ulam – Hyers – Rassias stable with respect to* $\Psi\in C(J,\Re^+)$ *if there exists a nonzero positive real number* $\aleph$ *such that for each* $\varepsilon>0$ *and for each solution* $w\in C^1(J,\Re)$ *of the inequality*

$$|^{c}D^{\alpha}w(t)-\Theta(t,w(t),^{c}D^{\alpha}w(t))|\leq\varepsilon\Psi(t),\qquad t\in J, \tag{3}$$

there exists a solution $z\in C^1(J,\Re)$ *of the equation* (1) *such that*

$$|w(t)-z(t)|\leq\aleph\varepsilon\Psi(t),\qquad t\in J.$$

Definition 6 [24]. *The equation* (1) *is said to be generalized Ulam – Hyers – Rassias stable with respect to* $\Psi \in C(J, \Re)$, *if there exists a real number* $\aleph_\Psi > 0$ *such that for each solution* $w \in \in C^1(J, \Re)$ *of the inequality*

$$|^c D^\alpha w(t) - \Theta(t, w(t), ^c D^\alpha w(t))| \leq \Psi(t), \qquad t \in J, \tag{4}$$

there exists a solution $z \in C^1(J, \Re)$ *of the equation* (1) *such that* $|w(t) - z(t)| \leq \aleph_\Psi \Psi(t)$, $t \in J$.

***Remark* 1.** A function $w \in C^1(J, \Re)$ is a solution of the inequality (2) if there exists a function $h \in C(J, \Re)$ (dependent on z) such that

(I) $|h(t)| \leq \varepsilon$ for all $t \in J$;

(II) $^c D^\alpha w(t) = \Theta(t, w(t), ^c D^\alpha w(t)) + h(t)$, $t \in J$.

Definition 7. *A function* $x \in C^1(J)$ *is said to be a solution of the problem* (1) *if* x *satisfies* (1) *and the boundary conditions on* J.

3. Existence and stability analysis. The concerned section is devoted to establish conditions for the existence of at least one solution to BVP (1) and also to discuss the four different kinds of stability for the afore said problem.

Theorem 1. *Let* $h \in C(J, \Re)$, *then the equivalent Fredholm integral equation of the given BVP* (1) *is* $z(t) = \int_0^1 \mathcal{H}(t, s) h(s) ds$, *where* $\mathcal{H}(t, s)$ *is the Green's function given by*

$$\mathcal{H}(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{\delta t}{\Delta}(\vartheta - s)^{\alpha - 1} - \dfrac{t}{\Delta}(1 - s)^{\alpha - 1}, & 0 \leq t \leq s \leq \vartheta \leq 1, \\ \dfrac{\delta t}{\Delta}(\vartheta - s)^{\alpha - 1} - \dfrac{t}{\Delta}(1 - s)^{\alpha - 1} + (t - s)^{\alpha - 1}, & 0 \leq s \leq t \leq \vartheta \leq 1, \\ -\dfrac{t}{\Delta}(1 - s)^{\alpha - 1}, & 0 \leq \vartheta \leq t \leq s \leq 1, \\ -\dfrac{t}{\Delta}(1 - s)^{\alpha - 1} + (t - s)^{\alpha - 1}, & 0 \leq \vartheta \leq s \leq t \leq 1, \end{cases}$$

where $\Delta = 1 - \delta\vartheta$.

Proof. Let us consider a linear BVP given by

$$^c D^\alpha z(t) = h(t), \qquad 1 \leq \alpha < 2, \qquad t \in [0, 1]. \tag{5}$$

Applying Lemma 1, we have

$$z(t) = b_0 + b_1 t + I^\alpha h(t). \tag{6}$$

By using initial and boundary conditions $z(0) = 0$ and $z(1) = \delta z(\eta)$, we get $b_0 = 0$ and

$$b_1 = \frac{1}{\Delta}[\delta I^\alpha h(\vartheta) - I^\alpha h(1)].$$

Inserting these values of b_0 and b_1 in equation (6), we have

$$z(t) = \frac{t}{\Delta}[\delta I^\alpha h(\vartheta) - I^\alpha h(1)] + I^\alpha h(t) =$$

$$= \frac{\delta t}{\Gamma\alpha}\int_0^{\vartheta}(\vartheta - s)^{\alpha-1}h(s)ds - \frac{t}{\Delta\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}h(s)ds +$$

$$+\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

which implies that

$$z(t) = \int_0^1 \mathcal{H}(t,s)h(s)ds,$$

where $\mathcal{H}(t,s)$ is the Green's function.

Therefore, in view of above theorem, our considered problem becomes

$$z(t) = \int_0^1 \mathcal{H}(t,s)\Theta(s, z(s), {}^c D^{\alpha} z(s))ds, \qquad t \in [0,1]. \tag{7}$$

Theorem 1 is proved.

The given assumptions are useful in the proof of the following theorems. Assume that there exist $\varpi(t) \in C(J, \Re+)$ and a continuous nondecreasing function $\varphi\colon [0,\infty) \to (0,\infty)$ such that

(A_1) $\Theta : J \times \Re \times \Re \to \Re$ is continuous;

(A_2) $|\Theta(t,z,w)| \le \varpi(t)\varphi(w)$ for $z, w \in \Re$;

(A_3) $\varphi(\varpi)\varpi^*\mathcal{H}^* \le \varpi$, where $\varpi^* = \sup\{\varpi(s) : s \in J\}$ and $\mathcal{H}^* = \max_{t\in[0,1]} \int_0^1 |\mathcal{H}(t,s)|ds$;

(A_4) there exists a constant $\lambda > 0$ such that for each $t \in J$ and for all $z, w, \bar{z}, \bar{w} \in \Re$, we have

$$|\Theta(t,z,w) - \Theta(t,\bar{z},\bar{w})| \le \lambda(|z-\bar{z}| + |w-\bar{w}|).$$

Theorem 2. *Under the assumptions $(A_1) - (A_3)$, there exists at least one solution of the concerned BVP* (1).

Proof. To prove the required result, we use Schauder fixed point theorem. Let z_n be a sequence such that $z_n \to z$, where $z \in (J, \Re)$. Let $\sigma > 0$ such that $\|z_n\| \le \sigma$ for each $t \in J$. Then considered a bounded set

$$D = \big\{z \in C(J,\Re) : \|z\| \le \varpi\big\} \subset C(J \times \Re, \Re),$$

and defined an operator

$$\digamma : D \to D \ \text{ by } \digamma z(t) = z(t), \qquad t \in J.$$

We have to show that the operator has at least one fixed point. To prove this, consider

$$|\digamma z_n(t) - \digamma z(t)| = \left|\int_0^1 \mathcal{H}(t,s)[\Theta(s, z_n(s), {}^c D^{\alpha} z_n(s)) - \Theta(s, z(s), {}^c D^{\alpha} z(s))]ds\right| \le$$

$$\le \int_0^1 |\mathcal{H}(t,s)||\Theta(s, z_n(s), {}^c D^{\alpha} z_n(s)) - \Theta(s, z(s), {}^c D^{\alpha} z(s))|ds.$$

By using Lebesgue dominated convergent theorem, we have $|\digamma z_n(t) - \digamma z(t)|_{n\to\infty} \to 0$, which implies that $\digamma$ is continuous.

Next, we show that $\digamma$ is bounded. For this we will show that $\digamma(D) \subseteq D$. Let $z \in D \in$ and consider

$$|\digamma z(t)| = \left|\int_0^1 \mathcal{H}(t,s)\Theta(s,z(s),^c D^\alpha z(s)ds\right| \leq$$

$$\leq \int_0^1 |\mathcal{H}(t,s)||\Theta(s,z(s),^c D^\alpha z(s)|ds \leq$$

$$\leq \varpi(t)\varphi|w| \max_{t\in[0,1]} \int_0^1 |\mathcal{H}(t,s)|ds \leq$$

$$\leq \varpi(t)\varphi|w|\mathcal{H}^* \leq$$

$$\leq \varpi^*\varphi(\varpi)\mathcal{H}^* \leq \varpi.$$

Thus, $|\digamma z(t)| \leq \varpi$.

This shows that $\digamma$ is bounded and hence $\digamma(D) \subseteq D$. For showing that $\digamma$ is equicontinuous, let $t_1,\ t_2 \in J$ with $t_1 < t_2$, consider

$$|\digamma z(t_2) - \digamma z(t_1)| = \left|\int_0^1 \mathcal{H}(t_2,s) - \mathcal{H}(t_1,s)\Theta(s,z(s),^c D^\alpha z(s)ds\right| \leq$$

$$\leq \int_0^1 |\mathcal{H}(t_2,s) - \mathcal{H}(t_1,s)||\Theta(s,z(s),^c D^\alpha \alpha z(s)|ds \leq$$

$$\leq \varpi(t)\varphi(w) \int_0^1 |\mathcal{H}(t_2,s) - \mathcal{H}(t_1,s)|ds.$$

Now, if $t_2 \to t_1$, then $\varpi(t)\varphi(w)\int_0^1 |\mathcal{H}(t_2,s) - \mathcal{H}(t_1,s)|ds \to 0$, consequently, $|\digamma z(t_2) - \digamma z(t_1)| \to \to 0$, which implies that $\digamma$ is equicontinuous. So by Arzelá – Ascoli theorem, $\digamma$ has at least one fixed point and hence the corresponding BVP(1) has at least one solution.

Theorem 2 is proved.

Theorem 3. *Under the assumptions* (A_1) *and* (A_4) *with the additional condition* $2\mathcal{H}^*\lambda < 1$, *the BVP* (1) *has a unique solution.*

Proof. To prove the required result, we use Banach contraction principle. Define a mapping $\digamma : (J \times \Re, \Re) \to C(J \times \Re, \Re)$ by

$$\digamma z(t) = z(t) = \int_0^1 \mathcal{H}(t,s)\Theta(s,z(s))ds.$$

Obviously, $\digamma z(t)$ is continuous, because $\mathcal{H}(t,s)$ and Θ are continuous. Let $z,\bar{z}\in C(J,\Re)$ and $t\in J$, consider

$$|\digamma z(t)-\digamma \bar{z}(t)|=\left|\int\limits_0^1\mathcal{H}(t,s)[\Theta(s,z(s),^cD^\alpha z(s))-\Theta(s,\bar{z}(s),^cD^\alpha\bar{z}(s))]ds\right|\leq$$

$$\leq\int\limits_0^1|\mathcal{H}(t,s)||\Theta(s,z(s),^cD^\alpha z(s))-\Theta(s,\bar{z}(s),^cD^\alpha\bar{z}(s))|ds\leq$$

$$\leq\mathcal{H}^*\lambda(\|z-\bar{z}\|_\infty+\|^cD^\alpha z-^cD^\alpha\bar{z}\|_\infty)\Rightarrow$$

$$\Rightarrow|\digamma z(t)-\digamma\bar{z}(t)|\leq 2\mathcal{H}^*\lambda\|z-\bar{z}\|_\infty.$$

Here,

$$\mathcal{H}^*=\max_{t\in[0,1]}\int\limits_0^1|\mathcal{H}(t,s)|ds.$$

Since $2\mathcal{H}^*\lambda<1$, so by Banach contraction theorem $\digamma$ is contraction and so has a unique fixed poind and hence the corresponding BVP (1) has a unique solution.

Theorem 3 is proved.

Theorem 4. *If the assumptions* (A_1), (A_4) *along with the conditions* $\mathcal{H}^*\lambda\neq 1-\lambda$ *and* $\lambda\neq 1$ *hold, then the BVP*(1) *is Ulam – Hyers stable.*

Proof. Let (A_1), (A_4) and the conditions $\mathcal{H}^*\lambda\neq 1-\lambda$ and $\lambda\neq 1$ hold. Let $w\in C(J,\Re)$ be a solution of the inequality (1) and $z\in(J,\Re)$ be a unique solution of the Cauchy problem

$$^cD^\alpha z(t)=\Theta(t,z(t),^cD^\alpha z(t))\quad\text{for all}\quad t\in J,\quad 1\leq\alpha<2.$$

By Theorem 1, we have

$$z(t)=\int\limits_0^1\mathcal{H}(t,s)h(s)ds,$$

where $h\in C(J,\Re)$ satisfies the functional equation

$$y(t)=\Theta\left(t,\int\limits_0^1\mathcal{H}(t,s)h(s)ds,h(t)\right).$$

Hence, we take

$$\left|w(t)-\int\limits_0^1\mathcal{H}(t,s)h_w(s)ds\right|\leq\varepsilon. \tag{8}$$

On the other hand, we get, for $t\in J$,

$$|w(t)-z(t)|=\left|w(t)-\int\limits_0^1\mathcal{H}(t,s)h_z(s)ds\right|=$$

$$= \left| w(t) - \int\limits_0^1 \mathcal{H}(t,s)h_w(s)ds + \int\limits_0^1 \mathcal{H}(t,s)h_w(s)ds - \int\limits_0^1 \mathcal{H}(t,s)h_z(s)ds \right| \leq$$

$$\leq \left| w(t) - \int\limits_0^1 \mathcal{H}(t,s)h_w(s)ds \right| + \left| \int\limits_0^1 \mathcal{H}(t,s)[h_w(s) - h_z(s)]ds \right| \leq$$

$$\leq \varepsilon + \int\limits_0^1 |\mathcal{H}(t,s)||h_w(s) - h_z(s)|ds \quad \text{(via using the inequality (8))}, \tag{9}$$

where $h_w(t) = \Theta(t, w(t), h_w(t))$ and $h_z(t) = \Theta(t, z(t), h_z(t))$. We have, for all $t \in J$,

$$|h_w(t) - h_z(t)| = |\Theta(t, w(t), h_w(t)) - \Theta(t, z(t), h_z(t))| \leq$$

$$\leq \lambda|w(t) - z(t)| + \lambda|h_w(t) - y_z(t)| \text{ (by using } (A_4)) \leq$$

$$\leq \frac{\lambda}{1-\lambda}|w(t) - z(t)|.$$

Hence from above inequality (9), we obtain

$$|w(t) - z(t)| \leq \varepsilon + \frac{\mathcal{H}^*\lambda}{1-\lambda}|w(t) - z(t)| \Rightarrow$$

$$\Rightarrow |w(t) - z(t)| \leq \frac{\varepsilon}{1 - \dfrac{\mathcal{H}^*\lambda}{1-\lambda}}, \qquad \mathcal{H}^*\lambda \neq 1 - \lambda \quad \text{and} \quad \lambda \neq 1 \Rightarrow$$

$$\Rightarrow |w(t) - z(t)| \leq C\varepsilon,$$

where $C = \dfrac{1}{1 - \dfrac{\mathcal{H}^*\lambda}{1-\lambda}}$ with $\mathcal{H}^*\lambda \neq 1 - \lambda$ and $\lambda \neq 1$.

So, equation (1) is Ulam–Hyers stable. By putting $\Psi(\varepsilon) = C\varepsilon$, $\Psi(0) = 0$, in this case the equation (1) is generalized Ulam–Hyers stable.

Theorem 4 is proved.

Theorem 5. *Assume that* $(A_1), (A_4)$ *hold, then the equation* (1) *is Ulam–Hyers–Rassias stable if* $\mathcal{H}^*\lambda \neq 1 - \lambda$ *and* $\lambda \neq 1$.

Proof. Let $w \in J$ be any solution of the inequality

$$|^c D^\alpha w(t) - \Theta(t, w(t), {}^c D^\alpha w(t))| \leq \varepsilon\Phi(t), \qquad t \in J, \tag{10}$$

and $z \in J$ be the unique solution of the considered Cauchy problem (1). Then, for $\varepsilon > 0$, we get

$$|w(t) - z(t)| \leq \varepsilon\Phi(t). \tag{11}$$

In view of Theorem 1, we get

$$z(t) = \int\limits_0^1 \mathcal{H}(t,s)h(s)ds,$$

where $y \in C(J, \mathbb{R})$ satisfies the functional equation

$$h(t) = \Theta\left(t, \int_0^1 \mathcal{H}(t,s)ds, h(t)\right).$$

Hence, we obtain, from inequality (11),

$$\left| w(t) - \int_0^1 \mathcal{H}(t,s)h_w(s)ds \right| \le \varepsilon\Phi(t). \tag{12}$$

Also, we have, for $t \in J$,

$$|w(t) - z(t)| = \left| w(t) - \int_0^1 \mathcal{H}(t,s)h_z(s)ds \right| \le \varepsilon\Phi(t) =$$

$$= \left| w(t) - \int_0^1 \mathcal{H}(t,s)h_w(s)ds + \int_0^1 \mathcal{H}(t,s)h_w(s)ds - \int_0^1 \mathcal{H}(t,s)h_z(s)ds \right| \le$$

$$\le \left| w(t) - \int_0^1 \mathcal{H}(t,s)h_w(s)ds \right| + \left| \int_0^1 \mathcal{H}(t,s)[h_w(s) - h_z(s)]ds \right|.$$

By using inequality (12), we get

$$|w(t) - z(t)| \le \varepsilon\Phi(t) + \int_0^1 |\mathcal{H}(t,s)||h_w(s) - h_z(s)|ds, \tag{13}$$

where $h_w(t) = \Theta(t, w(t), h_w(t))$ and $h_z(t) = \Theta(t, z(t), h_z(t))$. So, we have, for all $t \in J$,

$$|h_w(t) - h_z(t)| = |\Theta(t, w(t), h_w(t)) - \Theta(t, z(t), h_z(t))| \le$$

$$\le \lambda|w(t) - z(t)| + \lambda|h_w(t) - h_z(t)| \quad \text{(by using } (A_4)\text{)} \quad \Rightarrow \quad |h_w(t) - h_z(t)| \le$$

$$\le \frac{\lambda}{1-\lambda}|w(t) - z(t)|.$$

So, inequality (13) becomes

$$|w(t) - z(t)| \le \varepsilon\Phi(t) + \int_0^1 |\mathcal{H}(t,s)| \frac{\lambda}{1-\lambda}|w(t) - z(t)| \le$$

$$\le \varepsilon\Phi(t) + \mathcal{H}^* \frac{\lambda}{1-\lambda}|w(t) - z(t)| \Rightarrow$$

$$\Rightarrow |w(t) - z(t)| \le \frac{\varepsilon\Phi(t)}{1 - \dfrac{\mathcal{H}^*\lambda}{1-\lambda}} \quad \left(\text{where } \frac{\mathcal{H}^*\lambda}{1-\lambda} \ne 1\right) \quad \Rightarrow$$

$$\Rightarrow |w(t)-z(t)| \le \aleph\varepsilon\Phi(t) \qquad \left(\text{where} \quad \aleph = \frac{1}{1-\dfrac{\mathcal{H}^*\lambda}{1-\lambda}}\right)$$

with $\mathcal{H}^*\lambda \neq 1-\lambda$ and $\lambda \neq 1$. So, equation (1) is Ulam – Hyers – Rassias stable. By taking $\Psi(\varepsilon) = \aleph\varepsilon\Phi(t)$, we have $\Psi(0)=0$. This shows that equation (1) is generalized Ulam – Hyers – Rassias stable.

Theorem 5 is proved.

4. Examples. To demonstrate the established results in previous section, we provide the following examples.

***Example* 1.** We consider

$$\begin{gathered} {}^cD^{\frac{3}{2}}z(t) = \frac{1}{80}\Big(t\sin z(t) - z(t)\cos t\Big) + \frac{|{}^cD^{\frac{3}{2}}z(t)|}{40+|{}^cD^{\frac{3}{2}}z(t)|}, \qquad t\in[0,1], \\ z(0)=0, \qquad z(1)=\frac{1}{4}z\left(\frac{1}{3}\right). \end{gathered} \tag{14}$$

From the BVP (1), we see that $\alpha=\dfrac{3}{2}$, $\delta=\dfrac{1}{4}$, $\vartheta=\dfrac{1}{3}$ and the nonlinear function

$$\Theta(t,z,w) = \frac{1}{80}\Big(t\sin z(t) - z(t)\cos t\Big) + \frac{|{}^cD^{\frac{3}{2}}z(t)|}{40+|{}^cD^{\frac{3}{2}}z(t)|}$$

is clearly continuous and the Green's function is

$$\mathcal{H}(t,s) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}\begin{cases} \dfrac{3t}{11}\left(\dfrac{1}{3}-s\right)^{\frac{1}{2}} - \dfrac{12t}{11}(1-s)^{\frac{1}{2}}, & 0\le t\le s\le \dfrac{1}{3}<1, \\ \dfrac{3t}{11}\left(\dfrac{1}{3}-s\right)^{\frac{1}{2}} - \dfrac{12t}{11}(1-s)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}}, & 0\le s\le t\le \dfrac{1}{3}<1, \\ -\dfrac{12t}{11}(1-s)^{\frac{1}{2}}, & 0<\dfrac{1}{3}\le t\le s\le 1, \\ -\dfrac{12t}{11}(1-s)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}}, & 0<\dfrac{1}{3}\le s\le t\le 1. \end{cases}$$

Now for any $z,\ \bar{z},\ w,\ \bar{w}\in\Re$ and $t\in[0,1]$, we get

$$\begin{gathered} |\Theta(t,z,w)-\Theta(t,\bar z,\bar w)| \le \\ \le \frac{1}{80}|t||\sin z - \sin\bar z| + \frac{1}{80}|\cos t||z-\bar z| + \left|\frac{|w|}{40+|w|} - \frac{|\bar w|}{40+|\bar w|}\right| \le \\ \le \frac{1}{40}|z-\bar z| + \frac{1}{40}|w-\bar w|. \end{gathered}$$

Therefore, we have $\lambda = \dfrac{1}{40}$ and computing

$$\mathcal{H}^* = \max_{t\in[0,1]} \int_0^1 |\mathcal{H}(t,s)|ds \le \frac{2}{11\Gamma\left(\frac{3}{2}\right)} \int_0^1 (1-s)^{\frac{1}{2}} ds = \frac{8}{33\sqrt{\pi}}.$$

Now using Theorem 3, we see that $2\mathcal{H}^*\lambda = \dfrac{2}{165\sqrt{\pi}} < 1$. Hence, the BVP (1) has a unique solution. Further, as $\mathcal{H}^*\lambda \ne 1-\lambda$ and $\lambda \ne 1$ are also satisfied, hence, by Theorem 4, the given BVP (1) is Ulam – Hyers stable and, hence, generalized Ulam – Hyers stable. Also it can be easily derived that the given BVP is Ulam – Hyers – Rassias stable and, hence, generalized Ulam – Hyers – Rassias stable by applying Theorem 5, because it is obvious that $\aleph = \dfrac{1}{1-\dfrac{\mathcal{H}^*\lambda}{1-\lambda}} \ne 0$.

***Example* 2.** We consider

$$
\begin{gathered}
{}^cD^{\frac{3}{2}} z(t) = \frac{2 + |z(t)| + |{}^cD^{\frac{3}{2}} z(t)|}{120\, e^{2t}\big(1 + |z(t)| + |{}^cD^{\frac{3}{2}} z(t)|\big)}, \qquad t \in [0,1], \\
z(0) = 0, \\
z(1) = \frac{1}{3} z\left(\frac{1}{2}\right).
\end{gathered} \tag{15}
$$

From the BVP (2), we see that $\alpha = \dfrac{3}{2}$, $\delta = \dfrac{1}{3}$, $\vartheta = \dfrac{1}{2}$ and the nonlinear function

$$\Theta(t,z,w) = \frac{2 + |z| + |w|}{120\, e^{2t}(1 + |z| + |w|)}$$

is clearly continuous and the Green's function is

$$
\mathcal{H}(t,s) = \frac{1}{\Gamma\left(\frac{3}{2}\right)}
\begin{cases}
\dfrac{2t}{5}\left(\dfrac{1}{2} - s\right)^{\frac{1}{2}} - \dfrac{6t}{5}(1-s)^{\frac{1}{2}}, & 0 \le t \le s \le \dfrac{1}{2} < 1, \\[2mm]
\dfrac{2t}{5}\left(\dfrac{1}{3} - s\right)^{\frac{1}{2}} - \dfrac{6t}{5}(1-s)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}}, & 0 \le s \le t \le \dfrac{1}{2} < 1, \\[2mm]
-\dfrac{6t}{5}(1-s)^{\frac{1}{2}}, & 0 \le \dfrac{1}{2} \le t \le s \le 1, \\[2mm]
-\dfrac{6t}{5}(1-s)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}}, & 0 < \dfrac{1}{2} \le s \le t \le 1.
\end{cases}
$$

Now, for any $z, \bar{z}, w, \bar{w} \in \Re$ and $t \in [0,1]$, we get

$$|\Theta(t,z,w) - \Theta(t,\bar{z},\bar{w})| = \frac{1}{120e^{2t}}\left|\frac{2 + |z| + |w|}{1 + |z| + |w|} - \frac{2 + |\bar{z} + |\bar{w}|}{1 + |\bar{z}| + |\bar{w}|}\right| \le$$

$$\le \frac{1}{120}\left|\frac{2 + |z| + |w|}{1 + |z| + |w|} - \frac{2 + |\bar{z}| + |\bar{w}|}{1 + |\bar{z}| + |\bar{w}|}\right| =$$

$$= \frac{1}{120}\left|\frac{|z| - |\bar{z}| + |w| - |\bar{w}|}{(1+|z|+|w|)(1+|\bar{z}|+|\bar{w}|)}\right| \leq$$

$$\leq \frac{1}{120}\left|\frac{|z-\bar{z}| + |w-\bar{w}|}{(1+|z|+|w|)(1+|\bar{z}|+|\bar{w}|)}\right| \leq$$

$$\leq \frac{1}{120}\Big||z-\bar{z}| + |w-\bar{w}|\Big| \leq$$

$$\leq \frac{1}{120}|z-\bar{z}| + \frac{1}{120}|w-\bar{w}|.$$

Therefore, we have $\lambda = \dfrac{1}{120}$ and computing

$$\mathcal{H}^* = \max_{t\in[0,1]} \int_0^1 |\mathcal{H}(t,s)|ds \leq \int_0^1 \mathcal{H}(1,s)ds \leq \frac{1}{5\Gamma(3/2)} \int_0^1 (1-s)^{\frac{1}{2}}ds = \frac{4}{15\sqrt{\pi}}.$$

Now using Theorem 3, we see that $2\mathcal{H}^*\lambda \leq \dfrac{1}{225\sqrt{\pi}} < 1$, hence, the BVP (2) has a unique solution. Further, as $\mathcal{H}^*\lambda \neq 1-\lambda$ and $\lambda \neq 1$ are also satisfied, hence, by Theorem 4, the given BVP (2) is Ulam – Hyers stable and, hence, generalized Ulam – Hyers stable. Also it can be easily derived that the given BVP is Ulam – Hyers – Rassias stable and, hence, generalized Ulam – Hyers – Rassias stable by applying Theorem 5, because it is obvious that $\aleph = \dfrac{1}{1 - \dfrac{\mathcal{H}^*\lambda}{1-\lambda}} \neq 0$.

5. Conclusion. By use of Arzelá – Ascoli theorem, Lebesgue's dominated convergent theorem and Banach contraction principle, we have deduced the sufficient conditions for existence and uniqueness of solution for our considered problem (1). Also under certain assumptions and conditions, we have deduced the Ulam – Hyers stability results for the solution of the said problem by adopting the definitions from [24].

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