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L. Rempulska, Z. Walczak (Inst. Math. Poznań Univ. Technol., Poland)

ON MODIFIED PICARD AND GAUSS – WEIERSTRASS SINGULAR INTEGRALS

ПРО МОДИФІКОВАНІ СИНГУЛЯРНІ ІНТЕГРАЛИ ПІКАРА ТА ГАУССА – ВЕЙЄРШТРАССА

We introduce certain modification of the Picard and Gauss – Weierstrass singular integrals and we prove approximation theorems for them.

Введено деяку модифікацію сингулярних інтегралів Пікара та Гаусса – Вейєрштрасса, а також доведено апроксимаційні теореми для цих інтегралів.

1. Introduction. ***1.1.*** Let $L^p \equiv L^p(R)$, with fixed $1 \le p \le \infty$, be the space of all real-valued functions, Lebesgue integrable with p-th power over $R := (-\infty, +\infty)$ if $1 \le p < \infty$ and uniformly continuous and bounded on R if $p = \infty$. We define the norm in L^p, as usual, by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \begin{cases} \left(\displaystyle\int_R |f(x)|^p dx \right)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \sup\limits_{x \in R} |f(x)| & \text{if} \quad p = \infty, \end{cases} \tag{1}$$

where $\displaystyle\int_R \equiv \int_{-\infty}^{+\infty}$.

Denote (as usual) by $\omega_1(f; L^p; \cdot)$ and $\omega_2(f; L^p; \cdot)$ the modulus of continuity and the second modulus of smoothness of $f \in L^p$, respectively, i.e.,

$$\omega_i(f; L^p; t) := \sup_{0 \le h \le t} \|\Delta_h^i f(\cdot)\|_p, \qquad t \ge 0, \quad i = 1, 2, \tag{2}$$

where $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^2 f(x) = f(x+h) + f(x-h) - 2f(x)$.

It is known [1] that for $f \in L^p$, $1 \le p \le \infty$, and $i = 1, 2$ the following conditions are satisfied:

i) $\omega_i(f; L^p; \lambda t) \le (1+\lambda)^i \omega_i(f; L^p; t)$ for $\lambda, t \ge 0$;

ii) $\lim_{t \to 0+} \omega_i(f; L^p; t) = 0$;

iii) $\omega_2(f; L^p; t) \le 2\omega_1(f; L^p; t)$ for $t \ge 0$.

1.2. Let $P_r(f; \cdot)$ and $W_r(f; \cdot)$ be the Picard singular integral and the Gauss – Weierstrass singular integral of function $f \in L^p$, respectively, i.e.,

$$P_r(f; x) := \frac{1}{2r} \int_R f(x+t) \exp\left(-\frac{|t|}{r}\right) dt, \tag{3}$$

$$W_r(f; x) := \frac{1}{\sqrt{4\pi r}} \int_R f(x+t) \exp\left(-\frac{t^2}{4r}\right) dt \tag{4}$$

for $x \in R$, $r > 0$ and $r \to 0+$. It is known [1] that these singular integrals are well defined on every space L^p and $P_r(f)$, $W_r(f)$ with every fixed $r > 0$ are linear positive operators from the space L^p to L^p.

The fundamental approximation property of integrals (3) and (4) gives the following theorem.

Theorem A [1–3]. *Let* $f \in L^p$, $1 \leq p \leq \infty$. *Then*

$$\|P_r(f;\cdot) - f(\cdot)\|_p \leq \frac{5}{2}\omega_2(f;L^p;r),$$

$$\|W_r(f;\cdot) - f(\cdot)\|_p \leq \frac{7}{2}\omega_2(f;L^p;\sqrt{r}\,)$$

for all $r > 0$.

The limit properties (as $r \to 0+$) of these integrals were given in many papers and monographs (e.g., [1–3]).

1.3. The order of approximation given in Theorem A can be improved by certain modification of formulas (3) and (4).

Let $N := \{1, 2, \ldots\}$ and $N_0 := N \cup \{0\}$. For fixed $n \in N_0$ and $1 \leq p \leq \infty$, we denote by $L^{p,n}$ the set of all $f \in L^p$ which derivatives $f', \ldots, f^{(n)}$ belong also to L^p. The norm in these $L^{p,n}$ $(n \in N_0,\, 1 \leq p \leq \infty)$ is defined by (1), i.e., for $f \in L^{p,n}$ we have $\|f\|_{p,n} \equiv \|f\|_p$, where $\|f\|_p$ is defined by (1). Moreover, for $f \in L^{p,n}$ there exists norms $\|f^{(k)}\|_p$, $0 \leq k \leq n$, defined analogously to (1). Clearly, $L^{p,0} \equiv L^p$.

Definition. *Let* $f \in L^{p,n}$ *with fixed* $n \in N_0$ *and* $1 \leq p \leq \infty$. *We define the modified Picard and Gauss–Weierstrass singular integrals by formulas*

$$P_{r;n}(f;x) := \frac{1}{2r}\int\limits_R \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x-t)^j \exp\left(-\frac{|t-x|}{r}\right)dt, \tag{5}$$

$$W_{r;n}(f;x) := \frac{1}{\sqrt{4\pi r}}\int\limits_R \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x-t)^j \exp\left(-\frac{(t-x)^2}{4r}\right)dt, \tag{6}$$

for $x \in R$ *and* $r > 0$.

In particular, we have $P_{r;0}(f;\cdot) \equiv P_r(f;\cdot)$ *and* $W_{r;0}(f;\cdot) \equiv W_r(f;\cdot)$ *for* $f \in L^p$.

In Section 2, we shall give some elementary properties of integrals (5) and (6). In Section 3, we shall prove two approximation theorems.

2. Auxiliary results. It is obvious that formulas (5) and (6) can be written in the following form:

$$P_{r;n}(f;x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}\frac{1}{2r}\int\limits_R f^{(j)}(t+x)\, t^j e^{-|t|/r}dt, \tag{5$'$}$$

$$W_{r;n}(f;x) := \sum_{j=0}^{n} \frac{(-1)^j}{j!}\frac{1}{\sqrt{4\pi r}}\int\limits_R f^{(j)}(t+x)\, t^j e^{-t^2/4r}dt \tag{6$'$}$$

for every $f \in L^{p,n}$, $x \in R$, and $r > 0$.

By elementary calculations, we can prove the following lemma.

Lemma 1. *For every* $n \in N_0$ *and* $r > 0$, *we have*

$$I_n := \frac{1}{r}\int\limits_0^{+\infty} t^n e^{-t/r}dt = n!r^n, \tag{7}$$

$$I_n^* := \frac{1}{\sqrt{4\pi r}} \int\limits_0^{+\infty} t^n e^{-t^2/4r} dt =$$

$$= \begin{cases} \dfrac{1}{2} & if \quad n = 0, \\ 2^{k-1}(2k-1)!!r^k & if \quad n = 2k \geq 2, \\ \dfrac{4^k k! r^{k+1/2}}{\sqrt{\pi}} & if \quad n = 2k+1 \geq 1, \end{cases} \tag{8}$$

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)$ *for* $k \in N$.

Applying Lemma 1, we shall prove the main lemma.

Lemma 2. *Let* $n \in N_0$ *and* $1 \leq p \leq \infty$ *be fixed numbers. Then for every* $f \in L^{p,n}$ *and* $r > 0$ *we have*

$$\|P_{r;n}(f;\cdot)\|_p \leq \sum_{j=0}^{n} \frac{\|f^{(j)}\|_p}{j!} I_j = \sum_{j=0}^{n} r^j \|f^{(j)}\|_p, \tag{9}$$

$$\|W_{r;n}(f;\cdot)\|_p \leq \sum_{j=0}^{n} \frac{\|f^{(j)}\|_p}{j!} I_j^*, \tag{10}$$

where I_j *and* I_j^* *are given by* (7) *and* (8).

Formulas (5) *and* (6) *and inequalities* (9) *and* (10) *show that the integrals* $P_{r;n}(f)$ *and* $W_{r;n}(f)$, *with fixed* $n \in N_0$ *and* $r > 0$, *are linear operators from the space* $L^{p,n}$ *into* L^p.

Proof. Inequalities (9) and (10) for $n = 0$ are given in [1].

If $n \in N$ and $p = \infty$, then by (5′), (1), and (7) we get

$$\|P_{r;n}(f;\cdot)\|_\infty \leq \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!} \frac{1}{2r} \int\limits_R |t|^j e^{-|t|/r} dt =$$

$$= \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!} I_j = \sum_{j=0}^{n} \|f^{(j)}\|_\infty r^j, \quad r > 0.$$

If $n \in N$ and $1 \leq p < \infty$, then by (5′), (1), (7), and by Fubini inequality [4] we get

$$\|P_{r;n}(f;\cdot)\|_p = \left\| \sum_{j=0}^{n} \frac{(-1)^j}{j! 2r} \int\limits_R f^{(j)}(t+\cdot)\, t^j e^{-|t|/r} dt \right\|_p \leq$$

$$\leq \sum_{j=0}^{n} \frac{1}{j! 2r} \left(\int\limits_R \left| \int\limits_R f^{(j)}(t+x)\, t^j e^{-|t|/r} dt \right|^p dx \right)^{1/p} \leq$$

$$\leq \sum_{j=0}^{n} \frac{1}{j! 2r} \int\limits_R |t|^j e^{-|t|/r} \left(\int\limits_R \left| f^{(j)}(t+x) \right|^p dx \right)^{1/p} dt =$$

$$= \sum_{j=0}^{n} \frac{\|f^{(j)}\|_p}{j!} I_j = \sum_{j=0}^{n} r^j \|f^{(j)}\|_p, \quad r > 0.$$

Hence, the proof of (9) is completed. The proof of inequality (10) is analogous.

3. Theorems. ***3.1.*** First, we shall prove the theorem on the order of approximation of $f \in L^{p,n}$ by $P_{r;n}(f)$ and $W_{r;n}(f)$.

Theorem 1. *Suppose that $f \in L^{p,n}$ with fixed $n \in N$ and $1 \le p \le \infty$. Then*

$$\|P_{r;n}(f;\cdot) - f(\cdot)\|_p \le (n+2)r^n \omega_1\left(f^{(n)}; L^p; r\right), \tag{11}$$

$$\|W_{r;n}(f;\cdot) - f(\cdot)\|_p \le \frac{2}{n!}\left(I_n^* + r^{-1/2} I_{n+1}^*\right)\omega_1\left(f^{(n)}; L^p; r^{1/2}\right) \le$$

$$\le M_1(n) r^{n/2} \omega_1\left(f^{(n)}; L^p; r^{1/2}\right) \quad \text{for all} \quad r > 0, \tag{12}$$

where I_n^* is given by (8), *$M_1(n)$ is positive constant depending only on n, and $\omega_1\big(f^{(n)}; L^p; \cdot\big)$ is defined by* (2).

Proof. We shall prove only (11), because by (5) and (6) and Lemma 1 the proof of (12) is analogous.

We shall apply the following modified Taylor formula of $f \in L^{p,n}$ with $n \in N$:

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x-t)^j +$$

$$+\frac{(x-t)^n}{(n-1)!}\int_0^1 (1-u)^{n-1}\left\{f^{(n)}(t+u(x-t)) - f^{(n)}(t)\right\} du \tag{13}$$

for a fixed $t \in R$ and every $x \in R$.

Since $\int_R e^{-|t-x|/r} dt = 2r$ for $r > 0$ and $x \in R$, we have by (13) and (5):

$$f(x) = \frac{1}{2r}\int_R f(x) e^{-|t-x|/r} dt = P_{r;n}(f;x) +$$

$$+\frac{1}{2r}\int_R \left(\frac{(x-t)^n}{(n-1)!}\int_0^1 (1-u)^{n-1}\Delta^1_{u(x-t)} f^{(n)}(t)\, du\right) e^{-|t-x|/r} dt \tag{14}$$

for $x \in R$ and $r > 0$.

1. Let $p = \infty$. Then by (2) and the properties of $\omega_1(f; L^\infty; \cdot)$ we have

$$|\Delta^1_{u(x-t)} f^{(n)}(t)| \le \omega_1\left(f^{(n)}; L^\infty; |u(x-t)|\right) \le$$

$$\le \omega_1\left(f^{(n)}; L^\infty; |t-x|\right) \le \left(1 + r^{-1}|t-x|\right)\omega_1\left(f^{(n)}; L^\infty; r\right)$$

for $0 \le u \le 1$ and $t, x \in R$. From this and by (14) we get

$$|f(x) - P_{r;n}(f;x)| \le$$

$$\le \frac{\omega_1\left(f^{(n)}; L^\infty; r\right)}{n! 2r}\int_R \left(|t-x|^n + r^{-1}|t-x|^{n+1}\right) e^{-|t-x|/r} dt =$$

$$= \frac{\omega_1\left(f^{(n)}; L^\infty; r\right)}{n!}\left(I_n + r^{-1} I_{n+1}\right)$$

for $x \in R$ and $r > 0,$ where I_n is given by (7). Applying equality (7), we obtain (11) for $p = \infty$.

2. If $1 \le p < \infty,$ then from (14) we deduce that

$$f(x) - P_{r;n}(f;x) = \frac{1}{2r}\int\limits_R \frac{t^n}{(n-1)!}\left(\int\limits_0^1 (1-u)^{n-1}\Delta^1_{ut} f^{(n)}(x-t)\,du\right)e^{-|t|/r}dt.$$

Applying (similarly as in the proof of Lemma 2) the Fubini inequality, we get

$$\|f(\cdot) - P_{r;n}(f;\cdot)\|_p =$$

$$= \frac{1}{2r(n-1)!}\left(\int\limits_R\left|\int\limits_R t^n e^{-|t|/r}\left(\int\limits_0^1 (1-u)^{n-1}\Delta^1_{ut}f^{(n)}(x-t)\,du\right)dt\right|^p dx\right)^{1/p} \le$$

$$\le \frac{1}{2r(n-1)!}\int\limits_R |t|^n e^{-|t|/r}\left(\int\limits_R\left|\int\limits_0^1 (1-u)^{n-1}\Delta^1_{ut}f^{(n)}(x-t)\,du\right|^p dx\right)^{1/p} dt \le$$

$$\le \frac{1}{2r(n-1)!}\int\limits_R |t|^n e^{-|t|/r}\left(\int\limits_0^1 (1-u)^{n-1}\left(\int\limits_R\left|\Delta^1_{ut}f^{(n)}(x-t)\right|^p dx\right)^{1/p} du\right)dt \le$$

$$\le \frac{1}{2r(n-1)!}\int\limits_R |t|^n e^{-|t|/r}\left(\int\limits_0^1 (1-u)^{n-1}\omega_1\left(f^{(n)};L^p;|ut|\right)du\right)dt \le$$

$$\le \frac{1}{2rn!}\omega_1\left(f^{(n)};L^p;r\right)\int\limits_R |t|^n e^{-|t|/r}\left(1 + r^{-1}|t|\right)dt =$$

$$= \frac{1}{n!}\left(I_n + r^{-1}I_{n+1}\right)\omega_1\left(f^{(n)};L^p;r\right) \quad \text{for} \quad r > 0,$$

where I_n is given by (7). Using (7), we immediately obtain (11) for $1 \le p < \infty$. Thus the proof is completed.

From Theorem 1 and Theorem A we derive the following two corollaries.

***Corollary* 1.** *For every* $f \in L^{p,n}$, $n \in N_0$, $1 \le p \le \infty$, *we have*

$$\lim_{r\to 0+} r^{-n}\|P_{r;n}(f;\cdot) - f(\cdot)\|_p = 0,$$

$$\lim_{r\to 0+} r^{-n/2}\|W_{r;n}(f;\cdot) - f(\cdot)\|_p = 0.$$

***Corollary* 2.** *Let* $f \in L^{p,n}$, $n \in N_0$, $1 \le p \le \infty$, *and let* $f^{(n)} \in \operatorname{Lip}(\alpha; L^p)$ *with a fixed* $0 < \alpha \le 1$, *i.e.,* $\omega_1\left(f^{(n)};L^p;t\right) = O(t^\alpha)$, $t > 0$. *Then*

$$\|P_{r;n}(f;\cdot) - f(\cdot)\|_p = O(r^{n+\alpha}),$$

$$\|W_{r;n}(f;\cdot) - f(\cdot)\|_p = O(r^{(n+\alpha)/2})$$

for $r > 0$.

***Remark* 1.** Theorem 1 shows that the order of approximation of function $f \in L^{p,n}$, with $n \ge 2$ and $1 \le p \le \infty$, by the integrals $P_{r;n}(f)$ and $W_{r;n}(f)$ is better than for $P_r(f)$ and $W_r(f)$ given in Theorem A.

3.2. Now we shall prove the Voronovskaya-type theorem for integrals $P_{r;n}(f)$ and $W_{r;n}(f)$.

Theorem 2. *Suppose that* $f \in L^{\infty,n+2}$ *with a fixed* $n \in N_0$. *Then*

$$P_{r;n}(f;x) - f(x) = \frac{(-1)^n - 1}{2} r^{n+1} f^{(n+1)}(x) +$$

$$+ \frac{1 + (-1)^n}{2}(n+1) r^{n+2} f^{(n+2)}(x) + o\left(r^{n+2}\right) \quad as \quad r \to 0+ \tag{15}$$

and

$$W_{r;n}(f;x) - f(x) = \frac{(-1)^n - 1}{(n+1)!} I^*_{n+1} f^{(n+1)}(x) +$$

$$+ \frac{((-1)^n + 1)(n+1)}{(n+2)!} I^*_{n+2} f^{(n+2)}(x) + o\left(r^{1+n/2}\right) \quad as \quad r \to 0+ \tag{16}$$

for every $x \in R$, *where* I^*_n *is given by* (8).

Proof. Fix $x \in R$ and $f \in L^{\infty,n+2}$. Then $f^{(j)} \in L^{\infty,n+2-j}$, $0 \le j \le n$, and by the Taylor formula we can write

$$f^{(j)}(t) = \sum_{i=0}^{n+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t;x)(t-x)^{n+2-j} \tag{17}$$

for $t \in R$, where $\varphi_j(t) \equiv \varphi_j(t;x)$ is a function such that $\varphi_j(t) t^{n+2-j}$ belongs to L^∞ and $\lim_{t \to x} \varphi_j(t) = \varphi_j(x) = 0$ for every $0 \le j \le n$. Using (17) to formula (5), we get

$$P_{r;n}(f;x) = \frac{1}{2r} \int\limits_R e^{-|t-x|/r} \sum_{j=0}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{n+2-j} \frac{f^{(j+i)}(x)}{i!} (t-x)^{j+i} dt +$$

$$+ \frac{1}{2r} \int\limits_R e^{-|t-x|/r} (t-x)^{n+2} \sum_{j=0}^{n} \frac{(-1)^j}{j!} \varphi_j(t) dt := A_{r;n}(x) + B_{r;n}(x), \quad r > 0. \tag{18}$$

Further, by elementary calculations we have

$$A_{r;n}(x) = \frac{1}{2r} \int_R e^{-|t-x|/r} \left(\sum_{j=0}^{n} \frac{(-1)^j}{j!} \sum_{l=j}^{n} \frac{f^{(l)}(x)}{(l-j)!} (t-x)^l + \right.$$

$$+ \frac{f^{(n+1)}(x)(t-x)^{n+1}}{(n+1)!} \sum_{j=0}^{n} \binom{n+1}{j} (-1)^j +$$

$$\left. + \frac{f^{(n+2)}(x)(t-x)^{n+2}}{(n+2)!} \sum_{j=0}^{n} \binom{n+2}{j} (-1)^j \right) dt$$

and

$$\sum_{j=0}^{n} \frac{(-1)^j}{j!} \sum_{l=j}^{n} \frac{f^{(l)}(x)}{(l-j)!} (t-x)^l =$$

$$= \sum_{l=0}^{n} \frac{f^{(l)}(x)(t-x)^l}{l!} \sum_{j=0}^{l} \binom{l}{j} (-1)^j = f(x) \quad \text{for} \quad n \in N_0,$$

because

$$\sum_{j=0}^{l}\binom{l}{j}(-1)^j=\begin{cases}1 & \text{if} \quad l=0,\\ 0 & \text{if} \quad l\geq 1.\end{cases}$$

Moreover, for $n\in N_0$ we have

$$\sum_{j=0}^{n}\binom{n+1}{j}(-1)^j=(-1)^n, \qquad \sum_{j=0}^{n}\binom{n+2}{j}(-1)^j=(n+1)(-1)^n.$$

Consequently,

$$A_{r;n}(x)=f(x)\frac{1}{2r}\int\limits_R e^{-|t-x|/r}dt+$$

$$+\frac{(-1)^n f^{(n+1)}(x)}{(n+1)!}\frac{1}{2r}\int\limits_R (t-x)^{n+1}e^{-|t-x|/r}dt+$$

$$+\frac{(-1)^n(n+1)f^{(n+2)}(x)}{(n+2)!}\frac{1}{2r}\int\limits_R (t-x)^{n+2}e^{-|t-x|/r}dt.$$

Applying (7), we get for $q\in N_0$:

$$\frac{1}{2r}\int\limits_R (t-x)^q e^{-|t-x|/r}dt=\frac{1}{2r}\int\limits_R t^q e^{-|t|/r}dt=$$

$$=\frac{1+(-1)^q}{2}I_q=\frac{1+(-1)^q}{2}q!r^q.$$

From the above we obtain

$$A_{r;n}(f;x)=f(x)+\frac{(-1)^n-1}{2}r^{n+1}f^{(n+1)}(x)+$$

$$+\frac{1+(-1)^n}{2}(n+1)r^{n+2}f^{(n+2)}(x), \quad r>0. \tag{19}$$

Denoting by

$$\Phi_n(t):=\sum_{j=0}^{n}\frac{(-1)^j}{j!}\varphi_j(t), \quad t\in R,$$

we have $\Phi_n\in L^\infty$ and $\lim_{t\to x}\Phi_n(t)=\Phi_n(x)=0$. Hence, by (18) we get

$$B_{r;n}(x)=\frac{1}{2r}\int\limits_R \Phi_n(t)(t-x)^{n+2}e^{-|t-x|/r}dt,$$

which by the Hölder inequality and (5) and (7) implies that

$$|B_{r;n}(x)|\leq\left\{\frac{1}{2r}\int_R (t-x)^{2n+4}e^{-|t-x|/r}dt\right\}^{1/2}\left\{P_{r;0}\left(\Phi_n^2;x\right)\right\}^{1/2}\equiv$$

$$\equiv\{I_{2n+4}\}^{1/2}\left\{P_{r;0}\left(\Phi_n^2;x\right)\right\}^{1/2}, \quad r>0.$$

Applying Corollary 1 and the properties of $\Phi_n(\cdot)$, we can write

$$\lim_{r\to 0+} P_{r;n}(\Phi_n^2;x) = \Phi_n^2(x) = 0,$$

uniformly on R. From the above and (7) we deduce that

$$B_{r;x}(x) = o\left(r^{n+2}\right) \quad \text{as} \quad r \to 0+ \tag{20}$$

uniformly for $x \in R$.

Collecting (18) – (20), we obtain the desired assertion (15).

The proof of (16) is analogous.

From Theorem 2 we derive the following corollary.

***Corollary* 3.** *Let $f \in L^{\infty,n+2}$ with $n \in N_0$. Then for every $x \in R$ we have:*

$$\lim_{r\to 0+} r^{-n-2}\{P_{r;n}(f;x) - f(x)\} = f^{(n+2)}(x), \quad \textit{if} \quad n \quad \textit{is even number},$$

and

$$\lim_{r\to 0+} r^{-n-1}\{P_{r;n}(f;x) - f(x)\} = -f^{(n+1)}(x), \quad \textit{if} \quad n \quad \textit{is odd number}.$$

The similar equalities hold for $W_{r;n}(f)$.

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