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T. B. Fastovska (Karazin Nat. Univ., Kharkiv)

INVARIANT MANIFOLDS FOR COUPLED NONLINEAR PARABOLIC-HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

ІНВАРІАНТНІ МНОГОВИДИ ПОВ'ЯЗАНИХ НЕЛІНІЙНИХ ПАРАБОЛІКО-ГІПЕРБОЛІЧНИХ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ

We consider an abstract system of parabolic-hyperbolic coupled nonlinear partial differential equations. This system describes, for instance, thermoelastic phenomena in various physical bodies. Several results on the existence of invariant exponentially attracting manifolds for similar problems have been obtained earlier. In the present paper, we prove the existence of this invariant manifold under less restrictive conditions for a wider class of problems.

Розглянуто абстрактну систему параболіко-гіперболічних пов'язаних нелінійних рівнянь з частинними похідними. Ця система описує, наприклад, термопружні явища в різних фізичних тілах. Деякі результати щодо існування інваріантних многовидів, що експоненціально притягують, для задач подібного типу було отримано раніше. В даній роботі доведено існування цього інваріантного многовиду за менш обмежувальних умов для більш широкого класу задач.

Introduction. We consider an abstract system of coupled parabolic-hyperbolic differential equations

$$\Gamma w_{tt} + Aw = F(w, w_t, \theta), \quad t > 0, \quad \text{in} \quad H, \tag{1}$$

$$\theta_t + \eta L\theta = G(w, w_t, \theta) + K(w, w_t), \quad t > 0, \quad \text{in} \quad E, \tag{2}$$

where H and E are infinite-dimensional separable real Hilbert spaces and η is a positive constant. We assume the following hypotheses to hold.

A_1. Γ and A are linear positive self-adjoint operators in H with domains $D(\Gamma)$ and $D(A)$, respectively, such that

$$D\left(A^{1/2}\right) \subset D\left(\Gamma^{1/2}\right).$$

A_2. L is a linear positive self-adjoint operator in E with discrete spectrum, i.e., there exists an orthonormal basis $\{e_k\}$ in E such that

$$Le_k = \lambda_k e_k, \quad 0 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{k\to\infty} \lambda_k = \infty. \tag{3}$$

A_3. F and G are nonlinear globally Lipschitz mappings

$$F\colon D\left(A^{1/2}\right) \times D\left(\Gamma^{1/2}\right) \times D(L^{\alpha}) \to \left[D\left(\Gamma^{1/2}\right)\right]^*,$$

$$G\colon D\left(A^{1/2}\right) \times D\left(\Gamma^{1/2}\right) \times D(L^{\alpha}) \to E$$

for some $0 \le \alpha < 1$, i.e., there exist positive constants M_F and M_G such that

$$\left\| F(w_0, w_1, \theta) - F\left(\tilde{w}_0, \tilde{w}_1, \tilde{\theta}\right) \right\|_{\left[D\left(\Gamma^{1/2}\right)\right]^*} \le$$

$$\le M_F \left(\left\| A^{1/2}(w_0 - \tilde{w}_0) \right\|_H^2 + \left\| \Gamma^{1/2}(w_1 - \tilde{w}_1) \right\|_H^2 + \left\| L^{\alpha}(\theta - \tilde{\theta}) \right\|_E^2 \right)^{1/2} \tag{4}$$

and

$$\left\| G(w_0, w_1, \theta) - G\left(\tilde{w}_0, \tilde{w}_1, \tilde{\theta}\right) \right\|_E \le$$

$$\le M_G \left(\left\| A^{1/2}(w_0 - \tilde{w}_0) \right\|_H^2 + \left\| \Gamma^{1/2}(w_1 - \tilde{w}_1) \right\|_H^2 + \left\| L^{\alpha}(\theta - \tilde{\theta}) \right\|_E^2 \right)^{1/2}. \tag{5}$$

A_4. The mapping

$$K\colon\ D\left(A^{1/2}\right)\times D\left(\Gamma^{1/2}\right)\times\left[D(L^{\beta})\right]^{*}$$

possesses the property

$$\left\| L^{-\beta}(K(w_0, w_1) - K(\tilde{w}_0, \tilde{w}_1))\right\|_E \le$$

$$\le\ M_K\left(\left\|A^{1/2}(w_0-\tilde{w}_0)\right\|_H^2+\left\|\Gamma^{1/2}(w_1-\tilde{w}_1)\right\|_H^2\right)^{1/2} \tag{6}$$

for some $0\le\beta\le 1-\alpha$, where M_K is a positive constant.

System (1), (2) is an abstract representation of certain models of thermoelasticity (see, e.g., [1]).

The goal of this paper is to find sufficient conditions for existence of asymptotically stable invariant manifold of the dynamical system generated by (1), (2) and relying on this fact to formulate a reduction principle for the system considered. This principle allows us to show that the long-time behavior of system (1), (2) is completely determined by a nonlinear elastic system and (possibly) a *finite-dimensional* heat-conduction equation. For a discussion of a general idea of reduction principles we refer to [2].

A similar problem for coupled parabolic-hyperbolic partial differential equations was studied in [3] (but under rather restrictive hypotheses in the right-hand sides) and in [4] under the condition $\Gamma = I$. For instance, in [4] it was proved that the exponentially attracting invariant surface of the form

$$\mathcal{M} = \left\{(w, \overline{w}, \Phi(w, \overline{w}))\colon (w, \overline{w}) \in D\left(A^{1/2}\right)\times D\left(\Gamma^{1/2}\right), \Phi(w, \overline{w}) \in E\right\} \tag{7}$$

exists provided that

$$\eta > \frac{2}{\lambda_1}\left[M_F + \lambda_1^{\alpha} M_G + \lambda_1^{\alpha+\beta} M_K\right], \tag{8}$$

where λ_1 is the minimal point of the spectrum of L.

Our goal is to find less restrictive condition on the constants of the problem which, nevertheless, allows to prove reduction principle. Instead of relations between the diffusivity parameter η and Lipschitz constants in (4) – (6), we obtain a spectral condition. It allows to dispose of any conditions on the relation between η and M_F. In some cases (for certain parameters α and β and space E), we also have no conditions on M_G/η and M_K/η.

The paper is organized as follows. Section 1 contains the statement of our main result on the existence of invariant manifold (Theorem 1), in Section 2 we prove some auxiliary lemmas and the main theorem, in Section 3 we apply the results obtained to some parabolic-hyperbolic problems.

1. Statement of main result. Rewrite system (1), (2) in the following way:

$$\frac{d}{dt}V + \mathfrak{A}V = \mathfrak{B}(V), \quad t > 0, \tag{9}$$

where $V = (w(t), w_t(t), \theta(t))^T$,

$$\mathfrak{A} = \begin{pmatrix} 0 & -I & 0 \\ \Gamma^{-1}A & 0 & 0 \\ 0 & 0 & \eta L \end{pmatrix}$$

is an operator with the domain $D\left(A^{1/2}\right) \times D\left(A^{1/2}\right) \times D(L)$, and

$$\mathfrak{B}(V) = \begin{pmatrix} 0 \\ \Gamma^{-1}F(w, w_t, \theta) \\ G(w, w_t, \theta) + K(w, w_t) \end{pmatrix}.$$

We consider equation (9) with the initial condition

$$V\big|_{t=0} = V_0 \tag{10}$$

in the scale of spaces $\mathcal{H}_\sigma = D\left(A^{1/2}\right) \times D\left(\Gamma^{1/2}\right) \times D(L^\sigma)$ (where $\sigma \in \mathbb{R}$) equipped with the norms

$$|V|_\sigma = \left(\left\|A^{1/2}w_0\right\|_H^2 + \left\|\Gamma^{1/2}w_1\right\|_H^2 + \left\|L^\sigma\theta_0\right\|_E^2\right)^{1/2},$$

where $V = (w_0, w_1, \theta_0)$. The linear problem

$$\frac{d}{dt}V + \mathfrak{A}V = 0 \tag{11}$$

generates in the spaces $\mathcal{H}_\sigma$ C_0-semigroup

$$e^{-\mathfrak{A}t} = \begin{pmatrix} U_t & 0 \\ 0 & e^{-\eta L t} \end{pmatrix}, \tag{12}$$

where U_t is a C_0-group in the space $D\left(A^{1/2}\right) \times D\left(\Gamma^{1/2}\right)$ generated by the equation

$$\Gamma w_{tt} + Aw = 0, \quad t > 0, \quad \text{in} \quad H. \tag{13}$$

Definition 1. *A function $V(t)$ is said to be a mild solution to problem* (9), (10) *on the interval* $[0, T]$ *if* $V(t) \in C\left([0, T]; \mathcal{H}_{\alpha-1/2}\right) \cap L([0, T]; \mathcal{H}_\alpha)$, $V(0) = V_0$ *and for almost all* $t \in [0, T]$

$$V(t) = e^{-\mathfrak{A}t}V_0 + \int_0^t e^{-\mathfrak{A}(t-\tau)}\mathfrak{B}(V(\tau))\,d\tau.$$

The contraction principle allows to establish the following result on the existence of mild solution to (9), (10) (see [4] for the case $\Gamma = I$):

Proposition 1. *Let* $V_0 \in \mathcal{H}_{\alpha-1/2}$ *and let one of the following assertions take place*: $\alpha + \beta < 1$ *or* $\alpha + \beta = 1$ *and* $M_K < \eta$, *where* M_K *is the constant from* (6).

Then there exists the unique mild solution of the Cauchy problem (9), (10) *and, for any* σ *such that* $\alpha - \frac{1}{2} \le \sigma < \min\left(1 - \beta, \frac{1}{2}\right)$,

$$V_t \in C([0, T]; \mathcal{H}_\sigma). \tag{14}$$

It follows from Proposition 1 that for any $\alpha - \frac{1}{2} \le \sigma < \min\left(1 - \beta, \frac{1}{2}\right)$ problem (9), (10) generates a dynamical system $(\mathcal{H}_\sigma, S_t)$ with the evolution operators $S_tV = V(t)$, where $V(t) = (w(t), w_t(t), \theta(t))$ is a mild solution to (1), (2) with the initial data $V = (w_0, w_1, \theta_0)$.

Let P_n be an orthoprojector onto $\operatorname{lin}\{e_1, e_2, \ldots, e_n\}$ and $Q_n = I - P_n$, where $n \in \mathbb{N}$ and $\{e_i\}$ is an orthonormal basis of eigenfunctions of L. We can define such orthoprojectors in each of the spaces $D(L^\sigma)$ due to the fact that these spaces can be identified with the spaces of formal rows $\left\{\sum_{k=1}^\infty c_k e_k \colon \sum_{k=1}^\infty c_k^2 \lambda_k^{2\sigma} < \infty\right\}$. We also denote $P_0 = 0$ and $\lambda_0 = 0$.

Now we state our main result.

Theorem 1. *Let in addition to hypotheses* $A_1 - A_4$ *the following spectral condition hold:*

A_5. *There exists* $N \in \mathbb{N} \cup \{0\}$ *such that*

$$\eta > \frac{2M_F}{\lambda_{N+1} + \lambda_N} + \frac{2M_G\left(\lambda_{N+1}^{\alpha} + \lambda_N^{\alpha}\right)}{\lambda_{N+1} - \lambda_N} + \frac{2M_K\left(\lambda_{N+1}^{\alpha+\beta} + \lambda_N^{\alpha+\beta}\right)}{\lambda_{N+1} - \lambda_N}. \tag{15}$$

We define the orthoprojector

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_N \end{pmatrix}$$

in the space $\mathcal{H}_0$ *and the orthoprojector* $Q = I - P$. *Then, by any* σ *satisfying the inequality* $\alpha - \frac{1}{2} \le \sigma < \min\left(1 - \beta, \frac{1}{2}\right)$, *there exists the function* $\Phi\colon D\left(A^{1/2}\right) \times \times D\left(\Gamma^{1/2}\right) \times P_N D(L^{\sigma}) \to Q_N D(L^{\sigma})$ *such that*

$$\left\| L^{\sigma}(\Phi(W_1) - \Phi(W_2)) \right\|_E \le C_{\sigma} \| W_1 - W_2 \|_{\mathcal{H}_{\sigma}} \tag{16}$$

for any $W_1, W_2 \in \mathcal{H}_{\sigma}$, *where* C_{σ} *is a positive constant. The surface*

$$\mathcal{M} = \Big\{ (w, \overline{w}, \theta + \Phi(w, \overline{w}, \theta)) \colon (w, \overline{w}, \theta) \in D\left(A^{1/2}\right) \times D\left(\Gamma^{1/2}\right) \times P_N D(L^{\sigma}),$$
$$\Phi(w, \overline{w}, \theta) \in Q_N D(L^{\sigma}) \Big\}, \tag{17}$$

is invariant with respect to the semigroup S_t *in the space* H_{σ}, *i.e.*, $S_t\mathcal{M} \subseteq \mathcal{M}$ *and exponentially attracting, i.e., for any mild solution* $V(t)$ *to problem* (9) *there exists* $V_{\mathcal{M}} \in \mathcal{M}$ *such that*

$$\int_0^{\infty} e^{2\mu t} | V(t) - S_t V_{\mathcal{M}} |_{\alpha}^2 < C\left(1 + | V_0 |_{\sigma}\right) \tag{18}$$

and

$$| V(t) - S_t V_{\mathcal{M}} |_{\sigma} < C e^{-\mu t}\left(1 + | V_0 |_{\sigma}\right), \quad t > 0, \tag{19}$$

where $\mu = \eta \dfrac{\lambda_N + \lambda_{N+1}}{2}$.

***Remark* 1.** In the case of $N = 0$, condition (15) coincides with (8) and the manifold $\mathcal{M}$ given by (17) transforms to form (7). Therefore, our Theorem 1 is a generalization of the result from [4] for the case $\Gamma \ne I$.

We also note that if $\alpha = \beta = 0$ and $\lambda_{N+1} - \lambda_N \to \infty$, there exists N_0 such that condition (15) holds for any $N \ge N_0$. Thus, in this case an exponentially attracting invariant manifold exists for any choice of parameters of the problem.

2. Construction of the invariant manifold. To construct an invariant manifold, we consider (following [5]) the integral equation

$$V(t) = B_W[V](t), \quad t \le 0, \tag{20}$$

where $B_W[V] = J_W[\mathfrak{B}(V)]$. Here, $J_W[V]$ is as follows:

$$J_W[V](t) = e^{-\mathfrak{A}t}W - \int_t^0 e^{-\mathfrak{A}(t-\tau)}PV(\tau)d\tau + \int_{-\infty}^t e^{-\mathfrak{A}(t-\tau)}QV(\tau)d\tau \qquad (21)$$

and $W \in P\mathcal{H}_\alpha$.

We seek the solution of equation (20) in the space

$$Y_\alpha = \left\{V\colon e^{\mu t}V \in L^2\left((-\infty, 0]; \mathcal{H}_\alpha\right)\right\},$$

where $\eta\lambda_N < \mu < \eta\lambda_{N+1}$.

To prove Theorem 1, we need some preliminaries.

Lemma 1.

$$\left\| U_t(w_0, w_1)^T \right\|_{D\left(A^{1/2}\right)\times D\left(\Gamma^{1/2}\right)} = \left\| (w_0, w_1)^T \right\|_{D\left(A^{1/2}\right)\times D\left(\Gamma^{1/2}\right)}, \quad t \in \mathbb{R}, \qquad (22)$$

$$\left\| e^{-\eta L t}P_N\theta \right\|_{D(L^\sigma)} \le \lambda_N^\sigma e^{\eta\lambda_N|t|}\| P_N\theta\|_E, \quad t \in \mathbb{R}, \quad \sigma \ge 0, \qquad (23)$$

$$\left\| e^{-\eta L t}P_N\theta \right\|_{D(L^\sigma)} \le \left[\lambda_N^\sigma e^{-\eta\lambda_N t} + \lambda_1^\sigma e^{-\eta\lambda_1 t}\right]\| P_N\theta\|_E, \quad t \in \mathbb{R}, \quad \sigma < 0, \qquad (24)$$

$$\left| e^{-\mathfrak{A}t}QV \right|_\sigma \le \frac{1}{\eta^\sigma}\left[\left(\frac{\sigma}{t}\right)^\sigma + \left(\eta\lambda_{N+1}\right)^\sigma\right]e^{-\eta\lambda_{N+1}t}|QV|_0, \quad t > 0, \quad \sigma > 0, \qquad (25)$$

$$\left| e^{-\mathfrak{A}t}QV \right|_\sigma \le \lambda_{N+1}^\sigma e^{-\eta\lambda_{N+1}t}|QV|_0, \quad t > 0, \quad \sigma \le 0. \qquad (26)$$

The proof is standard. We refer to [5, p. 88] and [6, p. 425] for details.

Lemma 2. *Let a positive self-adjoint operator $\mathcal{L}$ be a generator of a strongly continuous semigroup $e^{-\mathcal{L}t}$ in a Hilbert space $\mathfrak{H}$. Assume that $\lambda_{\min} > 0$ is the minimal point of the spectrum of $\mathcal{L}$. Then, for any $0 \le \beta \le 1$ and $\mu \ge 0$, the mapping*

$$f(t) \to J_{\mathcal{L}}^\beta(f)(t) = \int_{-\infty}^t e^{-\mathcal{L}(t-\tau)}(\mathcal{L} + \mu I)^\beta f(\tau)d\tau$$

is continuous from $L^2(\mathbb{R}; \mathfrak{H})$ into $L^2\left(\mathbb{R}; D\left(\mathcal{L}^{1-\beta}\right)\right)$ and the estimate

$$\int_{\mathbb{R}} \left\| (\mathcal{L} + \mu I)^\alpha J_{\mathcal{L}}^\beta(f)(t) \right\|_{\mathfrak{H}}^2 dt \le \frac{(\lambda_{\min} + \mu)^{2(\alpha+\beta)}}{\lambda_{\min}^2} \int_{\mathbb{R}} \| f(t)\|_{\mathfrak{H}}^2 dt$$

holds for any $-\beta \le \alpha \le 1 - \beta$.

For the proof of the lemma we refer to [4].

Lemma 3. *For every $W \in P\mathcal{H}_\sigma$ and $\sigma \in [0, 1]$, the operator J_W is continuous from Y_0 into Y_σ and, for any $V_1, V_2 \in Y_0$, the estimates*

$$| J_W[PV_1] - J_W[PV_2]|_{Y_\sigma} \le \left(\frac{1}{\mu} + \frac{\lambda_N^\sigma}{\mu - \eta\lambda_N}\right)| PV_1 - PV_2 |_{Y_0} \qquad (27)$$

and

$$| J_W[QV_1] - J_W[QV_2]|_{Y_\sigma} \le \frac{\lambda_{N+1}^\sigma}{\eta\lambda_{N+1} - \mu}| QV_1 - QV_2 |_{Y_0} \qquad (28)$$

hold.

Proof. First, we prove relation (27):

$$|J_W[PV_1] - J_W[PV_2]|_{Y_\sigma} \le \left(\int_{-\infty}^{0} \left(\int_{t}^{0} e^{\mu(t-\tau)} e^{\mu\tau} |PV_1 - PV_2|_0 d\tau \right)^2 dt \right)^{1/2} +$$

$$+ \lambda_N^\sigma \left(\int_{-\infty}^{0} \left(\int_{t}^{0} e^{(\mu-\eta\lambda_N)(t-\tau)} e^{\mu\tau} |PV_1 - PV_2|_0 d\tau \right)^2 dt \right)^{1/2}.$$

Each of the integrals on the right-hand side has the form $\int_{\mathbb{R}} (e * f)(\tau) d\tau$, where $e(t) = e^{\delta t}$ for $t \le 0$ and $e(t) \equiv 0$ for $t > 0$. Hence, using the Fourier transformation and Plancherel formula, we get (27) (for a similar argument see also [4] (Lemma 2.2)).

Similarly, using Lemma 2, we obtain

$$|J_W[QV_1] - J_W[QV_2]|_{Y_\sigma} \le$$

$$\le \frac{1}{\eta^\sigma} \left(\int_{-\infty}^{0} \left(\int_{-\infty}^{t} (\eta L)^\sigma e^{-(\eta L - \mu)(t-\tau)} e^{\mu\tau} |QV_1 - QV_2|_0 d\tau \right)^2 dt \right)^{1/2} \le$$

$$\le \frac{\lambda_{N+1}^\sigma}{\eta\lambda_{N+1} - \mu} |QV_1 - QV_2|_{Y_0}.$$

It is easy to see that relations (27) and (28) entail continuity of J_W.

Lemma 4. *For every $W \in P\mathcal{H}_\alpha$, the operator B_W is continuous from Y_α into itself and*

$$\left| B_{W_1}[V_1] - B_{W_2}[V_2] \right|_{Y_\alpha} \le \left(1 + \lambda_N^{\alpha-\sigma} + \lambda_1^{\alpha-\sigma} \right) |W_1 - W_2|_\sigma + \rho(\mu) |V_1 - V_2|_{Y_\alpha}$$

for every $W_1, W_2 \in P\mathcal{H}_\alpha$ and $V_1, V_2 \in Y_\alpha$, where

$$\rho(\mu) = \frac{M_F}{\mu} + \frac{\lambda_N^\alpha M_G}{\mu - \eta\lambda_N} + \frac{\lambda_N^{\alpha+\beta} M_K}{\mu - \eta\lambda_N} + \frac{\lambda_{N+1}^\alpha M_G}{\eta\lambda_{N+1} - \mu} + \frac{\lambda_{N+1}^{\alpha+\beta} M_K}{\eta\lambda_{N+1} - \mu}.$$

Proof. We can rewrite B_W in the following way:

$$B_W[V] = J_W\left(0, \Gamma^{-1} F[V], P_N G[V] + \frac{1}{\eta^\beta} (\eta L)^\beta \left[L^{-\beta} P_N K[w, \overline{w}] \right] \right) +$$

$$+ \frac{1}{\eta^\beta} \left(0, 0, e^{-\mu t} J_{\mathcal{L}}^\beta \left(e^{\mu t} L^{-\beta} Q_N K \right) \right) + \left(0, 0, e^{-\mu t} J_{\mathcal{L}}^0 \left(e^{\mu t} Q_N G[V] \right) \right),$$

where

$$\mathcal{L} = \eta L - \mu I.$$

Consequently,

$$\left| B_{W_1}[V_1] - B_{W_2}[V_2] \right|_{Y_\alpha} \le$$

$$\le \left(\int_{-\infty}^{0} \left[\lambda_1^{\alpha-\sigma} e^{(\mu-\eta\lambda_1)t} + \lambda_N^{\alpha-\sigma} e^{(\mu-\eta\lambda_N)t} \right]^2 |P_N\theta_1 - P_N\theta_2|_\sigma^2 \right)^{1/2} +$$

$$+ \left(\int_{-\infty}^{0} e^{2\mu t} |(w_1, \overline{w}_1) - (w_2, \overline{w}_2)|^2 \right)^{1/2} +$$

$$+ \left| J_W\left(0, \Gamma^{-1}F[V_1], P_N G[V_1] + \frac{1}{\eta^\beta}(\eta L)^\beta\left[L^{-\beta}P_N K[w_1, \overline{w}_1]\right]\right) - \right.$$

$$\left. - J_W\left(0, \Gamma^{-1}F[V_2], P_N G[V_2] + \frac{1}{\eta^\beta}(\eta L)^\beta\left[L^{-\beta}P_N K[w_2, \overline{w}_2]\right]\right)\right|_{Y_\alpha} +$$

$$+ \frac{1}{\eta^\beta}\left(\int_{-\infty}^{0}\left|J^\beta\left(e^{\mu t}L^{-\beta}Q_N(K[w_1, \overline{w}_1] - K[w_2, \overline{w}_2])\right)\right|^2\right)^{1/2} +$$

$$+ \left(\int_{-\infty}^{0}\left|J^0\left(e^{\mu t}Q_N(G[V_1] - G[V_2])\right)\right|^2\right)^{1/2}.$$

It follows from Lemma 2 and (6) that

$$\frac{1}{\eta^\beta}\left(\int_{-\infty}^{0}\left|J^\beta\left(e^{\mu t}L^{-\beta}Q_N(K[w_1, \overline{w}_1] - K[w_2, \overline{w}_2])\right)\right|^2\right)^{1/2} \le \frac{M_K\,\lambda_{N+1}^{\alpha+\beta}}{\eta\lambda_{N+1} - \mu}|V_1 - V_2|_{Y_\alpha}. \tag{29}$$

Similarly, we get from (5) that

$$\left(\int_{-\infty}^{0}\left|J^0\left(e^{\mu t}Q_N(G[V_1] - G[V_2])\right)\right|^2\right)^{1/2} \le \frac{M_G\,\lambda_{N+1}^{\alpha}}{\eta\lambda_{N+1} - \mu}|V_1 - V_2|_{Y_\alpha}. \tag{30}$$

The statement of the lemma can be easily deduced from (27), (29), and (30).

Lemma 5. *Let* $\mu = \dfrac{\eta\left(\lambda_N + \lambda_{N+1}\right)}{2}$ *and hypotheses* $A_1 - A_5$ *hold. Then equation* (20) *has the unique solution* $V(t; W)$ *in the space* Y_α. *For any* σ *such that* $\sigma < \min\left(1 - \beta, \dfrac{1}{2}\right)$, *this solution possesses the properties*

$$V(t) \in C\left((-\infty, 0], \mathcal{H}_\sigma\right), \tag{31}$$

and

$$\sup_{t \le 0}\left\{e^{\mu t}|V_1 - V_2|_\sigma\right\} \le C_\sigma|W_1 - W_2|_\sigma \tag{32}$$

for any $W_1, W_2 \in P\mathcal{H}_\alpha$, *where* C_σ *is a positive constant. Moreover, for every* $s \in (-\infty, 0)$ *and for almost* $t \in [s, 0]$, *the function* $V(t)$ *satisfies*

$$V(t) = e^{-\mathfrak{A}(t-s)}V(s) + \int_s^t e^{-\mathfrak{A}(t-s)}\mathfrak{B}(V(\tau))\,d\tau. \tag{33}$$

Proof. By Lemma 3,

$$|B_W[V_1] - B_W[V_2]|_{Y_\alpha} \le \rho\left(\frac{\eta\left(\lambda_N + \lambda_{N+1}\right)}{2}\right)|V_1 - V_2|_{Y_\alpha}$$

for every $V_1, V_2 \in Y_\alpha$. Since hypothesis A_5 holds and $\rho\left(\dfrac{\eta\left(\lambda_N + \lambda_{N+1}\right)}{2}\right) < 1$, by contraction principle we have that equation (20) has the unique solution in Y_α.

Relation (33) can be obtained by direct calculation and (31) follows from this representation, therefore, we prove here only (32). As

$$V_1 - V_2 = B_{W_1}[V_1](t) - B_{W_2}[V_2](t) = e^{-\mathfrak{A}t}(W_1 - W_2) + [B_0[V_1](t) - B_0[V_2](t)],$$

we have

$$e^{\mu t}|PV_1 - PV_2|_\sigma \le e^{\mu t}\|(w_1, \overline{w}_1) - (w_2, \overline{w}_2)\|_{D(A^{1/2})\times D(\Gamma^{1/2})} +$$

$$+ e^{(\mu - \eta\lambda_N)t}\|\theta_1 - \theta_2\|_{D(L^\sigma)} + M_F \int_t^0 e^{\mu(t-\tau)} e^{\mu\tau}|V_1 - V_2|_\alpha \, d\tau +$$

$$+ \left(M_G \lambda_N^\sigma + M_K \lambda_N^{\sigma+\beta}\right)\int_t^0 e^{(\mu-\eta\lambda_N)(t-\tau)} e^{\mu\tau}|V_1 - V_2|_\alpha \, d\tau +$$

$$+ \left(M_G \lambda_1^\sigma + M_K \lambda_1^{\sigma+\beta}\right)\int_t^0 e^{(\mu-\eta\lambda_1)(t-\tau)} e^{\mu\tau}|V_1 - V_2|_\alpha \, d\tau.$$

Hence, using the Hölder inequality, we get

$$\sup_{t\le 0}\left\{e^{\mu t}|PV_1 - PV_2|_\sigma\right\} \le |W_1 - W_2|_\sigma + \sup_{t\le 0}\left[M_F\left(\int_t^0 e^{2\mu(t-\tau)}\,d\tau\right)^{1/2} +\right.$$

$$\left. + \left(M_G \lambda_N^\sigma + M_K \lambda_N^{\sigma+\beta}\right)\left(\int_t^0 e^{2(\mu-\eta\lambda_N)(t-\tau)}\,d\tau\right)^{1/2}\right]|V_1 - V_2|_{Y_\alpha} +$$

$$+ \left(M_G \lambda_1^\sigma + M_K \lambda_1^{\sigma+\beta}\right)\left(\int_t^0 e^{2(\mu-\eta\lambda_1)(t-\tau)}\,d\tau\right)^{1/2}|V_1 - V_2|_{Y_\alpha}.$$

This estimate implies that

$$\sup_{t\le 0}\left\{e^{\mu t}|PV_1 - PV_2|_\sigma\right\} \le$$

$$\le |W_1 - W_2|_\sigma + \left(\frac{M_F}{\sqrt{2\mu}} + \frac{M_G\lambda_N^\sigma + M_K\lambda_N^{\sigma+\beta}}{\sqrt{2(\mu - \eta\lambda_N)}} + \frac{M_G\lambda_N^\sigma + M_K\lambda_N^{\sigma+\beta}}{\sqrt{2(\mu - \eta\lambda_N)}}\right)|V_1 - V_2|_{Y_\alpha}. \tag{34}$$

Similarly, we obtain

$$e^{\mu t}|QV_1 - QV_2|_\sigma \le M_G \int_{-\infty}^t \left\|L^\sigma e^{-(\eta L - \mu)(t-\tau)}Q\right\| e^{\mu\tau}|V_1 - V_2|_\alpha \, d\tau +$$

$$+ M_K \int_{-\infty}^t \left\|L^{\sigma+\beta} e^{-(\eta L - \mu)(t-\tau)}Q\right\| e^{\mu\tau}|PV_1 - PV_2|_\alpha \, d\tau.$$

Therefore,

$$\sup_{t\le 0}\left\{e^{\mu t}|QV_1 - QV_2|_\sigma\right\} \le a_1(t)|V_1 - V_2|_{Y_\alpha} + a_2(t)\sup_{t\le 0}\left\{e^{\mu t}|PV_1 - PV_2|_\sigma\right\}, \tag{35}$$

where

$$a_1(t) = M_G\left[\int_{-\infty}^t \left\|L^\sigma e^{-(\eta L - \mu)(t-\tau)}Q\right\|^2 d\tau\right]^{1/2},$$

$$a_2(t) = M_K \int_{-\infty}^t \left\|L^{\sigma+\beta} e^{-(\eta L - \mu)(t-\tau)}Q\right\| d\tau.$$

Using Lemma 1, we have

$$a_1(t) \le$$

$$\le \frac{M_G}{\eta^\sigma}\left[\int_{-\infty}^{t} 2\left(\eta\lambda_{N+1}\right)^{2\sigma} e^{2(\eta\lambda_{N+1}-\mu)(t-\tau)}\,d\tau + \int_{-\infty}^{t} 2\left(\frac{\sigma}{t-\tau}\right)^{2\sigma} e^{2(\eta\lambda_{N+1}-\mu)(t-\tau)}\,d\tau\right]^{1/2} \le$$

$$\le \frac{M_G}{\eta^\sigma}\left[\frac{\left(\eta\lambda_{N+1}\right)^{2\sigma}}{\eta\lambda_{N+1}-\mu} + \frac{\kappa_{2\sigma}}{\eta\lambda_{N+1}-\mu^{1-2\sigma}}\right]^{1/2} \equiv a_1 \tag{36}$$

for any $0<\sigma<1$, where $\kappa_s = s^s\int_0^\infty \tau^{-s}e^{-\tau}d\tau$ for $0<s<1$ and $\kappa_0 = 0$. Since $\kappa_s > 0$, we can also estimate $a_1(t) \le a_1$ for $\sigma \le 0$.

Similarly,

$$a_2(t) \le \frac{M_K}{\eta^{\sigma+\beta}}\int_{-\infty}^{t}\left(\left(\eta\lambda_{N+1}\right)^{\sigma+\beta} + \left(\frac{\sigma+\beta}{t-\tau}\right)^{\sigma+\beta}\right)e^{(\eta\lambda_{N+1}-\mu)(t-\tau)}\,d\tau \le$$

$$\le \frac{M_K}{\eta^{\sigma+\beta}}\left[\frac{\kappa_{\sigma+\beta}}{\left(\eta\lambda_{N+1}-\mu\right)^{1-(\sigma+\beta)}} + \frac{\left(\eta\lambda_{N+1}\right)^{\sigma+\beta}}{\eta\lambda_{N+1}-\mu}\right] \equiv a_2. \tag{37}$$

From (34) and (35) we obtain

$$\sup_{t\le 0}\left\{e^{\mu t}|V_1-V_2|_\sigma\right\} \le (1+a_2)|W_1-W_2|_\sigma +$$

$$+\left(a_1 + (a_2+1)\left(\frac{M_F}{\sqrt{2\mu}} + \frac{M_G\lambda_N^\sigma + M_K\lambda_N^{\sigma+\beta}}{\sqrt{2(\mu-\eta\lambda_N)}} + \frac{M_G\lambda_1^\sigma + M_K\lambda_1^{\sigma+\beta}}{\sqrt{2(\mu-\eta\lambda_1)}}\right)\right)|V_1-V_2|_\alpha. \tag{38}$$

It follows from Lemma 3 that

$$|V_1-V_2|_{Y_\alpha} \le (1-\rho)^{-1}\left(1+\lambda_N^{\alpha-\sigma}+\lambda_1^{\alpha-\sigma}\right)|W_1-W_2|_\sigma.$$

Hence, (38) implies (32).

***Proof of Theorem* 1.** We define the mapping

$$\Phi(W) = \int_{-\infty}^{0} e^{-\eta\mathfrak{A}\tau}Q(G(V(\tau)) + K(w,\overline{w}))\,d\tau = V(0) - W,$$

where $V(t) = (w,\overline{w},\theta)$ is the solution to (20). Relation (32) and standard arguments (see, e.g., [5]) imply that the manifold $\mathcal{M}$ generated by Φ is forward invariant and possesses the property (16). To prove the tracking properties, we rely on the idea applied by Miclavčič [7] in the theory of inertial manifolds. We extend the mild solution $V(t)$ to (9), (10) on the semiaxis $(-\infty, 0]$ by the formula $V(t) = \left(w_0, w_1, (1+|t|A)^{-1}\theta_0\right)$.

Consider the function

$$Z_0(t) = \begin{cases} -V(t) + B_{PV(0)}[V](t), & t \le 0, \\ e^{-\mathfrak{A}t}\left[-V(0) + B_{PV(0)}[V](0)\right], & t > 0, \end{cases}$$

in the space

$$\mathcal{Z} = \left\{ Z(t) \colon |Z|_{\mathcal{Z}}^2 = \int_{-\infty}^{\infty} e^{2\mu t} |Z(t)|_{\alpha}^2 \, dt < \infty \right\}.$$

It is easy to see that

$$|V|_{Y_\alpha}^2 \le C |V_0|_\sigma^2, \quad V(t) \in C((-\infty, 0], \mathcal{H}_\sigma)$$

and

$$|Z_0|_{\mathcal{Z}} \le C(1 + |V_0|_\sigma), \qquad \sup_{t \in \mathbb{R}} \left\{ e^{\mu t} |Z_0|_\sigma \right\} \le C(1 + |V_0|_\sigma)$$

for any σ such that $\alpha - \dfrac{1}{2} \le \sigma < \min\left(1 - \beta, \dfrac{1}{2}\right)$.

Define the integral operator $\mathfrak{R} \colon \mathcal{Z} \to \mathcal{Z}$ by the formula

$$\mathfrak{R}[Z](t) = Z_0(t) - \int_t^{\infty} e^{-\mathfrak{A}(t-\tau)} P[\mathfrak{B}(V(\tau) + Z(\tau)) - \mathfrak{B}(V(\tau))] d\tau +$$

$$+ \int_{-\infty}^{t} e^{-\mathfrak{A}(t-\tau)} Q[\mathfrak{B}(V(\tau) + Z(\tau)) - \mathfrak{B}(V(\tau))] d\tau.$$

The reasons analogous to given in Lemma 4 for the operator $B_W[V]$ lead to the estimate

$$|\mathfrak{R}[Z_1] - \mathfrak{R}[Z_2]|_{\mathcal{Z}} \le q |Z_1 - Z_2|_{\mathcal{Z}}$$

for every $Z_1, Z_2 \in \mathcal{Z}$, i.e., $\mathfrak{R}$ is a contraction in $\mathcal{Z}$. This implies that the equation $Z = \mathfrak{R}[Z]$ has the unique solution $Z \in \mathcal{Z}$ and the estimate

$$|Z|_{\mathcal{Z}} \le (1 - q)^{-1} |Z_0|_{\mathcal{Z}} \le C(1 + |V_0|_\sigma) \tag{39}$$

holds.

Using the same arguments as in Lemma 5, we obtain

$$\sup_{t \in \mathbb{R}} \left\{ e^{\mu t} |Z|_\sigma \right\} \le C(1 + |V_0|_\sigma). \tag{40}$$

It follows from (39) and (40) that $V(t) + Z(t)$ is the desired trajectory emanating from $V^* = V(0) + Z(0)$ [4].

In the case of $\alpha < \min\left(1 - \beta, \dfrac{1}{2}\right)$, we can apply the results obtained to reduce system (1), (2), to the following system:

$$\Gamma w_{tt} + A w = F(w, w_t, v + \Phi(w, w_t, v)), \quad t > 0, \quad \text{in} \quad H, \tag{41}$$

$$v_t + \eta L v = P_N G(w, w_t, v + \Phi(w, w_t, v)) + P_N K(w, w_t), \quad t > 0, \quad \text{in} \quad P_N E \tag{42}$$

with initial conditions

$$w|_{t=0} = w_0, \qquad w_t|_{t=0} = w_1, \qquad v|_{t=0} = v_0. \tag{43}$$

Denote

$$\mathfrak{B}^{\Phi}(w(\tau), w_t(\tau), v(t)) =$$

$$= \begin{bmatrix} 0 \\ F(w(\tau), w_t(\tau), v + \Phi(w(\tau), w_t(\tau), v(t))) \\ P_N G(w(\tau), w_t(\tau), v + \Phi(w(\tau), w_t(\tau), v(\tau))) + P_N K(w(\tau), w_t(\tau)) \end{bmatrix}.$$

A mild solution to this system on the interval $[0, T]$ is a function

$$W(t) = \big(w(t), w_t(t), v\big) \in C\Big([0, T], D\big(A^{1/2}\big) \times D\big(\Gamma^{1/2}\big) \times P_N D(L^{\alpha})\Big) \tag{44}$$

such that

$$\int_0^T \big\| L^{\alpha} \Phi(w(t), w_t(t), v) \big\|_E^2 \, dt < \infty \tag{45}$$

and

$$W(t) = S_t \begin{bmatrix} w_0 \\ w_1 \\ v_0 \end{bmatrix} + \int_0^t S_{t-\tau} B^{\Phi}\big(w(\tau), w_t(\tau), v(\tau)\big) d\tau$$

for almost all $t \in [0, T]$.

Proposition 2. *Let* $(w_0, w_1, v_0) \in D\big(A^{1/2}\big) \times D\big(\Gamma^{1/2}\big) \times P_N D\big(L^{\alpha-1/2}\big)$ *and let conditions of Theorem* 1 *hold. Then problem* (41) – (43) *has a mild solution. If* $\alpha < \min\left(1-\beta, \frac{1}{2}\right)$, *then the solution is unique and any mild solution* $\tilde{W}(t) = (\tilde{w}, \tilde{w}_t, v)$ *to this problem generates the mild solution to problem* (1), (2) *by the formula*

$$\big(w(t), w_t(t), \theta(t)\big) = \big(\tilde{w}(t), \tilde{w}_t(t), v + \Phi(\tilde{w}(t), \tilde{w}_t(t), v)\big). \tag{46}$$

Proof. Let $V(t) = \big(w(t), w_t(t), \theta(t)\big)$ be a mild solution to (9) with initial conditions $V_0 = \big(w_0, w_1, v_0 + \Phi(w_0, w_1, v_0)\big)$. The property (44) holds by Proposition 1. We get (45) from the fact that $PV(t) = W(t)$, $QV(t) = \Phi(W(t))$ and

$$\int_0^T |QV(t)|_{\alpha}^2 \, dt \le \int_0^T |V(t)|_{\alpha}^2 \, dt < \infty,$$

consequently, $W(t)$ is a mild solution to (41) – (43).

Let $W_1 = (w_1, \overline{w}_1, v_1)$, $W_2 = (w_2, \overline{w}_2, v_2) \in D\big(A^{1/2}\big) \times D\big(\Gamma^{1/2}\big) \times P_N D(L^{\alpha})$. If $\alpha < \min\left(1-\beta, \frac{1}{2}\right)$, we use the Lipschitz properties (16) and (4) – (6) to get

$$\begin{aligned}
&\big\| \mathfrak{B}^{\Phi}(W_1) - \mathfrak{B}^{\Phi}(W_2) \big\|_{\mathcal{H}_\alpha}^2 \le \\
&\le \Big[\big(M_F + M_G \big\| L^{\alpha} P_N \big\|\big) \sqrt{\big(1 + C_\alpha^2\big)} + M_K \big\| L^{\alpha+\beta} P_N \big\|\Big] \times \\
&\times \Big(\| w_1 - w_2 \|_{D(A^{1/2})}^2 + \| \overline{w}_1 - \overline{w}_2 \|_{D(\Gamma^{1/2})}^2 + \| v_1 - v_2 \|_{D(L^{\alpha})}^2 \Big)^{1/2} \le \\
&\le \Big(M_F + \lambda_N M_G + \lambda_N^{\alpha+\beta} M_K\Big) \sqrt{\big(1 + C_\alpha^2\big)} \| W_1 - W_2 \|_{\mathcal{H}_\alpha}.
\end{aligned}$$

The uniqueness of the solution to reduced system and relation (46) follow immediately from the estimate obtained.

Theorem 1 and Proposition 2 allow us to obtain a reduction principle for problem (1), (2). The point is that by Theorem 1 for any mild solution $V(t) = \big(w(t), w_t(t), \theta(t)\big)$ to problem (1), (2) with initial data $V_0 \in \mathcal{H}_\sigma$, where $\alpha - \frac{1}{2} \le \sigma < \min\left(1-\beta, \frac{1}{2}\right)$, there exists a mild solution $W(t) = \big(\tilde{w}(t), \tilde{w}_t(t), v(t)\big)$ to reduced system such that

$$\left\| A^{1/2}\left(w(t)-\tilde{w}(t)\right)\right\|_H^2 + \left\|\Gamma^{1/2}\left(w_t(t)-\tilde{w}_t(t)\right)\right\|_H^2 +$$
$$+\left\|\theta(t)-v(t)-\Phi\left(\tilde{w}(t),\tilde{w}_t(t),v(t)\right)\right\|_E^2 \le Ce^{-\mu t}$$

for any $t \ge 0$, where $C, \mu > 0$. Therefore, under conditions of Theorem 1, the long-time behavior of solutions to (1), (2) can be described by solutions to the reduced problem. Moreover, if $\alpha < \min\left(1-\beta, \frac{1}{2}\right)$, then by Proposition 2 every limiting regime of the reduced system appears in system (1), (2).

3. Application to the thermoelastic models. In this section, we give examples of several thermoelastic models with various α and β to illustrate the results obtained.

Consider a one-dimensional Mindlin – Timoshenko system

$$\rho(x)v_{tt} + \beta_0 v_t - \alpha_0 v_{xx} + \mu_0 v + \mu_0 u_x + \delta_0\theta_x = T(v,u,v_x,u_x,v_t,u_t,\theta,\theta_x),$$
$$\rho(x)u_{tt} + \beta_1 u_t - \mu_0 u_{xx} - \mu_0 v_x = D\left(v,u,v_x,u_x,v_t,u_t,\theta,\theta_x\right), \quad t>0, \quad x\in(0,l),$$
$$\theta_t - \eta\theta_{xx} + \delta v_{tx} = G\left(v,u,v_x,u_x,v_t,u_t,\theta,\theta_x\right),$$

with Dirichlet boundary conditions

$$v(0,t) = v(l,t) = u(0,t) = u(l,t) = \theta(0,t) = \theta(l,t) = 0.$$

This system describes the dynamics of heat conductive elastic beam. Here, $v(x,t)$ and $u(x,t)$ are, respectively, the angle of slope of the transverse section and deflection averaged with respect to the thickness of the beam, $\theta(x,t)$ is the temperature variation, $\rho(x)$ is a strictly positive continuous function. For details concerning Mindlin – Timoshenko hypotheses see [8]. We assume that $T, D, G\colon \mathbb{R}^8 \to \mathbb{R}$ are globally Lipshitz functions. Denote $w = (v(x,t), u(x,t))^T$ and rewrite the system in the following way:

$$\begin{gathered}\rho(x)w_{tt} - \alpha_0 w_{xx} = F(w,w_x,w_t,\theta,\theta_x), \quad t>0, \quad x\in(0,l),\\ \theta_t - \eta\theta_{xx} = G(w,w_x,w_t,\theta,\theta_x) - K(w,w_t).\end{gathered} \tag{47}$$

Note that system (47) satisfies conditions A_3, A_4 with $\alpha=\beta=\frac{1}{2}$, $H = \left[L^2(0,l)\right]^2$, $E = L^2(0,l)$, the operators Γ, L and A are given by $\Gamma w = \rho(x)w$, $A = -\alpha_0\,\partial_x^2$ and $L = -\partial_x^2$.

To use Theorem 1, we should check the spectral condition (15). The spectrum of the operator $-\partial_x^2$ with the Dirichlet boundary conditions has the form $\left\{\left(\frac{\pi n}{l}\right)^2 : n\in\mathbb{R}\right\}$, therefore, (15) looks like

$$\frac{2l^2 M_F}{\eta\pi^2\left(2N^2+2N+1\right)} + \frac{2lM_G}{\eta\pi} + \frac{2M_K\left(2N^2+2N+1\right)}{\eta(2N+1)} < 1.$$

This condition holds, for instance, if each item is less then $\frac{1}{3}$. The first one tends to zero if $N\to\infty$, thus, we can find $N_0 \in \mathbb{N}\cup\{0\}$ such that condition

$$\frac{2l^2 M_F}{\eta\pi^2\left(2N^2+2N+1\right)} < \frac{1}{3}$$

holds for any $N > N_0$, namely, $N_0 = [r] + 1$, where $r = -\frac{1}{2} + \sqrt{\frac{3M_F l^2}{\eta\pi^2} - \frac{1}{4}}$ provided $r > 0$ ($[r]$ denotes the integer part of r), otherwise, $N_0 = 0$. Therefore, we exclude M_F from the condition on the constants of the system given in [4]. Thus, we have obtained the following result.

Proposition 3. *Assume that conditions on the parameters of the problem* (47)

$$l < \frac{\pi\eta}{6M_G}$$

and

$$\frac{M_K}{\eta} < \frac{2N+1}{2N^2+2N+1}$$

hold for some $N > N_0$. *Then, for any* σ *such that* $0 \le \sigma < \frac{1}{2}$, *there exists the mapping* $\Phi : \left[L^2(0,l)\right]^2 \times L^2(0,l) \times P_N H_0^{2\sigma}(0,l) \to Q_N H_0^{2\sigma}(0,l)$ *holds and the surface* (17) *is forward invariant in* $\mathcal{H}_\sigma = \left[L^2(0,l)\right]^2 \times L^2(0,l) \times H_0^{2\sigma}(0,l)$ *and exponentially attracting*.

The case of $\alpha = 0$ and $\beta = 0$ in conditions A_3, A_4 is exemplified in a system appearing in classical linear two-dimensional thermoelasticity (for the statement of the problem see, e.g., [1]). We use the well-known decomposition of the displacement vector into potential and solenoidal part $v = \nabla w + \operatorname{rot} u$, where w, u are scalar functions and the rotation of u is defined by $\operatorname{rot} u = (\partial_2 u, -\partial_1 u)$. The function u satisfies a linear wave equation and the problem is reduced to the following coupled system:

$$w_{tt} - \alpha_0 \Delta w + \delta_0 \theta = 0, \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2,$$
$$\theta_t - \eta\Delta\theta + \delta\Delta w_t = 0,$$

with Dirichlet boundary conditions. For this system, condition (15) has the form

$$\frac{2\delta_0}{\eta\left(\lambda_{N+1} + \lambda_N\right)} + \frac{2\delta\left(\lambda_{N+1} + \lambda_N\right)}{\eta\left(\lambda_{N+1} - \lambda_N\right)} < 1. \tag{48}$$

Due to the fact that $\lim_{N\to\infty} \lambda_N = \infty$, the first item tends to zero as $N \to \infty$ and there exists $N_0 \in \mathbb{N}$ such that the first item in (48) is less then $\frac{1}{2}$. Hence, we obtain the existence of the mapping Φ: $\left[L^2(\Omega)\right]^2 \times L^2(\Omega) \times P_N H^{2\sigma}(\Omega) \to Q_N H^{2\sigma}(\Omega)$ for any $-\frac{1}{2} \le \sigma < 0$ such that surface (17) is forward invariant and exponentially attracting provided that

$$\frac{2\delta\left(\lambda_{N_0+1} + \lambda_{N_0}\right)}{\eta\left(\lambda_{N_0+1} - \lambda_{N_0}\right)} < \frac{1}{2}.$$

In two cases considered, Theorem 1 gives us a good sufficient condition for establishing the existence of invariant manifold if M_F is large and M_K is small. Then condition (8) may not be satisfied, while condition (15) holds.

It appears that, in the case of $\alpha + \beta = 0$ and a rectangular domain Ω in $\mathbb{R}^2$, Theorem 1 holds for any choice of the parameters of problem (1), (2). Consider a two-dimensional problem

$$\rho(x)w_{tt} - \alpha_0 \Delta w = F(w, \nabla w, w_t, \theta), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2,$$
$$\theta_t - \eta \Delta\theta = G(w, \nabla w, w_t, \theta),$$

with globally Lipschitz functions F and G. Here, $H = E = L^2(\Omega)$, the operators Γ, L and A are given by $\Gamma w = \rho(x)w$, $A = -\alpha_0\Delta$ and $L = -\Delta$, and the spectral condition turns into

$$\frac{2M_F}{\eta(\lambda_{N+1} + \lambda_N)} + \frac{2M_G}{\eta(\lambda_{N+1} - \lambda_N)} + \frac{2M_K}{\eta(\lambda_{N+1} - \lambda_N)} < 1 \tag{49}$$

As it have been already note, the item including M_F tends to zero when N goes to infinity. We assume that $\Omega = (0, l_1) \times (0, l_2)$ is a rectangle with $\frac{l_1}{l_2}$ rational. For such form of the domain, there is a spectral gap limit result, i.e., $\lambda_{N(k)+1} - \lambda_{N(k)} \to \infty$ when $k \to \infty$ for some subsequence $N(k)$ (see [9]). Therefore, by the choice of N, the expression on the left-hand side of (49) can be made arbitrary small and we establish the existence of invariant exponentially attracting manifold in the space $\mathcal{H}_\sigma$ for $-\frac{1}{2} \le \sigma < \frac{1}{2}$.

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