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Nguyen Manh Hung, Pham Trieu Duong (Hanoi Univ. Educ., Vietnam)

ON THE SMOOTHNESS WITH RESPECT TO TIME VARIABLE OF GENERALIZED SOLUTION OF THE FIRS INITIAL BOUNDARY-VALUE PROBLEM FOR STRONGLY PARABOLIC SYSTEMS IN THE CYLINDER WITH NONSMOOTH BASE

ПРО ГЛАДКІСТЬ ВІДНОСНО ЧАСОВОЇ ЗМІННОЇ УЗАГАЛЬНЕНОГО РОЗВ'ЯЗКУ ПЕРШОЇ ПОЧАТКОВОЇ КРАЙОВОЇ ЗАДАЧІ ДЛЯ СИЛЬНО ПАРАБОЛІЧНИХ СИСТЕМ НА ЦИЛІНДРІ З НЕГЛАДКОЮ ОСНОВОЮ

We consider the first initial boundary-value problem for strongly parabolic systems in infinite cylinder with nonsmooth boundary. We establish conditions for the existence of generalized solutions, an estimate of this solutions, and an estimate of derivative of the solution.

Досліджується перша початкова крайова задача для сильно параболічних систем на нескінченному циліндрі з негладкою межею. Встановлено умови існування узагальненого розв'язку та його оцінку, а також оцінку похідної розв'язку.

1. Introduction. The boundary-value problems for a system of partial differential equations in domains with smooth boundary in nowadays are well studied. General boundary problems for elliptic equations and systems in domain with nonsmooth boundary were considered by V. A. Kondratiev [1], V. G. Mazya, B. A. Plamenevsky [2]. In [3, 4], the single solvability of boundary problems for parabolic equations was established and it was shown that if the right-hand side, the initial, and the boundary functions are infinitely differentiable, then the solution is also infinitely differentiable.

In this paper, we consider the first initial boundary-value problems for strongly parabolic systems in infinite cylinder with nonsmooth boundary.

2. Smoothness of generalized solutions. Let Ω be the bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\Omega_T = \Omega \times (0, T)$, $0 < T \leq \infty$.

We introduce some following spaces:

$H^{l,k}(\Omega_T)$ is a space which consists of functions $u = (u_1, \ldots, u_s)$ from $L_2(\Omega_T)$ such that they have generalized derivatives up to order l with respect to x and up to order k with respect to t belonging to $L_2(\Omega_T)$. The norm in this spaces is defined as follows:

$$\|u\|_{H^{l,k}(\Omega_T)} = \left(\int\limits_{\Omega_T} \sum_{|\alpha|=0}^{l} \left| D^\alpha u \right|^2 dx\,dt + \int\limits_{\Omega_T} \sum_{j=1}^{k} \left| u_{t^j} \right|^2 dx\,dt \right)^{1/2},$$

where

$$|D^\alpha u| = \sum_{i=1}^{s} \left| D^\alpha u_i \right|^2, \quad D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad |u_{t^j}|^2 = \sum_{i=1}^{s} \left| \frac{\partial^j u_i}{\partial t^j} \right|^2,$$

$\overset{\circ}{H}{}^{l,k}(\Omega_T)$ is the closure in $H^{l,k}(\Omega_T)$ of the set consisting of all functions infinitely differentiable in Ω_T which vanish near $S_T = \partial\Omega \times (0, T)$. Let $T = \infty$. We assume that $\Omega_\infty = \Omega \times (0, \infty) \cdot H^{l,k}(e^{-\gamma t}, \Omega_\infty)$ is the space consisting of all functions $u(x, t)$ which have generalized derivatives $D^\alpha u_i$, $\dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \leq l$, $1 \leq i \leq s$, $1 \leq j \leq k$, satisfying

$$\|u\|^2_{H^{l,k}(e^{-\gamma t},\Omega_\infty)} = \int_{\Omega_\infty} \sum_{|\alpha|=0}^{l} \left|D^\alpha u\right| e^{-2\gamma t}\, dx\, dt + \int_{\Omega_\infty} \sum_{j=1}^{k} \left|u_{t^j}\right|^2 e^{-2\gamma t}\, dx\, dt < \infty;$$

$\overset{\circ}{H}{}^{l,k}(e^{-\gamma t}, \Omega_\infty)$ is a closure in $H^{l,k}(\Omega_\infty)$ of the set consisting of all functions infinitely differentiable in Ω_∞ and vanishing near $S_\infty = \partial\Omega \times (0, \infty)$.

Let $\varepsilon = \text{const} > 0$, $L^\infty(0, \infty; \varepsilon, L_2(\Omega))$ be the space consisting of all measurable functions with the norm

$$\|u\|_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))} = \operatorname*{essup}_{t>0} \left\|\varepsilon^{-1} u(x, t)\right\|_{L_2(\Omega)}.$$

In the domain Ω_∞, we consider the first initial boundary-value problem for the following system:

$$(-1)^m \left[\sum_{|p|,|q|=1}^{m} D^p a_{pq}(x, t) D^q u + \sum_{|p|=1}^{m} a_p(x, t) D^p u + a(x, t) u \right] - u_t = f(x, t), \quad (1)$$

where a_{pq}, a_p, a are the bounded measurable complex-valued $s \times s$ matrices, $a_{pq} = (-1)^{|p|+|q|} a^*_{pq}$, a_{pq}, $|p| = |q| = m$, are uniformly continuous in $\overline{\Omega}_\infty = \overline{\Omega} \times [0, \infty)$.

We assume that the considered system (1) is strongly parabolic, i.e., for each $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{C}^S \setminus \{0\}$, we have

$$\sum_{|p|=|q|=m} a_{pq}(x, t) \xi^p \xi^q \eta \overline{\eta} > \mu_0 |\xi|^{2m} |\eta|^2, \qquad (x, t) \in \overline{\Omega}_\infty, \quad (2)$$

where $\xi^p = \xi_1^{p_1} \ldots \xi_n^{p_n}$, μ_0 is a positive constant.

Let (2) be satisfied. For any function $u(x, t)$ from $H^{m,0}(e^{-\gamma t}, \Omega_\infty)$, the inequality

$$\sum_{|p|,|q|=1}^{m} (-1)^{m+|q|} \int_\Omega a_{pq}(x, t) D^q u \overline{D^p u}\, dx + 2\operatorname{Re} \sum_{|p|=1}^{m} a_p D^p u \overline{u}\, dx \geq$$

$$\geq \mu_1 \sum_{|\alpha|=0}^{m} \int_\Omega \left|D^\alpha u\right|^2 dx - \lambda_1 \int_\Omega |u|^2\, dx \quad (3)$$

holds, where $\mu_1 = \text{const} > 0$, $\lambda_1 = \text{const} \geq 0$. This statement can be proved on the basis of the Garding inequality [5].

The function $u(x, t)$ is called a generalized solution of the first boundary-value problem for the system (1) in the space $\overset{\circ}{H}{}^{m,1}(e^{-\gamma t}, \Omega_\infty)$ if $u(x, 0) = 0$ and for each $T > 0$ we have

$$\int_{\Omega_T} \left[-u_t + \sum_{|p|,|q|=1}^{m} (-1)^{m-1+|p|} a_{pq} D^q u \overline{D^p \eta} + \right.$$

$$\left. + \sum_{|p|=1}^{m} (-1)^{m-1} a_p D^p u \overline{\eta} + (-1)^{m-1} a u \overline{\eta} \right] dx\, dt = \int_{\Omega_T} f \overline{\eta}\, dx\, dt \quad (4)$$

for all functions $\eta \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\eta(x, T) = 0$.

Theorem 1. *Let* $\left| \dfrac{\partial a_{pq}}{\partial t}, \dfrac{\partial a_p}{\partial t}, a_{pq}, a_p, a \right| \leq \mu$, $1 \leq |p|, |q| \leq m$, $(x, T) \in \overline{\Omega}_\infty$.

Then there exists a positive constant γ_0 *such that if* $f \in L^\infty(0, \infty; \varepsilon, L_2(\Omega))$ *then* *the first initial boundary-value problem for system* (1) *has a unique generalized solution* $u(x,t)\in \overset{\circ}{H}{}^{m,1}(e^{-(\gamma_0+\varepsilon)t}, L_2(\Omega))$ *and*

$$\|u\|^2_{H^{m,1}(e^{-(\gamma_0+\varepsilon)t},\,\Omega_\infty)} \le C\|f\|^2_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))},$$

where $C=\text{const}$ *is independent of* u, f.

Proof. Suppose that the problem has two solutions u_1, u_2. We denote $u=u_1-u_2$ and set

$$\eta(x,t) = \begin{cases} \int\limits_b^t u(x,\tau)\,d\tau, & 0\le t\le b;\\ 0, & b\le t\le T,\end{cases}$$

then (4) takes the form

$$-\int\limits_{\Omega_b}\eta_{tt}\overline{\eta}\,dx\,dt + \int\limits_{\Omega_b}\Bigg[\sum_{|p|,|q|=1}^{m}(-1)^{m-1+|p|}a_{pq}D^q\eta_t\overline{D^p\eta} + (-1)^{m-1}\sum_{|p|=1}^{m}a_pD^p\eta_t\overline{\eta} + (-1)^{m-1}a\eta_t\overline{\eta}\Bigg]dx\,dt = 0.$$

Let $a_1 = a-(-1)^m\lambda_1 I$. Thus,

$$(-1)^{m-1}\int\limits_{\Omega_b}\Bigg[\sum_{|p|,|q|=1}^{m}(-1)^{|p|}a_{pq}D^q\eta_t\overline{D^p\eta} + (-1)^m\lambda_1\eta_t\overline{\eta} + \sum_{|p|=1}^{m}a_pD^p\eta_t\overline{\eta} + a_1\eta_t\overline{\eta}\Bigg]dx\,dt - \int\limits_{\Omega_b}\eta_{tt}\overline{\eta}\,dx\,dt = 0. \tag{5}$$

Since $a_{pq} = (-1)^{|p|+|q|}a^*_{qp}$, we obtain

$$2\operatorname{Re}\Bigg[\sum_{|p|,|q|=1}^{m}(-1)^{|p|}a_{pq}D^q\eta_t\overline{D^p\eta} + (-1)^m\eta_t\overline{\eta}\Bigg] =$$

$$= \sum_{|p|,|q|=1}^{m}(-1)^{|p|}\frac{\partial}{\partial t}(a_{pq}D^q\eta\overline{D^p\eta}) - \sum_{|p|,|q|=1}^{m}(-1)^{|p|}\frac{\partial a_{pq}}{\partial t}D^q\eta\overline{D^p\eta} + (-1)^m\lambda_1\frac{\partial(\eta\overline{\eta})}{\partial t}.$$

The integration of the real part of (5) gives

$$2\int\limits_{\Omega_b}|\eta_t|^2\,dx\,dt + (-1)^m\int\limits_{\Omega}\sum_{|p|,|q|=1}^{m}(-1)^{|p|}a_{pq}D^q\eta\overline{D^p\eta}\Big|_{t=0}dx + (-1)^m 2\operatorname{Re}\sum_{|p|=1}^{m}\int\limits_{\Omega}a_pD^p\eta\overline{\eta}\Big|_{t=0}dx + \lambda_1\int\limits_{\Omega}|\eta|^2\Big|_{t=0}dx =$$

$$= (-1)^{m-1}\int\limits_{\Omega_b}\sum_{|p|,|q|=1}^{m}(-1)^{|p|}\frac{\partial a_{pq}}{\partial t}D^q\eta\overline{D^p\eta}\,dx\,dt + (-1)^{m-1}2\operatorname{Re}\int\limits_{\Omega_b}\Bigg\{\sum_{|p|=1}^{m}\Bigg[\frac{\partial a_p}{\partial t}D^p\eta\overline{\eta} + a_pD^p\eta\overline{\eta}_t\Bigg] + a_1\eta_t\cdot\overline{\eta}\Bigg\}dx\,dt.$$

Since $\dfrac{\partial a_{pq}}{\partial t}$, $\dfrac{\partial a_p}{\partial t}$ are bounded, by using the Cauchy inequality, we obtain

$$\|\eta_t\|^2_{L_2(\Omega_b)} + \|\eta(x,0)\|^2_{H^m(\Omega)} \le C_1 \sum_{|p|=0}^{m} \left|D^p\eta\right|^2 dx\,dt + C_2\|\eta_t\|^2_{L_2(\Omega_b)}$$

or

$$\|\eta(x,0)\|^2_{H^m(\Omega)} \le C \sum_{|p|=0}^{m} \int_{\Omega_b} \left|D^p\eta\right|^2 dx\,dt, \tag{6}$$

where $C = C(\mu, \mu_1, \lambda_1) = \text{const.}$

Putting

$$v_p(x,t) = \int_t^0 D^p u(x,\tau)\,d\tau, \quad 0 < t < b,$$

we can write

$$D^p\eta(x,t) = \int_b^t D^p u(x,\tau)\,d\tau = v_p(x,b) - v_p(x,t).$$

By virtue of (6),

$$\|\eta(x,0)\|^2_{H^m(\Omega)} = \sum_{|p|=0}^{m} \int_{\Omega} \left|v_p(x,b)\right|^2 dx \le$$

$$\le C \sum_{|p|=0}^{m} \int_{\Omega_b} \left|D^p\eta\right|^2 dx\,dt \le C \sum_{|p|=0}^{m} \int_{\Omega_b} \left[\left|v_b(x,b)\right|^2 + \left|v_b(x,t)\right|\right] dx\,dt =$$

$$= Cb \sum_{|p|=0}^{m} \int_{\Omega} \left|v_p(x,b)\right|^2 dx + C \sum_{|p|=0}^{m} \int_{\Omega_b} \left|v_p(x,t)\right|^2 dx\,dt. \tag{7}$$

Settings

$$J(t) = \sum_{|p|=0}^{m} \int_{\Omega} \left|v_p(x,t)\right|^2 dx$$

and using (7), we get

$$(1 - Cb)J(b) \le C \int_0^b J(t)\,dt, \quad b \in \left[0, \frac{1}{2C}\right].$$

According to the Gronwall – Bellman inequality, one has $J(t) \equiv 0$. This implies that $\eta_t \equiv 0$, i.e., $u_1 \equiv u_2 \quad \forall t \in \left[0, \frac{1}{2C}\right]$. By using the same argument as before for functions u_1, u_2 on $\left[\frac{1}{2C}, T\right]$, we can prove that, after finite steps, $u_1 \equiv u_2 \quad \forall t \in [0, T]$. Since $T > 0$ is arbitrary, $u_1 \equiv u_2 \quad \forall t \in [0, \infty)$. This completes the proof of the uniqueness.

Let us prove the existence by the Galerkin method. A function $u(x)$ belongs to $H^m(\Omega)$ if it has generalized derivatives of all orders α with $|\alpha| \le m$ and

$$\|u\|_{H^m(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha|=0}^{m} \left|D^\alpha u(x)\right|^2 dx\right)^{1/2} < \infty,$$

$\overset{\circ}{H}{}^m(\Omega)$ is the closure of $\overset{\circ}{C}{}^\infty(\Omega)$ in the norm of the space $H^m(\Omega)$.

Let $\{\varphi_k(x)\}_{k=1}^{\infty} \subset \overset{\circ}{C}^{\infty}(\Omega)$ be an orthonormal system in $L_2(\Omega)$, linear closure of which in $H^m(\Omega)$ is the space $\overset{\circ}{H}^m(\Omega)$. For each natural number N, let us consider the function

$$u^N(x, t) = \sum_{k=1}^{N} C_k^N(t)\varphi_k(x), \tag{8}$$

where C_k^N satisfy

$$\int_{\Omega} \left[u_t^N \overline{\varphi_l} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p \varphi_l} + \lambda_1 u^N \overline{\varphi_l} \right] dx + \\ + (-1)^m \int_{\Omega} \left(\sum_{|p|=1}^{m} a_p D^p u^N \overline{\varphi_l} + a_0 u^N \overline{\varphi_l} \right) dx = \\ = -\int_{\Omega} f \overline{\varphi_l}\, dx, \quad l = 1, 2, \ldots, N, \tag{9}$$

$C_k^N(0) = 0$, with $a_0 = a - (-1)^m \lambda_1 I$. After multiplying (9) by $\dfrac{dC_l^N}{dl}$, taking the sum in l from 1 to N, integrating the real part of obtained equality in t from 0 to T, and using $a_{pq} = (-1)^{|p|+|q|} a_{qp}^*$, we get

$$2 \int_{\Omega_T} \left|u_t^N\right|^2 dx\, dt + \left[\int_{\Omega} \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p u^N} + \lambda_1 \left|u^N\right|^2 \right]_{t=T} dx + \\ + 2(-1)^m \operatorname{Re} \int_{\Omega_T} \left[\sum_{|p|=1}^{m} a_p D^p u^N u_t^N + a_0 u^N u_t^N \right] dx\, dt - \\ - \int_{\Omega_T} \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} \frac{\partial a_{pq}}{\partial t} D^q u^N \overline{D^p u^N}\, dx\, dt = -2 \operatorname{Re} \int_{\Omega_T} f u_t^N\, dx\, dt.$$

Inequality (3) and the Cauchy inequality imply

$$2\left\|u_t^N\right\|_{L_2(\Omega_T)}^2 + \mu_1 \left\|u^N(x, t)\right\|_{H^m(\Omega)} \le \\ \le \left(\mu \frac{1}{\varepsilon_1} + M_\mu\right) \int_0^T \left\|u^N\right\|_{H^m(\Omega)}^2 dt + 2\varepsilon_1 \left\|u_t^N\right\|_{L_2(Q_T)}^2 + \frac{1}{\varepsilon_1} \int_0^t \|f(x, t)\|_{L_2(\Omega)}^2 dt, \tag{10}$$

where M is a number of multi-index p such that $|p| \le m$. So, for $0 < \varepsilon_1 < 1$, we have

$$\left\|u^N\right\|_{H^m(\Omega)}^2 \le \frac{\mu(M\varepsilon_1 + 1)}{\mu_1 \varepsilon_1} \int_0^T \left\|u^N\right\|_{H^m(\Omega)}^2 dt + \frac{1}{\varepsilon_1} \int_0^T \|f\|_{L_2(\Omega)}^2 dt. \tag{11}$$

According to the Gronwall – Bellman inequality, it follows from (10) that

$$\left\|u^N\right\|_{H^m(\Omega)}^2 \le \frac{\varepsilon}{\varepsilon_1} \|f\|_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))} e^{\frac{\mu(M\varepsilon_1 + 1)}{\mu_1 \varepsilon_1} T}. \tag{12}$$

For each given positive ε choose $\varepsilon_1 = \dfrac{\mu}{\mu M + 2\mu_1 \varepsilon}$. By multiplying both sides of (12) with $e^{-2(\mu M/\mu_1 + \varepsilon)T}$ and then integrating in T from 0 to ∞, we obtain

$$\left\| u^N \right\|^2_{H^{m,0}(e^{-(\mu M/\mu_1+\varepsilon)t},\Omega_\infty)} \le C\| f \|^2_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))}, \tag{13}$$

where C depends only on M, μ, μ_1.

Let us multiply (9) by $e^{-\lambda t}\dfrac{d}{dt}(\overline{C_l^N}(t)e^{-\lambda t})$ with $\lambda = \dfrac{M\mu}{\mu_1}+\varepsilon$ and take the sum in l from 1 to N. After integrating the result obtained in t from 0 to T, as in the proof for (10), we have

$$2\left\| u_t^N e^{-\lambda t} \right\|^2_{L_2(\Omega_T)} + \mu_1 \left\| u^N e^{-\lambda t} \right\|^2_{H^m(\Omega)} \le$$

$$\le \left(\mu\frac{1}{\varepsilon_2}+M\mu\right)\int_0^T \left\| u^N e^{-\lambda t} \right\|^2_{H^m(\Omega)} dt + 2\varepsilon_2 \left\| u_t^N e^{-\lambda t} \right\|^2_{L_2(\Omega_T)} +$$

$$+\frac{1}{\varepsilon_2}\int_0^T \| f \|^2_{L_2(\Omega)} e^{-\lambda t}\, dt.$$

By choosing $\varepsilon_2 = \dfrac{1}{2}$, we can see that

$$\left\| u_t^N e^{-\lambda t} \right\|^2_{L_2(\Omega_\infty)} \le D\left\| u^N \right\|^2_{H^{m,0}(e^{-\lambda t},\Omega_\infty)} + E\| f \|^2_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))} \le$$

$$\le (CD+E)\| f \|^2_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))}. \tag{14}$$

It follows from (13) and (14) that

$$\left\| u^N \right\|^2_{H^{m,1}(e^{-(\gamma_0+\varepsilon)t},\Omega_\infty)} \le C\| f \|^2_{L^\infty(0,\infty;\varepsilon,L_2(\Omega))},$$

where

$$\gamma_0 = \frac{M\mu}{\mu_1}. \tag{15}$$

Since the sequence of functions $\{u^N\}$ is uniformly bounded in $H^{m,1}(e^{-(\gamma_0+\varepsilon)t},\Omega_\infty)$, we can take a subsequence which is weakly convergent to some function $u(x,t) \in$ $\in H^{m,1}(e^{-(\gamma_0+\varepsilon)t},\Omega_\infty)$.

We will prove that $u(x,t)$ is a generalized solution. Since $u^N(x,0)=0$ on Ω and $u^N(x,t) \in \overset{\circ}{H}{}^{m,1}(e^{-(\gamma_0+\varepsilon)t},\Omega_\infty)$, it follows that $u(x,t)=0$ on Ω and $u(x,t) \in$ $\in \overset{\circ}{H}{}^{m,1}(e^{-(\gamma_0+\varepsilon)t},\Omega_\infty)$. Take $T>0$. Multiplying (9) by $d_l(t)=H^1(0,T)$, $d_l(T)=0$, taking the sum in l from 1 to N, then integrating in t from 0 to T, we obtain

$$\int_{\Omega_T} u_t^N \overline{\eta}\, dx\, dt + (-1)^m \int_{\Omega_T}\left(\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u^N \overline{D^p \eta} + \right.$$

$$\left. + \sum_{|p|=1}^{m} a_p D^p u^N \overline{\eta} + a u^N \overline{\eta} \right) dx\, dt = -\int_{\Omega_T} f\overline{\eta}\, dx\, dt. \tag{16}$$

Equality (16) is true for any function $\eta \in M_N$, where

$$M_N = \left\{ \eta = \sum_{i=1}^{N} d_i(t)\varphi_i(x) \,\middle|\, d_i(t) \in H^1(0,T),\, d_i(T)=0 \right\}.$$

For each $\eta \in M_N$, where $N \to \infty$, equality (16) implies

$$\int_{\Omega_T} u_t \overline{\eta}\, dx\, dt + (-1)^n \int_{\Omega_T} \Bigg[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \eta} + \sum_{|p|=1}^{m} a_p D^p \overline{\eta} + au\overline{\eta} \Bigg] dx\, dt = - \int_{\Omega_T} f \overline{\eta}\, dx\, dt. \quad (17)$$

It is easy to check that (17) holds for any function $\eta \in \overset{\circ}{H}{}^{m,0}(\Omega_T)$, $\eta(x, T) = 0$; i.e., $u(x, t)$ is a weak solution of the first initial boundary-value problem for (1) in the space $\overset{\circ}{H}{}^{m,1}(e^{-(\gamma_0+\varepsilon)t}, \Omega_\infty)$. Moreover, the weak convergence of $\{u^N\}$ and (15) imply that

$$\| u(x, t) \|^2_{H^{m,1}(e^{-(\gamma_0+\varepsilon)t}, \Omega_\infty)} \leq C \| f \|^2_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))},$$

where the constant C does not depend on u, f, and ε.

The theorem is completely proved.

Theorem 2. *Suppose that* $\left| \dfrac{\partial^k a_{pq}}{\partial t^k}, \dfrac{\partial^{k-1} a_p}{\partial t^{k-1}}, \dfrac{\partial^{k-1} a}{\partial t^{k-1}} \right| \leq \mu$, $1 \leq |p|$, $|q| \leq m$, $k \leq h+1$, $\mu = \text{const}$, $(x, t) \in \overline{\Omega_\infty}$. *Then there exists a positive constant* γ_h *such that if* $f_{t^k} \in L^\infty(0, \infty; \varepsilon, L_2(\Omega))$, $f_{t^k}(x, 0) = 0$, $0 \leq k \leq h$, *then the generalized solution* $u(x, t)$ *of the first initial boundary-value problem for system* (1) *has derivatives with respect to* t *of all orders* $k \leq h$ *and*

$$\| u_{t^h} \|^2_{H^{m,1}(e^{-(\gamma_h+\varepsilon)t}, \Omega_\infty)} \leq C \sum_{k=0}^{h} \| f_{t^k} \|^2_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))}, \quad (18)$$

where the constant C *does not depend on* u, f, *and* ε.

Proof. We prove that there exists a positive constant γ_h such that an inequality

$$\| u^N_{t^h} \|^2_{H^{m,1}(e^{-(\gamma_h+\varepsilon)t}, \Omega_\infty)} \leq C \sum_{k=0}^{h} \| f_{t^k} \|^2_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))} \quad (18_N)$$

holds, where u^N is the same as in (8), C is independent of N and f. We will apply induction on h.

If $h = 0$, (18_N) is true according to Theorem 1. Let now $j \geq 1$ and the inequality (18_N) be true for $h \leq j-1$. From the identity (9), we have

$$\int_{\Omega} \Bigg[u^N_{t^{j+1}} \overline{\varphi_l} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} \frac{\partial^j}{\partial t^j} (a_{pq} D^q u^N) \overline{D^p \varphi_l} \Bigg] dx + (-1)^m \int_{\Omega} \frac{\partial^j}{\partial t^j} \Bigg(\sum_{|p|=1}^{m} a_p D^p u^N + au^N \Bigg) \overline{\varphi_l}\, dx = - \int_{\Omega} f_{t^j} \overline{\varphi_l}\, dx.$$

After multiplying both sides of this identity by $\dfrac{d^{j+1} C_l^N}{dt^{j+1}}$ [C_l^N is determined by (8)], taking the sum in l from 1 to N, and integrating the result over $[0, T]$, we obtain

$$\int_{\Omega_T} \Bigg[u^N_{t^{j+1}} \overline{u^N_{t^{j+1}}} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} \frac{\partial^j}{\partial t^j} (a_{pq} D^q u^N) \overline{D^p u^N_{t^{j+1}}} \Bigg] dx\, dt + (-1)^m \int_{\Omega_T} \frac{\partial^j}{\partial t^j} \Bigg(\sum_{|p|=1}^{m} a_p D^p u^N + au^N \Bigg) \overline{u^N_{t^{j+1}}}\, dx\, dt = - \int_{\Omega_T} f_{t^j} \overline{u^N_{t^{j+1}}}\, dx\, dt.$$

Since $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$, we have

$$\int_{\Omega_T} \left[u^N_{t^{j+1}} \overline{u^N_{t^{j+1}}} + \sum_{|p|,|q|=1}^{m} (-1)^m \frac{\partial}{\partial t} (a_{pq} D^q u^N_{t^j}) \overline{D^p u^N_{t^j}} \right] dx\,dt =$$

$$= -2\,\mathrm{Re} \sum_{|p|,|q|=1} \sum_{s=1}^{j} (-1)^{m+|p|} \binom{i}{s} \int_{\Omega_T} \frac{\partial}{\partial t} \left(\frac{\partial^s a_{pq}}{\partial t^s} D^q u^N_{t^{j-s}} D^p u^N_{t^j} \right) +$$

$$+ \mathrm{Re} \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} \int_{\Omega_T} \frac{\partial a_{pq}}{\partial t} D^q u^N_{t^j} \overline{D^p u^N_{t^j}}\, dx\,dt +$$

$$+ 2\,\mathrm{Re} \sum_{|p|=1,|q|=1}^{m} \sum_{s=1}^{j} (-1)^{m+|p|} \binom{j}{s} \int_{\Omega_T} \frac{\partial^{s+1}}{\partial t^{s+1}} a_{pq} D^q u^N_{t^{j-s}} \overline{D^p u^N_{t^j}}\, dx\,dt +$$

$$+ 2\,\mathrm{Re} \sum_{|p|,|q|=1}^{m} \sum_{s=1}^{j} (-1)^{m+|p|} \binom{j}{s} \int_{\Omega_T} \frac{\partial^s}{\partial t^s} a_{pq} D^q u^N_{t^{j-s+1}} \overline{D^p u^N_{t^j}}\, dx\,dt -$$

$$- 2\,\mathrm{Re}(-1)^m \int_{\Omega_T} \frac{\partial^j}{\partial t^j} \left(\sum_{|p|=1}^{m} a_p D^q u^N + a u^N \right) \overline{u^N_{t^{j+1}}}\, dx\,dt - 2\,\mathrm{Re} \int_{\Omega_T} f_{t^j} \overline{u^N_{t^{j+1}}}\, dx\,dt,$$

where $\binom{j}{s} = \frac{j!}{s!(j-s)!}$. Let us integrate this result by parts. By using condition (3), and estimating the right-hand side of considered relation according to the Cauchy inequality, we obtain that

$$2 \left\| u^N_{t^{j+1}} \right\|^2_{L_2(\Omega_T)} + \left\| u^N_{t^j}(x, T) \right\|^2_{H^m(\Omega)} \le$$

$$\le \left((2j+1) \frac{M\mu}{\mu_1} + \varepsilon \right) \int_0^T \left\| u^N_{t^j} \right\|^2_{H^m(\Omega)} dx +$$

$$+ C \left(\sum_{k=1}^{j-1} \left\| f_{t^k} \right\|^2_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))} e^{(2j-1)\gamma_1 + \varepsilon} + \sum_{k=0}^{j-1} \left\| u_{t^k} \right\|^2_{H^{m,0}(\Omega_T)} + \right.$$

$$\left. + \varepsilon_1 \left\| u^N_{t^j} \right\|^2_{L_2(\Omega_T)} + \varepsilon_1^{-1} \int_0^T \left\| f_{t^j} \right\|^2_{L_2(\Omega)} dt. \right.$$

Let us choose $\varepsilon_1 < 2$, use the Gronwall – Bellman inequality and the inductive hypothesis by the same way as in the proof of Theorem 1. In general, we arrive at the inequality

$$\left\| u^N_{t^j} \right\|^2_{H^{m,1}(e^{-(\gamma_j + \varepsilon)t}, \Omega_\infty)} \le C \sum_{k=0}^{j} \left\| f_{t^k} \right\|^2_{L^\infty(0,\infty;\varepsilon, L_2(\Omega))},$$

where the constant C, γ_j are independent of N. Thus, we obtain inequality (18_N).

Since γ_h and C from (18_N) are independent of N, (18_N) implies (18) as $N \to \infty$. The theorem is proved.

3. Example. In this section, we will apply the previous results to the theory of elasticity.

Let u, f be the n-dimension vector functions with real components. Denote $u = (u_1, u_2, \ldots, u_n)$, $f = (f_1, f_2, \ldots, f_n)$.

We will consider the next differential operators of the form

$$L_s(x,t,D)u = \sum_{j,h,k=1}^{n} \frac{\partial}{\partial x_h}\left(a_{sh}^{jk}(x,t)\frac{\partial u^j}{\partial x_k}\right), \quad s = 1,\dots,n, \tag{19}$$

where a_{sh}^{jk} are continuous real functions bounded in $\overline{\Omega_\infty}$ and satisfying

$$a_{sh}^{jk} \equiv a_{hs}^{jk} \equiv a_{jk}^{sh}. \tag{20}$$

Denote

$$e_{sh} = \frac{1}{2}\left(\frac{\partial u_s}{\partial x_h} + \frac{\partial u_h}{\partial x_s}\right), \quad s,h = 1,\dots,n. \tag{21}$$

Assume that the elastic potential

$$W(x,t,e) = \frac{1}{2}\sum_{s,h,j,k} a_{sh}^{jk} e_{sh} e_{jk} \tag{22}$$

is positively defined quadric with respect to variables e_{sh}, $1 \le s \le h \le n$, for each $(x,t) \in \Omega_\infty$.

Consider the following problem:

$$\sum_{j,h,k=1}^{n} \frac{\partial}{\partial x_h}\left(a_{sh}^{jk}(x,t)\frac{\partial u^j}{\partial x_k}\right) - \frac{\partial u_s}{\partial t} = f_s, \quad s = 1,\dots,n, \tag{23}$$

with an initial condition

$$u\big|_{t=0} = 0 \tag{24}$$

and a boundary condition

$$u\big|_{S_\infty} = 0. \tag{25}$$

It is well-know that (see [5]), for any function $v \in \overset{\circ}{H}{}^1(\Omega)$, the following inequality holds

$$\sum_{s,h=1}^{n} \int_\Omega \left(\frac{\partial v_s}{\partial x_h} + \frac{\partial v_h}{\partial x_s}\right)^2 dx \ge C\|v\|^2_{H^1(\Omega)}, \tag{26}$$

where $C = \text{const} > 0$. By virfue of (22), this inequality is equivalent to the following one:

$$\sum_{s,h,j,k=1}^{n} \int_\Omega a_{sh}^{jk} \frac{\partial v_s}{\partial x_h}\frac{\partial v_j}{\partial x_k} dx \ge C\|v\|^2_{H^1(\Omega)}. \tag{27}$$

Inequality (27) in this case is equivalent to (3). In particular, when $a_{ss}^{ss} = \lambda + \mu$, $a_{sh}^{hs} = a_{hs}^{sh} = \mu$, $s \ne h$, $a_{jk}^{sh} = 0$; in the cases where $\mu > 0$, $\mu + \lambda > 0$, we obtain the Lame system

$$\mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,(\mathrm{div}\, u) - u_t = f. \tag{28}$$

We have

$$2W(x,t,e) = \sum_{s,h,j,k=1}^{n} a_{sh}^{jk} e_{sh} e_{jk} = (\lambda + 2\mu)\sum_{s=1}^{n} e_{ss}^2 + \lambda \sum_{h\ne s}^{1,n} e_{ss}e_{hh} + \mu \sum_{h\ne s}^{1,n} e_{hs}^2 =$$

$$= 2\mu \sum_{s=1}^{n} e_{ss}^2 + \lambda\left(\sum_{s=1}^{n} e_{ss}\right)^2 + \mu \sum_{h\ne s}^{1,n} e_{hs}^2. \tag{29}$$

Since $\mu > 0$ and $\mu + \lambda > 0$, it follows that there exists $\varepsilon > 0$ such that $\mu - \varepsilon > 0$ and $\mu + \lambda - \varepsilon \geq 0$. This and (29) imply

$$2W(x, t, e) = 2\varepsilon \sum_{s=1}^{n} e_{ss}^2 + 2(\mu - \varepsilon) \sum_{s=1}^{n} e_{ss}^2 + \lambda \left(\sum_{s=1}^{n} e_{ss} \right)^2 + \mu \sum_{h \neq s}^{1,n} e_{hs}^2 \geq$$

$$\geq 2\varepsilon \sum_{s=1}^{n} e_{ss}^2 + \mu \sum_{h \neq s}^{1,n} e_{hs}^2 + (\mu - \varepsilon) \left(\sum_{s=1}^{n} e_{ss} \right)^2 + \lambda \left(\sum_{s=1}^{n} e_{ss} \right)^2 =$$

$$= 2\varepsilon \sum_{s=1}^{n} e_{ss}^2 + (\mu + 1 - \varepsilon) \left(\sum_{s=1}^{n} e_{ss} \right)^2 + \mu \sum_{h \neq s}^{1,n} e_{hs}^2 .$$

Thus, we have

$$W(x, t, e) \geq \mu_0 \sum_{s,h=1}^{n} e_{sh}^2,$$

where $\mu_0 = \text{const} > 0$. Therefore, inequality (27) is also true for the Lame system.

We obtain the following results.

Proposition. *Suppose that*

i) $f_{t^k} \in L^\infty(0, \infty; 1, L_2(\Omega))$ *for* $0 \leq k \leq h$;

ii) $f_{t^k}(x, 0) = 0$, $0 \leq k \leq h$.

Then, for every $\gamma > 0$, *the unique generalized solution of problem* (28), (24) *and* (25) *from* $\overset{\circ}{H}{}^{1,1}(e^{-\gamma t}, \Omega_\infty)$ *has the derivatives with respect to* t *up to order* h *belonging to* $\overset{\circ}{H}{}^{1,1}(e^{-\gamma(2h+1)t}, \Omega_\infty)$ *and*

$$\left\| u_{t^h} \right\|^2_{H^{1,1}(e^{-\gamma(2h+1)t}, \Omega_\infty)} \leq C \sum_{k=0}^{h} \left\| f_{t^k} \right\|_{L^\infty(0,\infty;1,L_2(\Omega))},$$

where $C = \text{const} > 0$ *does not depend on* u *and* f.

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