UDC 517.9

A. K. Prykarpatsky (AGH Univ. Sci. and Technology, Krakow,
Poland and Inst. Appl. Problems Mech. and Math. Nat. Acad. Sci. Ukraine, Lviv),
A. M. Samoilenko (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv),
D. L. Blackmore (Univ. Heights, New Jersey, USA)

THE HOPF ALGEBRAS AND THE HEISENBERG – WEIL COALGEBRA RELATED INTEGRABLE FLOWS

АЛГЕБРИ ХОПФА ТА ІНТЕГРОВНІ ПОТОКИ, ПОВ'ЯЗАНІ З КОАЛГЕБРОЮ ХЕЙЗЕНБЕРГА – ВЕЙЛЯ

On the basis of the structure of Casimir elements associated with general Hopf algebras, we construct Liouville – Arnold integrable flows related with naturally induced Poisson structures on arbitrary coalgebra and their deformations. Some interesting special cases including the oscillatory Heisenberg – Weil algebra related coalgebra structures and adjoint with them integrable Hamiltonian systems are considered.

На основі структури елементів Казиміра, асоційованих із загальними алгебрами Хопфа, побудовано інтегровні потоки Ліувілля – Арнольда, що пов'язані з природно індукованими структурами Пуассона на довільній коалгебрі, та їх деформації. Розглянуто деякі цікаві спеціальні випадки, в тому числі коалгебраїчні структури, що пов'язані з осциляційною алгеброю Хейзенберга – Вейля, та спряжені з ними інтегровні гамільтонові системи.

1. Hopf algebras and coalgebras: main definitions. Consider a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with two special homomorphisms called coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, as well an antihomomorphism (antipode) $\nu : \mathcal{A} \to \mathcal{A}$, such that for any $a \in \mathcal{A}$

$$\begin{gathered}(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a),\\ (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes \mathrm{id})\Delta(a) = a, \\ m((\mathrm{id} \otimes \nu)\Delta(a)) = m((\nu \otimes \mathrm{id})\Delta(a)) = \varepsilon(a)I,\end{gathered} \tag{1.1}$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the usual multiplication mapping, that is for any $a, b \in \mathcal{A}$ $m(a \otimes b) = ab$. The conditions (1.1) were introduced by Hopf [1] in a cohomological context. Since most of the Hopf algebras properties depend on the coproduct operation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and related with it Casimir elements, below we shall dwell mainly on the objects called coalgebras endowed with this coproduct.

The most interesting examples of coalgebras are provided by the universal enveloping algebras $U(\mathcal{G})$ of Lie algebras $\mathcal{G}$. If, for instance, a Lie algebra $\mathcal{G}$ possesses generators $X_i \in \mathcal{G}$, $i = \overline{1,n}$, $n = \dim \mathcal{G}$, the corresponding enveloping algebra $U(\mathcal{G})$ can be naturally endowed with a Hopf algebra structure by defining

$$\begin{gathered}\Delta(X_i) = I \otimes X_i + X_i \otimes I, \qquad \Delta(I) = I \otimes I,\\ \varepsilon(X_i) = -X_i, \qquad \nu(I) = -I.\end{gathered} \tag{1.2}$$

These mappings acting only on the generators of $\mathcal{G}$ are straightforwardly extended to any monomial in $U(\mathcal{G})$ by means of the homomorphism condition $\Delta(XY) = \Delta(X)\Delta(Y)$ for any $X, Y \in \mathcal{G} \subset U(\mathcal{G})$. In general, an element $Y \in U(\mathcal{G})$ of a Hopf algebra such that $\Delta(Y) = I \otimes Y + Y \otimes I$ is called primitive, and the known Friedrichs theorem [2] ensures that, in $U(\mathcal{G})$, the only primitive elements are exactly generators $X_i \in \mathcal{G}$, $i = \overline{1,n}$.

On the other hand the homomorphism condition for the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ implies the compatibility of the coproduct with the Lie algebra commutator structure:

$$\left[\Delta(X_i), \Delta(X_j)\right]_{\mathcal{A}\otimes\mathcal{A}} = \Delta\left(\left[X_i, X_j\right]_{\mathcal{A}}\right) \tag{1.3}$$

for any $X_i, X_j \in \mathcal{G}$, $i, j = \overline{1, n}$. Since the Drinfeld report [3], the coalgebras defined above are also often called "quantum" groups due to their importance [4] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It was also observed (see, for instance, [4]) that the standard coalgebra structure (1.2) of the universal enveloping algebra $U(\mathcal{G})$ can be nontrivially extended making use of some its infinitesimal deformations, saving the coassociativity (1.3) of the deformed coproduct $\Delta : U_z(\mathcal{G}) \to U_z(\mathcal{G}) \otimes U_z(\mathcal{G})$ with $U_z(\mathcal{G})$ being the corresponding universal enveloping algebra deformation by means of a parameter $z \in \mathbb{C}$ such that $\lim_{z\to 0} U_z(\mathcal{G}) = U(\mathcal{G})$ subject to some natural topology on $U_z(\mathcal{G})$.

2. Casimir elements and their special properties. Take any Casimir element $C \in U_z(\mathcal{G})$, that is an element satisfying the condition $[C, U_z(\mathcal{G})] = 0$, and consider the action on it of the coproduct mapping Δ:

$$\Delta(C) = C(\{\Delta(X)\}),$$

where we put, by definition, $C := C(\{X\})$ with a set $\{X\} \subset \mathcal{G}$. It is a trivial consequence that for $\mathcal{A} := U_z(\mathcal{G})$

$$\left[\Delta(C), \Delta(X_i)\right]_{\mathcal{A}\otimes\mathcal{A}} = \Delta\left(\left[C, X_i\right]_{\mathcal{A}}\right) = 0$$

for any $X_i \in \mathcal{G}$, $i = \overline{1, n}$.

Define now recurrently the following N-th coproduct $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes} \mathcal{A}$ for any $N \in \mathbb{Z}_+$, where $\Delta^{(2)} := \Delta$ and $\Delta^{(1)} :=$ id and

$$\Delta^{(N)} := ((\mathrm{id}\otimes)^{N-2} \otimes \Delta) \cdot \Delta^{(N-1)},$$

or as

$$\Delta^{(N)} := (\Delta \otimes (\mathrm{id}\otimes)^{N-2} \otimes \mathrm{id} \otimes \mathrm{id}) \cdot \Delta^{(N-1)}.$$

One can straightforwardly verify that

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \cdot \Delta$$

for any $m = \overline{0, N}$, and the mapping $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes} \mathcal{A}$ is an algebras homomorphism, that is

$$\left[\Delta^{(N)}(X), \Delta^{(N)}(Y)\right]_{\overset{(N+1)}{\otimes}\mathcal{A}} = \Delta^{(N)}\left([X, Y]_{\mathcal{A}}\right)$$

for any $X, Y \in \mathcal{A}$. In a particular case if $\mathcal{A} = U(\mathcal{G})$, the following exact expression

$$\Delta^{(N)}(X) = X(\otimes\mathrm{id})^{N-1} \otimes \mathrm{id} + \mathrm{id} \otimes X(\otimes\mathrm{id})^{N-1} \otimes \mathrm{id} + \ldots$$
$$\ldots + (\otimes\mathrm{id})^{N-1} \otimes \mathrm{id} \otimes X$$

holds for any $X \in \mathcal{G}$.

3. Poisson coalgebras and their realizations. As is well known [5, 6], a Poisson algebra $\mathcal{P}$ is a vector space endowed with a commutative multiplication and a Lie bracket $\{.,.\}$ including a derivation on $\mathcal{P}$ in the form

$$\{a, bc\} = b\{a, c\} + \{a, b\}c$$

for any a, b and $c \in \mathcal{P}$. If $\mathcal{P}$ and $\mathcal{Q}$ are Poisson algebras, one can naturally define the following Poisson structure on $\mathcal{P} \otimes \mathcal{Q}$:

$$\{a \otimes b, c \otimes d\}_{\mathcal{P}\otimes\mathcal{Q}} = \{a, c\}_{\mathcal{P}} \otimes (bd) + (ac) \otimes \{b, d\}_{\mathcal{Q}}$$

for any $a, c \in \mathcal{P}$ and $b, d \in Q$. We shall also say that $(\mathcal{P}; \Delta)$ is a Poisson coalgebra if $\mathcal{P}$ is a Poisson algebra and $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is a Poisson algebras homomorphism, that is

$$\{\Delta(a), \Delta(b)\}_{\mathcal{P}\otimes\mathcal{P}} = \Delta\big(\{a, b\}_{\mathcal{P}}\big) \tag{3.1}$$

for any $a, b \in \mathcal{P}$.

It is useful to note here that any Lie algebra $\mathcal{G}$ generates naturally a Poisson coalgebra $(\mathcal{P}; \Delta)$ by defining a Poisson bracket on $\mathcal{P}$ by means of the following expression: for any $a, b \in \mathcal{P}$

$$\{a, b\}_{\mathcal{P}} = \langle \operatorname{grad} a, \vartheta \operatorname{grad} b \rangle.$$

Here $\mathcal{P} \simeq C^{\infty}(\mathbb{R}^n; \mathbb{R})$ is a space of smooth mappings linked with a base variables of the Lie algebra $\mathcal{G}$, $n = \dim \mathcal{G}$, and the implectic [6] matrix $\vartheta : T^*(\mathcal{P}) \to T(\mathcal{P})$ is given as

$$\vartheta(x) = \left\{ \sum_{k=1}^{n} c_{ij}^{k} x_k : i, j = \overline{1, n} \right\},$$

where c_{ij}^{k}, $i, j, k = \overline{1, n}$, are the corresponding structure constants of the Lie algebra $\mathcal{G}$ and $x \in \mathbb{R}^n$ are the corresponding linked coordinates. It is easy to check that the coproduct (1.2) is a Poisson algebras homomorphism between $\mathcal{P}$ and $\mathcal{P} \otimes \mathcal{P}$. If one can find a "quantum" deformation $U_z(\mathcal{G})$, then the corresponding Poisson coalgebra $\mathcal{P}_z$ can be constructed making use of the naturally deformed implectic matrix ϑ_z: $T^*(\mathcal{P}_z) \to T(\mathcal{P}_z)$. For instance, if $\mathcal{G} = \mathrm{so}(2, 1)$, there exists a deformation $U_z(\mathrm{so}(2, 1))$ defined by the following deformed commutator relations with a parameter $z \in \mathbb{C}$:

$$\begin{gathered} \big[\tilde{X}_2, \tilde{X}_1\big] = \tilde{X}_3, \quad \big[\tilde{X}_2, \tilde{X}_3\big] = -\tilde{X}_1, \\ \big[\tilde{X}_3, \tilde{X}_1\big] = \frac{1}{z}\sinh\big(z\tilde{X}_2\big), \end{gathered} \tag{3.2}$$

where at $z = 0$ elements

$$\tilde{X}_i \Big|_{z=0} = X_i \in \mathrm{so}(2, 1), \quad i = \overline{1, 3},$$

compile a base of generators of the Lie algebra so (2, 1). Then, based on expressions (3.2) one can easily construct the corresponding Poisson coalgebra $\mathcal{P}_z$, endowed with the implectic matrix

$$\vartheta_z(\tilde{x}) = \begin{pmatrix} 0 & -\tilde{x}_3 & -\frac{1}{z}\sinh(z\tilde{x}_2) \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ \frac{1}{z}\sinh(z\tilde{x}_2) & \tilde{x}_1 & 0 \end{pmatrix}$$

for any point $\tilde{x} \in \mathbb{R}^3$, linked naturally with the deformed generators $\tilde{X}_i$, $i = \overline{1, 3}$, taken above. Since the corresponding coproduct on $U_z(\mathrm{so}(2, 1))$ acts on this deformed base of generators as

$$\begin{aligned} \Delta(\tilde{X}_2) &= I \otimes \tilde{X}_2 + \tilde{X}_2 \otimes I, \\ \Delta(\tilde{X}_1) &= \exp\Big(-\frac{z}{2}\tilde{X}_2\Big) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\Big(\frac{z}{2}\tilde{X}_2\Big), \\ \Delta(\tilde{X}_2) &= \exp\Big(-\frac{z}{2}\tilde{X}_2\Big) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\Big(\frac{z}{2}\tilde{X}_2\Big), \end{aligned} \tag{3.3}$$

satisfying the main homomorphism property for the whole deformed universal enveloping algebra $U_z(\mathrm{so}(2,1))$.

Consider now some realization of the deformed generators $\tilde{X}_i \in U_z(\mathcal{G})$, $i = \overline{1,n}$, that is a homomorphism mapping $D_z : U_z(\mathcal{G}) \to \mathcal{P}(M)$, such that

$$D_z(\tilde{X}_i) = \tilde{e}_i, \qquad i = \overline{1,n}, \tag{3.4}$$

are some elements of a Poisson manifold $\mathcal{P}(M)$ realized as a space of functions on a finite-dimensional manifold M, satisfying the deformed commutator relationships

$$\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}),$$

where, by definition, expressions

$$[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \quad i,j = \overline{1,n},$$

generate a Poisson coalgebra structure on the function space $\mathcal{P}_z := \mathcal{P}_z(\mathcal{G})$ linked with a given Lie algebra $\mathcal{G}$. Making use of the homomorphism property (3.1) for the coproduct mapping $\Delta : \mathcal{P}_z(\mathcal{G}) \to \mathcal{P}_z(\mathcal{G}) \otimes \mathcal{P}_z(\mathcal{G})$, one finds that for all $i, j = \overline{1,n}$

$$\{\Delta(\tilde{x}_i), \Delta(\tilde{x}_j)\}_{\mathcal{P}_z(\mathcal{G})\otimes\mathcal{P}_z(\mathcal{G})} = \Delta\left(\{\tilde{x}_i, \tilde{x}_j\}_{\mathcal{P}_z(\mathcal{G})}\right) = \vartheta_{z,ij}(\Delta(\tilde{x}))$$

and for the corresponding coproduct $\Delta : \mathcal{P}(M) \to \mathcal{P}(M) \otimes \mathcal{P}(M)$ one gets similarly

$$\{\Delta(\tilde{e}_i), \Delta(\tilde{e}_j)\}_{\mathcal{P}(M)\otimes\mathcal{P}(M)} = \Delta\left(\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)}\right) = \vartheta_{z,ij}(\Delta(\tilde{e})), \tag{3.5}$$

where $\{.,.\}_{\mathcal{P}(M)}$ is some, eventually, canonical Poisson structure on a finite-dimensional manifold M.

Let $q \in M$ be a point of M. We consider its coordinates as elements of $\mathcal{P}(M)$. Then one can define the following elements:

$$q_j := (I\otimes)^{j-1} q (\otimes I)^{N-j} \in \overset{(N)}{\otimes} \mathcal{P}(M),$$

where $j = \overline{1,N}$ by means of which one can construct the corresponding N-tuple realization of the Poisson coalgebra structure (3.5) as follows:

$$\{\tilde{e}_i^{(N)}, \tilde{e}_j^{(N)}\}_{\overset{(N)}{\otimes}\mathcal{P}(M)} = \vartheta_{z,ij}\left(\tilde{e}^{(N)}\right),$$

with $i, j = \overline{1,n}$ and

$$\overset{(N)}{\otimes} D_z\left(\Delta^{(N-1)}(\tilde{e}_i)\right) := \tilde{e}_i^{(N)}(q_1, q_2, \ldots, q_N). \tag{3.6}$$

For instance, for the $U_z(\mathrm{so}(2,1))$ case (3.2), one can take [7] the realization Poisson manifold $\mathcal{P}(M) = \mathcal{P}(\mathbb{R}^2)$ with the standard canonical Heisenberg – Weil Poissonian structure on it:

$$\{q, q\}_{\mathcal{P}(\mathbb{R}^2)} = 0 = \{p, p\}_{\mathcal{P}(\mathbb{R}^2)}, \qquad \{p, q\}_{\mathcal{P}(\mathbb{R}^2)} = 1, \tag{3.7}$$

where $(q, p) \in \mathbb{R}^2$. Then expressions (3.6) for $N = 2$ give rise to the following relationships:

$$\tilde{e}_1^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_1) =$$

$$= 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\cos q_1 \exp\left(\frac{z}{2}p_1\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\cos q_2,$$

$$\tilde{e}_2^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_2) = p_1 + p_2,$$

$$\tilde{e}_3^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_3) =$$

$$= 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\sin q_1 \exp\left(\frac{z}{2}p_2\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\sin q_2,$$

where elements $(q_1, q_2, p_1, p_2) \in \mathcal{P}(\mathbb{R}^2) \otimes \mathcal{P}(\mathbb{R}^2)$. satisfy the induced by (3.7) Heisenberg – Weil commutator relations:

$$\{q_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = 0 = \{p_i, p_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)},$$

$$\{p_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = \delta_{ij}$$

for any $i, j = \overline{1, 2}$.

4. Casimir elements and the Heisenberg – Weil algebra related algebrais structures. Consider any Casimir element $\tilde{C} \in U_z(\mathcal{G})$ related with an $\mathbb{R} \ni z$-deformed Lie algebra $\mathcal{G}$ structure in the form

$$\left[\tilde{X}_i, \tilde{X}_j\right] = \vartheta_{z, ij}\left(\tilde{X}\right), \tag{4.1}$$

where $i, j = \overline{1, n}$, $n = \dim\mathcal{G}$, and, by definition, $\left[\tilde{C}, \tilde{X}_i\right] = 0$. The following general lemma holds.

Lemma 1. *Let $(U_z(\mathcal{G}); \Delta)$ be a coalgebra with generators satisfying* (4.1) *and $\tilde{C} \in U_z(\mathcal{G})$ be its Casimir element; then*

$$\left[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{X}_i)\right]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0 \tag{4.2}$$

for any $i = \overline{1, n}$ and $m = \overline{1, N}$.

As a simple corollary of this lemma one finds from (4.2) that

$$\left[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})\right]_{\overset{(N+1)}{\otimes} U_z(\mathcal{G})} = 0$$

for any $k, m \in \mathbb{Z}_+$.

Consider now some realization (3.4) of our deformed Poisson coalgebra structure (4.1) and check that

$$\left[\Delta^{(m)}(C(\tilde{e})), \Delta^{(N)}(\mathcal{H}(\tilde{e}))\right]_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0$$

for any $m = \overline{1, N}$, $N \in \mathbb{Z}_+$, if $C(\tilde{e}) \in I(\mathcal{P}(M))$, that is $\{C(\tilde{e}), q\}_{\mathcal{P}(M)} = 0$ for any $q \in M$. Since

$$\mathcal{H}^{(N)}(q) := \Delta^{(N-1)}(\mathcal{H}(\tilde{e})) \tag{4.3}$$

are, in general, smooth functions on $\overset{(N+1)}{\otimes} M$, which can be used as Hamilton ones subject to the Poisson structure on $\overset{(N+1)}{\otimes} \mathcal{P}(M)$, the expressions (4.3) mean nothing else that functions

$$\gamma^{(m)}(q) := \Delta^{(N)}(C(\tilde{e})) \tag{4.4}$$

are their invariants, that is

$$\left\{\gamma^{(m)}(q), \mathcal{H}^{(N)}(q)\right\}_{\overset{(N+1)}{\otimes} \mathcal{P}(M)} = 0 \tag{4.5}$$

for any $m = \overline{1, N}$. Thereby, the functions (4.3) and (4.4) generate under some additional but natural conditions a hierarchy of a priori Liouville – Arnold integrable Hamiltonian flows on the Poisson manifold $\overset{(N+1)}{\otimes} \mathcal{P}(M)$.

Consider now the case of a Poisson manifold $\mathcal{P}(M)$ and its coalgebra deformation $\mathcal{P}_z(\mathcal{G})$. Thus for any coordinate points $x_i \in \mathcal{P}(\mathcal{G})$, $i = \overline{1, n}$, the following relationships

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k := \vartheta_{ij}(x) \tag{4.6}$$

define a Poisson structure on $\mathcal{P}(\mathcal{G})$, related with the corresponding Lie algebra structure of $\mathcal{G}$, and there exits a representation (3.4) such that elements

$$\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x)$$

satisfy the relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\tilde{e})$ for any $i = \overline{1, n}$, with the limiting conditions

$$\lim_{z\to 0} \vartheta_{z,ij}(\tilde{e}) = \sum_{k=1}^{n} c_{ij}^k x_k, \qquad \lim_{z\to 0} \tilde{e}_i(x) = x_i \tag{4.7}$$

for any $i, j = \overline{1, n}$. For instance, take the Poisson coalgebra $\mathcal{P}_z(\mathrm{so}(2, 1))$ for which there exists a realization (3.4) in the following form:

$$\tilde{e}_1 := D_z(\tilde{X}_1) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2} x_1, \quad \tilde{e}_2 := D_z(\tilde{X}_2) = x_2,$$

$$\tilde{e}_3 := D_z(\tilde{X}_3) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2} x_3,$$

where $x_i \in \mathcal{P}(\mathrm{so}(2, 1))$, $i = \overline{1, 3}$, satisfy the $\mathrm{so}(2, 1)$-commutator relationships

$$\{x_2, x_1\}_{\mathcal{P}(\mathrm{so}(2,1))} = x_3, \quad \{x_2, x_3\}_{\mathcal{P}(\mathrm{so}(2,1))} = -x_1,$$

$$\{x_3, x_1\}_{\mathcal{P}(\mathrm{so}(2,1))} = x_2,$$

with the coproduct operator

$$\Delta : \mathcal{U}_z(\mathrm{so}(2, 1)) \to \mathcal{U}_z(\mathrm{so}(2, 1)) \otimes \mathcal{U}_z(\mathrm{so}(2, 1))$$

given by (3.3). It is easy to check that conditions (4.6) and (4.7) hold.

The next example is related with the coalgebra $U_z(\pi(1, 1))$ of the Poincare algebra $\pi(1, 1)$ for which the following nondeformed relationships

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = 0 \tag{4.8}$$

hold. The corresponding coproduct

$$\Delta : \mathcal{U}_z(\pi(1, 1)) \to \mathcal{U}_z(\pi(1, 1)) \otimes \mathcal{U}_z(\pi(1, 1))$$

is given by the Woronowicz [8] expressions

$$\Delta(\tilde{X}_1) = I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I,$$

$$\Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right), \tag{4.9}$$

$$\Delta(\tilde{X}_3) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right),$$

where $z \in \mathbb{R}$ is a parameter. Under the deformed expressions (4.9) the elements

$$\tilde{X}_j \in \mathcal{U}_z(\pi(1,1)), \qquad j = \overline{1,3},$$

satisfy still undeformed commutator relationships, that is

$$\vartheta_{z,ij}(\tilde{X}) = \vartheta_{ij}(X)\big|_{X \Rightarrow \tilde{X}}$$

for any $z \in \mathbb{R}$, $i, j = \overline{1,3}$, given by (4.8). As a result, we can state that

$$\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x) = x_i,$$

where for $x_i \in \mathcal{P}(\pi(1,1))$, $i = \overline{1,3}$, the following Poisson structure

$$\{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} = x_3, \qquad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2,$$
$$\{x_3, x_2\}_{\mathcal{P}(\pi(1,1))} = 0$$

holds. Moreover, since

$$C = x_2^2 - x_3^2 \in I(\mathcal{P}(\pi(1,1))),$$

that is

$$\{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0$$

for any $i = \overline{1,3}$, on can construct, making use of (4.3) and (4.4), integrable Hamiltonian systems on $\overset{(N)}{\otimes}\mathcal{P}(\pi(1,1))$. The same can be done for the discussed above Poisson coalgebra $\mathcal{P}_z(\mathrm{so}(2,1))$ realized by means of the Poisson manifold $\mathcal{P}(\mathrm{so}(2,1))$, taking into account that the following element

$$C = x_2^2 - x_1^2 - x_3^2 \in I(\mathcal{P}(\mathrm{so}(2,1)))$$

is a Casimir one.

Now we will consider a special extended Heisenberg – Weil coalgebra $\mathcal{U}_z(h_4)$, called still the oscillator coalgebra. The undeformed Lie algebra h_4 commutator relationships take the form:

$$\begin{gathered}[n, a_+] = a_+, \qquad [n, a_-] = -a_-, \\ [a_-, a_+] = m, \qquad [m, \cdot] = 0,\end{gathered} \tag{4.10}$$

where $\{n, a_\pm, m\} \subset h_4$ compile a basis of h_4, $\dim h_4 = 4$. The Poisson coalgebra $\mathcal{P}(h_4)$ is naturally endowed with the Poisson structure like (4.10) and admits its realization (3.4) on the Poisson manifold $\mathcal{P}(\mathbb{R}^2)$. Namely, on $\mathcal{P}(\mathbb{R}^2)$ one has

$$e_\pm = D(a_\pm) = \sqrt{p}\exp(\mp q),$$
$$e_1 = D(m) = 1, \qquad e_0 = D(n) = p,$$

where $(q, p) \in \mathbb{R}^2$ and the Poisson structure on $\mathcal{P}(\mathbb{R}^2)$ is canonical, that is the same as (3.7).

Closely related with the relationships (4.10) there is a generalized $\mathcal{U}_z(\mathrm{su}(2))$ coalgebra, for which

$$\begin{gathered}[x_3, x_\pm] = \pm x_\pm, \qquad [y_\pm, \cdot] = 0, \\ [x_+, x_-] = y_+ \sin(2zx_3) + y_- \cos(2zx_3)\frac{1}{\sin z},\end{gathered} \tag{4.11}$$

where $z \in \mathbb{C}$ is an arbitrary parameter. The coalgebra structure is given now as follows:

$$\Delta(x_\pm) = c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3},$$

$$\Delta(x_3) = I \otimes x_3 + x_3 \otimes I, \qquad \Delta(c_i^\pm) = c_i^\pm \otimes c_i^\pm,$$

$$\nu(x_\mp) = -\left(c^\pm_{1(2)}\right)^{-1} e^{-izx_3} x_\mp e^{izx_3} \left(c^\pm_{2(1)}\right)^{-1},$$

$$\nu(c_i^\pm) = \left(c_i^\pm\right)^{-1}, \qquad \nu\left(e^{\pm izx_3}\right) = e^{\mp izx_3},$$

where $c_i^\pm \in \mathcal{U}_z(\mathrm{su}(2))$, $i = \overline{1,2}$, are fixed elements. One can check that the Poisson structure on $\mathcal{P}_z(\mathrm{su}(2))$ corresponding to (4.11) can be realized by means of the canonical Poisson structure on the phase space $\mathcal{P}(\mathbb{R}^2)$ as follows:

$$[q,p] = i, \qquad D_z(x_3) = q, \qquad D_z(x_\mp) = e^{\pm ip} g_z(q),$$

$$g_z(q) = \big(k + \sin[z(s-q)]\big)\big(y_+ \sin[(q+s+1)] + y_- \cos[z(q+s+1)]\big)^{1/2} \frac{1}{\sin z},$$

where $k, s \in \mathbb{C}$ are constant parameters. Thereby making use of (4.4) and (4.5), one can construct a new class of Liouville integrable Hamiltonian flows.

5. The Heisenberg – Weil coalgebra structure and related integrable flows. Consider the Heisenberg – Weil algebra commutator relationships (4.10) and related with them the following homogenous quadratic forms

$$\left.\begin{aligned} x_1x_2 - x_2x_1 - \alpha x_3^2 &= 0, \\ x_1x_3 - x_3x_1 = 0, \quad x_2x_3 - x_3x_2 &= 0 \end{aligned}\right\} R(x), \tag{5.1}$$

where $\alpha \in \mathbb{C}$, $x_i \in A$, $i = \overline{1,3}$, are some elements of a free associative algebra A. The quadratic algebra $A / R(x)$ can be deformed via

$$\left.\begin{aligned} x_1x_2 - z_1x_2x_1 - \alpha x_3^2 &= 0, \\ x_1x_3 - z_2x_3x_1 = 0, \quad x_2x_3 - z_2^{-1}x_3x_2 &= 0 \end{aligned}\right\} R_z(x), \tag{5.2}$$

where $z_1, z_2 \in \mathbb{C}\setminus\{0\}$ are some parameters.

Let V be the vector space of columns $X := (x_1, x_2, x_3)^\top$ and define the following action

$$h_T\colon V \to (V \otimes V^*) \otimes V,$$

where, by definition, for any $X \in V$

$$h_T(T) = T \otimes X.$$

It is easy to check that conditions (5.2) will be satisfied if [9]

$$\begin{gathered} T_{11}T_{33} = T_{33}T_{11}, \quad T_{12}T_{33} = z_2^{-2}T_{33}T_{12}, \quad T_{21}T_{33} = z_1^2T_{33}T_{21}, \\ T_{22}T_{33} = T_{33}T_{22}, \quad T_{31}T_{33} = z_2T_{33}T_{31}, \quad T_{32}T_{33} = z_1^{-1}T_{33}T_{32}, \\ T_{11}T_{12} = z_1T_{12}T_{11}, \quad T_{21}T_{22} = z_1T_{22}T_{21}, \\ z_2T_{11}T_{32} - z_2T_{32}T_{11} = z_1z_2T_{12}T_{31} - T_{31}T_{12}, \\ T_{21}T_{32} - z_1z_2T_{32}T_{21} = z_1T_{22}T_{31} - z_2T_{31}T_{22}, \\ T_{11}T_{22} - T_{22}T_{11} = z_1T_{12}T_{21} - z_1^{-1}T_{21}T_{12}, \\ \left(T_{11}T_{22} - z_1T_{12}T_{21}\right) = \alpha T_{33}^2 - T_{31}T_{32} + z_1T_{32}T_{31}. \end{gathered} \tag{5.3}$$

Put mow for further convenience $z_1 = z_2^2 := z^2 \in \mathbb{C}$ and compute the "quantum" determinant $D(T)$ of the matrix $T: (A/R_z(x))^3 \to (A/R_z(x))^3$:

$$D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{5.4}$$

Note here that the determinant (5.4) is not central, that is

$$\begin{aligned} D^{-1}T_{11} &= T_{11}D^{-1}, \quad D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \\ D^{-1}T_{33} &= T_{33}D^{-1}, \quad z^{-6}D^{-1}T_{21} = T_{12}D^{-1}, \\ D^{-1}T_{22} &= T_{22}D^{-1}, \quad z^{-3}D^{-1}T_{31} = T_{31}D^{-1}, \\ & D^{-1}T_{32} = z^{-3}T_{32}D^{-1}. \end{aligned} \tag{5.5}$$

Taking into account properties (5.3) – (5.5), one can construct the Heisenberg – Weil related coalgebra $\mathcal{U}_z(h)$ being a Hopf algebra with the following coproduct Δ, counit ε and antipode ν:

$$\begin{gathered} \Delta(T) := T \otimes T, \quad \Delta(D^{-1}) := D^{-1} \otimes D^{-1}, \\ \varepsilon(T) := I, \quad \varepsilon(D^{-1}) := I, \quad \nu(T) := T^{-1}, \quad \nu(D) := D^{-1}. \end{gathered} \tag{5.6}$$

Based now on relationships (5.3), one can easily construct the Poisson tensor

$$\left\{\Delta(\tilde{T}), \Delta(\tilde{T})\right\}_{\mathcal{P}_z(h)\otimes\mathcal{P}_z(h)} = \Delta\left(\left\{\tilde{T}, \tilde{T}\right\}_{\mathcal{P}_z(h)}\right) := \vartheta_z\left(\Delta(\tilde{T})\right),$$

subject to which all of functionals (4.4) will be commuting to each other, and moreover, will be Casimir ones. Choosing some appropriate Hamiltonian functions

$$\mathcal{H}^{(N)}(\tilde{T}) := \Delta^{(N-1)}\left(\mathcal{H}(\tilde{T})\right)$$

for $N \in \mathbb{Z}_+$ one makes it possible to present a priori nontrivial integrable Hamiltonian systems. On the other hand, the coalgebra $\mathcal{U}_z(h)$ constructed by (5.5) and (5.6) possesses the following fundamental $\mathcal{R}$-matrix [4] property:

$$\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z)$$

for some complex-valued matrix $\mathcal{R}(z) \in \mathrm{Aut}(\mathbb{C}^3 \otimes \mathbb{C}^3)$, $z \in \mathbb{C}$. The latter, as is well known [4], gives rise to a regular procedure of constructing an infinite hierarchy of Liouville-integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces. We plan to consider their special cases interesting for applications elsewhere.

1. *Hopf H.* Noncommutative associative algebraic structures // Ann. Math. – 1941. – **42**, № 1. – P. 22.
2. *Postnikov M.* Lie groups and Lie algebras. – M.: Mir, 1982. – 447 p.
3. *Drinfeld V. G.* Quantum groups // Proc. Int. Congr. Math., MRSI Berkeley. – 1986. – P. 798.
4. *Korepin V., Bogoliubov N., Izergin A.* Quantum inverse scattering method and correlation functions. – Cambridge: Cambridge Univ. Press, 1993.
5. *Perelomov F.* Integrable systems of classical mechanics and Lie algebras. – Birkhauser Publ., 1990.
6. *Prykarpatsky A. K., Mykytyuk I. V.* Algebraic integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects. – Kluwer Acad. Publ., 1998.
7. *Ballesteros A., Ragnisco O.* A systematic construction of completely integrable Hamiltonian flows from coalgebras // solv-int / 9802008-6 Feb 1998. – 26 p.
8. *Woronowicz S. I.* Communs Math. Phys. – 1992. –**149** – P. 637.
9. *Bertrand J., Irac-Astaud M.* Invariance quantum groups of the deformed oscillator algebra // J. Phys. A: Math. and Gen. – 1997. – **30**. – P. 2021 – 2026.

Received 07.08.2003