UDC 517.9

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# SYSTEMS OF DIFFERENTIAL INEQUALITIES WITH INITIAL TIME DIFFERENCE

# СИСТЕМИ ДИФЕРЕНЦІАЛЬНИХ НЕРІВНОСТЕЙ З ПОЧАТКОВОЮ ЧАСОВОЮ РІЗНИЦЕЮ

Some comparison results are formulated for systems of differential inequalities starting from different initial points.

Сформульовано деякі порівнювальні результати для систем диференціальних нерівностей з різними початковими точками.

**1. Introduction.** Differential inequalities are important in the theory of differential equations to obtain, for example, some existence results or to discuss the problem of stability. The investigating of scalar differential inequalities with initial time difference was initiated in papers [1 – 3]. In this paper, we extend this discussion to finite systems of differential inequalities with initial time difference to obtain more general results. We need to use vectorial inequalities which are understood to mean that the same inequalities hold between their corresponding components. Also when we deal with systems of inequalities certain properties known as quasimonotone nondecreasing and nonincreasing properties are necessary.

Using such an approach we can compare two solutions of differential equations starting from different initial points. As we shall see there are several ways to obtain corresponding comparison results.

**2. Comparison results.** Put

$$C_1 = C^1(\mathbb{R}_+, \mathbb{R}^p), \qquad C_0 = C(\mathbb{R}_+ \times \mathbb{R}^{2p}, \mathbb{R}^p)$$

with $\mathbb{R}_+ = [0, \infty)$. First we start from a comparison result similar to the known result [4].

**Theorem 1.** *Let* $\alpha, \beta \in C_1$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$, $\Delta(t_0) = \tau_0$, $\Delta'(t) \geq 1$ *for* $t \geq t_0$, $\eta = \tau_0 - t_0 > 0$ *and* $F \in C_0$. *Let* $F(t, x, y)$ *be nondecreasing in* $t$ *for each* $(x, y)$. *Suppose that one of the following conditions hold*:

$(H_1)$:

(i) $F(t, x, y)$ *is quasimonotone nondecreasing in* $x$ *and* $y$ *for each* $t$ *and there exists* $L > 0$ *such that for each* $i = 1, 2, \ldots, p$

$$F_i(t, x_1, y_1) - F_i(t, x_2, y_2) \leq L \sum_{j=1}^{p} \left(x_{1j} - x_{2j} + y_{1j} - y_{2j}\right) \quad \textit{for each} \quad t$$

*when* $x_1 \geq x_2$, $y_1 \geq y_2$;

(ii)

$$\alpha'(t) \leq F(t, \alpha(t), \alpha(t)), \quad t \geq t_0 \geq 0, \quad \alpha(t_0) \leq x_0,$$

$$\beta'(t) \leq F(t, \beta(t), \beta(t)), \quad t \geq \tau_0 \geq 0, \quad \beta(\tau_0) \geq x_0;$$

(iii) *if* $\Delta'(t) > 1$ *for* $t \in \Omega \subset [t_0, \infty)$, *then* $F(\Delta(t), \beta(\Delta(t)), \beta(\Delta(t))) \geq 0$, $t \in \Omega$;

$(H_2)$:

(i) $F(t, x, y)$ *is quasimonotone nondecreasing in* $x$ *for each* $(t, y)$ *and nonincreasing in* $y$ *for each* $(t, x)$ *and there exists* $L > 0$ *such that for each* $i = 1, 2, \ldots, p$

$$F_i(t,x_1,y) - F_i(t,x_2,y) \le L\sum_{j=1}^{p}\left(x_{1j}-x_{2j}\right) \quad \textit{for each} \quad (t,y),$$

$$F_i(t,x,y_1) - F_i(t,x,y_2) \ge -L\sum_{j=1}^{p}\left(y_{1j}-y_{2j}\right) \quad \textit{for each} \quad (t,x)$$

*when* $x_1 \ge x_2,\ y_1 \ge y_2$;

(ii)

$$\alpha'(t) \le F(t,\alpha(t),\beta(\Delta(t))), \quad t \ge t_0 \ge 0, \quad \alpha(t_0) \le x_0,$$

$$\beta'(t) \ge F(t,\beta(t),\alpha(\Delta^{-1}(t))), \quad t \ge \tau_0 \ge 0, \quad \beta(\tau_0) \ge x_0;$$

(iii) *if* $\Delta'(t) > 1$ *for* $t \in \Omega \subset [t_0,\infty)$, *then* $F(\Delta(t),\beta(\Delta(t)),\alpha(t)) \ge 0,\ t \in \Omega$;

$(H_3)$:

(i) $F(t,x,y)$ *is quasimonotone nonincreasing in* $x$ *for each* $(t,y)$ *and nondecreasing in* $y$ *for each* $(t,x)$ *and there exists* $L > 0$ *such that for each* $i = 1,2,\ldots,p$

$$F_i(t,x_1,y) - F_i(t,x_2,y) \ge -L\sum_{j=1}^{p}\left(x_{1j}-x_{2j}\right) \quad \textit{for each} \quad (t,y),$$

$$F_i(t,x,y_1) - F_i(t,x,y_2) \le L\sum_{j=1}^{p}\left(y_{1j}-y_{2j}\right) \quad \textit{for each} \quad (t,x)$$

*when* $x_1 \ge x_2,\ y_1 \ge y_2$;

(ii)

$$\alpha'(t) \le F(t,\beta(\Delta(t)),\alpha(t)), \quad t \ge t_0 \ge 0, \quad \alpha(t_0) \le x_0,$$

$$\beta'(t) \ge F(t,\alpha(\Delta^{-1}(t)),\beta(t)), \quad t \ge \tau_0 \ge 0, \quad \beta(\tau_0) \ge x_0;$$

(iii) *if* $\Delta'(t) > 1$ *for* $t \in \Omega \subset [t_0,\infty)$, *then* $F(\Delta(t),\alpha(t),\beta(\Delta(t))) \ge 0,\ t \in \Omega$;

$(H_4)$:

(i) $F(t,x,y)$ *is quasimonotone nonincreasing in* $x$ *and* $y$ *for each* $t$, *and there exists* $L > 0$ *such that for each* $i = 1,2,\ldots,p$

$$F_i(t,x_1,y_1) - F_i(t,x_2,y_2) \ge -L\sum_{j=1}^{p}\left(x_{1j}-x_{2j}+y_{1j}-y_{2j}\right) \quad \textit{for each} \quad t,$$

*when* $x_1 \ge x_2,\ y_1 \ge y_2$;

(ii)

$$\alpha'(t) \le F(t,\beta(\Delta(t)),\beta(\Delta(t))), \quad t \ge t_0 \ge 0, \quad \alpha(t_0) \le x_0,$$

$$\beta'(t) \ge F(t,\alpha(\Delta^{-1}(t)),\alpha(\Delta^{-1}(t))), \quad t \ge \tau_0 \ge 0, \quad \beta(\tau_0) \ge x_0;$$

(iii) *if* $\Delta'(t) > 1$ *for* $t \in \Omega \subset [t_0,\infty)$, *then* $F(\Delta(t),\alpha(t),\alpha(t)) \ge 0,\ t \in \Omega$. *Then* (a) $\alpha(t) \le \beta(\Delta(t))$ *for* $t \ge t_0$, *and* (b) $\alpha(\Delta^{-1}(t)) \le \beta(t)$ *for* $t \ge \tau_0$.

***Proof.*** In view of the assumptions,

$$\Delta(t) = \Delta(t_0) + \Delta'(\xi)(t-t_0) \ge \tau_0 + t - t_0 = t + \eta > t, \quad t \ge t_0,$$

for some $\xi$, by the mean value theorem.

Suppose that Assumption $(H_1)$ holds. Put $\beta_0(t) = \beta(\Delta(t))$, so

$$\beta_0(t_0) = \beta(\Delta(t_0)) = \beta(\tau_0) \geq x_0 \geq \alpha(t_0).$$

Let

$$\overline{\beta}_0(t) = \beta_0(t) + \overline{\varepsilon} e^{(2p+1)Lt}, \quad t \geq t_0,$$

for some $\varepsilon > 0$ and $\overline{\varepsilon} = (\varepsilon, \varepsilon, \ldots, \varepsilon)$. Note that $\overline{\beta}_0(t) > \beta_0(t)$ for $t \geq t_0$, and $\overline{\beta}_0(t_0) > \beta_0(t_0) \geq \alpha(t_0)$. Hence, in view of Assumption $(H_1)$, for $i = 1, 2 \ldots, p$ we have

$$\overline{\beta}_{0i}(t) = \beta'_{0i}(\Delta(t))\Delta'(t) + (2p+1)L\varepsilon e^{(2p+1)Lt} \geq$$

$$\geq [\Delta'(t)-1]F_i(\Delta(t), \beta_0(t), \beta_0(t)) + F_i(\Delta(t), \beta_0(t), \beta_0(t)) + (2p+1)L\varepsilon e^{(2p+1)Lt} \geq$$

$$\geq F_i(\Delta(t), \beta_0(t), \beta_0(t)) - F_i\left(\Delta(t), \overline{\beta}_0(t), \overline{\beta}_0(t)\right) + (2p+1)L\varepsilon e^{(2p+1)Lt} +$$

$$+ F_i\left(\Delta(t), \overline{\beta}_0(t), \overline{\beta}_0(t)\right) \geq$$

$$\geq F_i\left(\Delta(t), \overline{\beta}_0(t), \overline{\beta}_0(t)\right) - 2L \sum_{j=1}^{p} \left[\overline{\beta}_{0j}(t) - \beta_{0j}(t)\right] + (2p+1) L\varepsilon e^{(2p+1)Lt} =$$

$$= F_i\left(\Delta(t), \overline{\beta}_0(t), \overline{\beta}_0(t)\right) + L\varepsilon e^{(2p+1)Lt} > F_i\left(\Delta(t), \overline{\beta}_0(t), \overline{\beta}_0(t)\right), \quad t \geq t_0.$$

We need to show that $\alpha(t) < \overline{\beta}_0(t)$ for $t \geq t_0$. Assume that it is false. Then there would exist an index $j$, $1 \leq j \leq p$ and $t_1 \geq t_0$ such that

$$\alpha_j(t_1) = \overline{\beta}_{0j}(t_1), \qquad \alpha_j(t) < \overline{\beta}_{0j}(t_1), \qquad t_0 \leq t < t_1,$$

and $\alpha_i(t_1) \leq \overline{\beta}_{0i}(t_1)$ for $i \neq j$. It then follows that $\alpha'_j(t_1) \geq \overline{\beta}'_{0j}(t_1)$. The above considerations and the quasimonotonicity of $F$ yield

$$F_j(t_1, \alpha(t_1), \alpha(t_1)) \geq \alpha'_j(t_1) \geq \overline{\beta}'_{0j}(t_1) > F_j\left(\Delta(t_1), \overline{\beta}_0(t_1), \overline{\beta}_0(t_1)\right) \geq$$

$$\geq F_j(\Delta(t_1), \alpha(t_1), \alpha(t_1)) \geq F_j(t_1, \alpha(t_1), \alpha(t_1)) \tag{1}$$

since $F$ is nondecreasing in $t$ and $\Delta(t) > t$, $t \geq t_0$. This leads to the contradiction which proves that $\alpha(t) < \overline{\beta}_0(t)$, $t \geq t_0$. If $\varepsilon \to 0$, we conclude that

$$\alpha(t) \leq \beta_0(t) = \beta(\Delta(t)), \quad t \geq t_0.$$

Conclusion (b) results from (a).

We shall next show that conclusion (a) and (b) hold when Assumption $(H_2)$ is satisfied. In this case the proof is similar to the proof of the previous case and therefore we only indicate needed changes. Note that

$$\overline{\beta}'_{0i}(t) \geq [\Delta'(t)-1]F_i(\Delta(t), \beta_0(t), \alpha(t)) + F_i(\Delta(t), \beta_0(t), \alpha(t)) -$$

$$- F_i\left(\Delta(t), \overline{\beta}_0(t), \alpha(t)\right) + F_i\left(\Delta(t), \overline{\beta}_0(t), \alpha(t)\right) + (2p+1)L\varepsilon e^{(2p+1)Lt} \geq$$

$$\geq F_i\left(\Delta(t), \overline{\beta}_0(t), \alpha(t)\right) + (p+1)L\varepsilon e^{(2p+1)Lt}.$$

Instead of (1) we now have

$$F_j(t_1, \alpha(t_1), \beta_0(t_1)) \geq \alpha'_j(t_1) \geq \overline{\beta}'_{0j}(t_1) >$$

$$> F_j\left(\Delta(t_1), \overline{\beta}_0(t_1), \alpha(t_1)\right) + (p+1)L\varepsilon e^{(2p+1)Lt_1} \geq$$

$$\geq F_j\left(\Delta(t_1), \alpha(t_1), \overline{\beta}_0(t_1)\right) + (p+1)L\varepsilon e^{(2p+1)Lt_1} \geq$$

$$\geq F_j(\Delta(t_1), \alpha(t_1), \beta_0(t_1)) + L\varepsilon e^{(2p+1)Lt_1} > F_j\left(t_1, \alpha(t_1), \beta_0(t_1)\right).$$

This proves that $\alpha(t) < \overline{\beta}_0(t)$, $t \geq t_0$ and if $\varepsilon \to 0$, we conclude that (a) holds. Indeed (b) results from (a).

The conclusion relative to $(H_3)$ and $(H_4)$ can be proved using similar arguments.

The theorem is proved.

***Remark* 1.** Let $a \geq 1$ and $\Delta(t) = a(t - t_0) + \tau_0$, $t \geq t_0$. Then $\Delta(t_0) = \tau_0$, $\Delta'(t) = a \geq 1$, $t \geq t_0$, and

$$\Delta^{-1}(t) = \frac{1}{a}(t - \tau_0) + t_0, \quad t \geq \tau_0.$$

In this case the assertion of Theorem 1 takes the form:

(a) $\alpha(t) \leq \beta[\alpha(t - t_0) + \tau_0]$, $t \geq \tau_0$,

(b) $\alpha\left[\frac{1}{a}(t - \tau_0) + t_0\right] \leq \beta(t)$, $t \geq \tau_0$.

If $p = a = 1$, $F(t, x, y) = F(t, x)$ and Assumption $(H_1)$ holds, then we have Theorem 2.1 [2] (see also [5]).

***Remark* 2.** Let

$$\Delta(t) = \begin{cases} t + \eta, & t \in [t_0, t_1); \\ -\dfrac{(t - t_1)^3}{3} + \dfrac{(t_2 - t_1)(t - t_1)^2}{2} + t + \eta, & t \in [t_1, t_2]; \\ t + \eta + \dfrac{1}{6}(t_2 - t_1)^3, & t > t_2, \end{cases}$$

for $\eta = \tau_0 - t_0 > 0$. Then $\Delta(t_0) = \tau_0$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$. Note that

$$\Delta'(t) = \begin{cases} 1, & t \in [t_0, t_1]; \\ -(t - t_1)^2 + (t_2 - t_1)(t - t_1) + 1, & t \in (t_1, t_2); \\ 1, & t \geq t_2, \end{cases}$$

so $\Delta'(t) = 1$ for $t \in [t_0, t_1] \cup [t_2, \infty)$, and $\Delta'(t) > 1$ for $t \in (t_1, t_2)$. Indeed, $\Delta^{-1}$ exists.

Also we can consider the case where $t_0 > \tau_0$. By $(H_1)'$, $(H_2)'$, $(H_3)'$, $(H_4)'$ we denote respectively Assumptions $(H_1)$, $(H_2)$, $(H_3)$, $(H_4)$ in which we replace condition (iii) respectively by

(iii)′ if $0 \leq \Delta'(t) < 1$ for $t \in \Omega$, then $F(\Delta(t), \beta(\Delta(t)), \beta(\Delta(t))) \leq 0$, $t \in \Omega$,

(iii)′ if $0 \leq \Delta'(t) < 1$ for $t \in \Omega$, then $F(\Delta(t), \beta(\Delta(t)), \alpha(t)) \leq 0$, $t \in \Omega$,

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(iii)′ if $0 \leq \Delta'(t) < 1$ for $t \in \Omega$, then $F(\Delta(t), \alpha(t), \alpha(t)) \leq 0$, $t \in \Omega$.

Then we have the following theorem.

**Theorem 2.** *Let* $\alpha, \beta \in C_1$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$, $\Delta(t_0) = \tau_0$, $0 \leq \Delta'(t) \leq 1$ *for* $t \geq t_0$, $\eta = \tau_0 - t_0 < 0$ *and* $F \in C_0$. *Let* $F(t, x, y)$ *be nonincreasing in* $t$ *for each* $(x, y)$. *If one of Assumptions* $(H_1)'$ *or* $(H_2)'$ *or* $(H_3)'$ *or* $(H_4)'$ *holds, then the conclusions* (a) *and* (b) *of Theorem* 1 *are valid.*

***Remark* 3.** Let the assumptions of Theorem 1 or the assumptions of Theorem 2 hold. Then, in any case of Assumptions $(H_i)$ or $(H_j)'$ for $i, j = 1, 2, 3, 4$, we have 16 possibilities of the systems of inequalities, for example,

$$\alpha'(t) \leq F(t, \alpha(t), \alpha(t)), \quad t \geq t_0 \geq 0, \quad \alpha(t_0) \leq x_0,$$
$$\beta'(t) \geq F(t, \beta(t), \beta(t)), \quad t \geq \tau_0 \geq 0, \quad \beta(\tau_0) \geq x_0,$$

or

$$\alpha'(t) \le F(t, \beta(\Delta(t)), \beta(\Delta(t))), \quad t \ge t_0 \ge 0, \quad \alpha(t_0) \le x_0,$$

$$\beta'(t) \ge F(t, \alpha(\Delta^{-1}(t)), \alpha(\Delta^{-1}(t))), \quad t \ge \tau_0 \ge 0, \quad \beta(\tau_0) \ge x_0.$$

This results from the quasimonotonicity of $F$ and conclusions (a) and (b) of Theorem 1.

In the next theorem we assume that there exist minimal or maximal solutions of corresponding equations instead of the assumption that $\alpha'$ and $\beta'$ satisfy some inequalities.

**Theorem 3.** *Let* $\alpha, \beta \in C_1$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$, $\Delta(t_0) = \tau_0$, $F \in C_0$. *Assume that one of the following assumptions hold:*

($H_5$): $\Delta'(t) \ge 1$ *for* $t \ge t_0$, $\eta = \tau_0 - t_0 > 0$, $F(t, x, y)$ *is nondecreasing in* $t$ *for each* $(x, y)$, *Assumption* ($H_1$) (i), (iii) *holds and*

(a) $\alpha'(t) \le F(t, \alpha(t), \alpha(t))$, $t \ge t_0 \ge 0$, $\alpha(t_0) \le x_0$, *and there exists for* $t \ge \ge \tau_0$ *the maximal solution* $\beta$ *of equation* $\beta'(t) = F(t, \beta(t), \beta(t))$, $t \ge \tau_0 \ge 0$, $\beta(\tau_0) \ge x_0$;

($H_6$): $\Delta'(t) \ge 1$ *for* $t \ge t_0$, $\eta = \tau_0 - t_0 > 0$, $F(t, x, y)$ *is nondecreasing in* $t$ *for each* $(x, y)$, *Assumption* ($H_1$) (i), (iii) *holds and*

(b) $\beta'(t) \ge F(t, \beta(t), \beta(t))$, $t \ge \tau_0 \ge 0$, $\beta(\tau_0) \ge x_0$, *and there exists for* $t \ge \ge t_0$ *the minimal solution* $\alpha$ *of equation* $\alpha'(t) = F(t, \alpha(t), \alpha(t))$, $t \ge t_0 \ge 0$, $\alpha(t_0) \le x_0$;

($H_7$): $0 \le \Delta'(t) \le 1$ *for* $t \ge t_0$, $\eta = \tau_0 - t_0 < 0$, $F(t, x, y)$ *is nonincreasing in* $t$ *for each* $(x, y)$, *Assumption* ($H_1$)′ (i)′, (iii)′ *and condition* (a) *hold;*

($H_8$): $0 \le \Delta'(t) \le 1$ *for* $t \ge t_0$, $\eta = \tau_0 - t_0 < 0$, $F(t, x, y)$ *is nonincreasing in* $t$ *for each* $(x, y)$, *Assumption* ($H_1$)′ (i)′, (iii)′ *and condition* (b) *hold.*

*Then conclusions* (a) *and* (b) *of Theorem* 1 *are valid.*

The special linear case of Theorem 3 is the Gronwall inequality in this framework which we state as a corollary (see also [2, 3]).

***Corollary* 1.** *Let* $m, \lambda \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$, $\Delta(t_0) = \tau_0$, *and*

$$m(t) \le m(t_0) + \int_{t_0}^{t} \lambda(s) m(s)\, ds, \quad t \ge t_0 \ge 0.$$

*If* $\Delta'(t) \ge 1$, $t \ge t_0$, $\eta = \tau_0 - t_0 > 0$ *and* $\lambda$ *is nondecreasing, then*

$$m(t) \le m(t_0) \exp\left[\int_{\tau_0}^{\Delta(t)} \lambda(s)\, ds\right] = m(t_0) \exp\left[\int_{t_0}^{t} \Delta'(s) \lambda(\Delta(s))\, ds\right], \quad t \ge t_0,$$

*and*

$$m\left(\Delta^{-1}(t)\right) \le m(t_0) \exp\left[\int_{\tau_0}^{t} \lambda(s)\, ds\right] \le m(t_0) \exp\left[\int_{t_0}^{t} \lambda(s)\, ds\right], \quad t \ge \tau_0.$$

*If* $\eta = \tau_0 - t_0 < 0$ *and* $\lambda$ *is nonincreasing, then*

$$m(t - \eta) \le m(t_0) \exp\left[\int_{\tau_0}^{t} \lambda(s)\, ds\right] \le m(t_0) \exp\left[\int_{\tau_0}^{t} \lambda(s + \eta)\, ds\right], \quad t \ge \tau_0,$$

*and*

$$m(t) \le m(t_0) \exp \left[ \int_{\tau_0}^{t} \lambda(s)\, ds \right], \quad t \ge t_0.$$

Also we can formulate the following corollary.

**Corollary 2.** *Let* $m, \lambda \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\Delta \in C^1([t_0, \infty), [\tau_0, \infty))$, $\Delta(t_0) = \tau_0$, *and*

$$m(t) \ge m(t_0) + \int_{\tau_0}^{t} \lambda(s) m(s)\, ds, \quad t \ge \tau_0 \ge 0.$$

*If* $\Delta'(t) \ge 1$, $t \ge t_0$, $\eta = \tau_0 - t_0 > 0$ *and* $\lambda$ *is nonincreasing, then*

$$m(\Delta(t)) \ge m(t_0) \exp \left[ \int_{t_0}^{t} \lambda(s)\, ds \right], \quad t \ge t_0,$$

*and*

$$m(t) \ge m(t_0) \exp \left[ \int_{t_0}^{\Delta^{-1}(t)} \lambda(s)\, ds \right] = m(t_0) \exp \left[ \int_{\tau_0}^{t} \left(\Delta^{-1}(s)\right)' \lambda\left(\Delta^{-1}(s)\right) ds \right]$$

*for* $t \ge \tau_0$. *If* $\eta = \tau_0 - t_0 < 0$ *and* $\lambda$ *is nonincreasing, then*

$$m(t + \eta) \ge m(t_0) \exp \left[ \int_{t_0}^{t} \lambda(s)\, ds \right], \quad t \ge \tau_0,$$

*and*

$$m(t) \ge m(t_0) \exp \left[ \int_{t_0}^{t-\eta} \lambda(s)\, ds \right] = m(t_0) \exp \left[ \int_{\tau_0}^{t} \lambda(s - \eta)\, ds \right], \quad t \ge \tau_0.$$

In Theorems 1 and 2, the monotonicity of $F$ with respect to the first argument was assumed. Now we formulate corresponding comparison results when we do not need this assumption.

**Theorem 4.** *Let* $\alpha, \beta \in C_1$, $\eta = \tau_0 - t_0 > 0$, *and* $F \in C_0$, *Suppose that Assumption* $(H_1)$ (i), (ii) *holds. Let the condition*

(iv) $\displaystyle\int_{t_0}^{\tau_0} F(s, \alpha(s), \alpha(s))\, ds \le 0$

*hold. Then* $\alpha(t) \le \beta(t)$, $t \ge \tau_0$.

*Proof.* Note that

$$\alpha(\tau_0) \le \alpha(t_0) + \int_{t_0}^{\tau_0} F(s, \alpha(s), \alpha(s))\, ds \le \alpha(t_0) \le \beta(\tau_0).$$

Now, if we put

$$\overline{\beta}(t) = \beta(t) + \overline{\varepsilon} e^{(2p+1)Lt}, \quad t \ge \tau_0,$$

then repeating the proof of Theorem 1, we obtain $\alpha(t) < \overline{\beta}(t)$, $t \ge \tau_0$. Letting $\varepsilon \to 0$ finishes the proof.

*Remark* 4. Let $p = 1$, $F(t, x, y) = F(t, x)$. Suppose that condition (iv) and Assumption $(H_1)$ (i), (ii) hold. Then we have Theorem 3.1 [1].

**Theorem 5.** *Let* $\alpha, \beta \in C_1$, $\eta = \tau_0 - t_0 < 0$, *and* $F \in C_0$. *Suppose that Assumption* $(H_1)$ (i), (ii) *holds. Let the condition*

(v) $\int_{\tau_0}^{t_0} F(s, \beta(s), \beta(s))\, ds \geq 0$

*hold. Then* $\alpha(t) \leq \beta(t)$, $t \geq t_0$.

*Proof.* Indeed, in this case we have

$$\beta(t_0) \geq \beta(\tau_0) + \int_{\tau_0}^{t_0} F(s, \beta(s), \beta(s))\, ds \geq \beta(\tau_0) \geq \alpha(t_0).$$

The rest of this proof is clear.

**Theorem 6.** **(A)** *Let the assumptions of Theorem* 1 (*without* $(H_1)$) *and condition* (iv) *hold. Then the conclusion of Theorem* 4 *holds.*

**(B)** *Let the assumptions of Theorem* 2 (*without* $(H_1)'$) *and condition* (v) *hold. Then the conclusion of Theorem* 5 *holds.*

*Proof.* This results from Remark 3 and Theorem 3 or Theorem 4, and therefore the proof is omitted.

The theory of differential inequalities is useful to discuss the method of upper and lower solutions, monotone iterative technique, global existence and stability criteria. Corresponding results for differential problems with initial conditions are considered, for example, in papers [1 – 3, 5, 6].


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Received 16.04.2002