UDC 517.5

B. Forster (Zentrum Math., Techn. Univ. München, Germany)

ON THE RELATION BETWEEN FOURIER AND LEONT'EV COEFFICIENTS WITH RESPECT TO SMIRNOV SPACES *

ПРО СПІВВІДНОШЕННЯ МІЖ КОЕФІЦІЄНТАМИ ФУР'Є ТА ЛЕОНТЬЄВА СТОСОВНО ПРОСТОРІВ СМІРНОВА

Yu. I. Mel'nik showed that the Leont'ev coefficients $\kappa_f(\lambda)$ in the Dirichlet series $f \sim \sum_{\lambda\in\Lambda} \kappa_f(\lambda) \dfrac{e^{\lambda\cdot}}{L'(\lambda)}$ of a function $f \in E^p(D)$, $1 < p < \infty$, are the Fourier coefficients of some function $F \in L^p([0, 2\pi])$ and that the first modulus of continuity of F can be estimated by first moduli and majorants in f. In the present paper, we extend his results to moduli of arbitrary order.

Ю. І. Мельник показав, що коефіцієнти Леонтьєва $\kappa_f(\lambda)$ в рядах Діріхле $f \sim \sum_{\lambda\in\Lambda} \kappa_f(\lambda) \times \dfrac{e^{\lambda\cdot}}{L'(\lambda)}$ для функції $f \in E^p(D)$, $1 < p < \infty$, є коефіцієнтами Фур'є для деякої функції $F \in L^p([0, 2\pi])$ і що перший модуль неперервності F можна оцінити першими модулями та мажорантами в f. У даній статті його результати поширено на модулі довільного порядку.

1. Introduction. Let $\overline{D}$ be a closed convex polygon with vertices $a_1, \ldots, a_N$, $N > 2$, D its open part, and $\partial D = \overline{D} \setminus D$ the boundary of D. We assume that the origin belongs to D. As is customary, we denote by $E^p(D)$, $1 < p < \infty$, the Banach space of all functions $f(z)$ which are analytic in D and satisfy

$$\|f\|_p := \sup_{n\in\mathbb{N}} \int_{\gamma_n} |f(z)|^p |dz| < \infty.$$

Here, $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of closed rectifiable Jordan contours $\gamma_n \subset D$ which converges to ∂D. The space $E^p(D)$ is called Smirnov space.

Consider the quasipolynomial $L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$, where $d_k \in \mathbb{C}\setminus\{0\}$ and a_k are the vertices of D, $k = 1, \ldots, N$. Let $\Lambda = (\lambda_m)_{m\in\mathbb{N}}$ be its sequence of zeros. We can expand functions $f \in E^p(D)$ with respect to the family $\mathcal{E}(\Lambda) := (e^{\lambda_m z})_{m\in\mathbb{N}}$ into a series of complex exponentials, the so-called Dirichlet series

$$f(z) \sim \sum_{\lambda_m\in\mathbb{N}} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)}, \tag{1}$$

where

$$\kappa_f(\lambda_m) = \sum_{k=1}^{N} d_k e^{a_k\lambda_m} \int_{a_j}^{a_k} f(\eta) e^{-\lambda_m\eta} d\eta. \tag{2}$$

The indexing of Λ is chosen such that $(|\lambda_m|)_{m\in\mathbb{N}}$ is nondecreasing. The coefficients $\kappa_f(\lambda_m)$ are called Leont'ev coefficients. Many results on these series are due to A. F. Leont'ev [1]. B. Ja. Levin and Ju. I. Ljubarskii showed in [2] that, for $p = 2$, the

* Supported by the "Deutsche Forschungsgemeinschaft" through the graduate program "Angewandte Algorithmische Mathematik", Technische Universität München.

family $\mathcal{E}(\Lambda)$ forms a Riesz basis of $E^2(D)$ and, hence, the series (1) converges unconditionally in norm. In [3], A. M. Sedleckii proved that, for arbitrary $1 < p < \infty$, the Dirichlet series (1) converges in norm since $\mathcal{E}(\Lambda)$ forms a Schauder basis in $E^p(D)$.

To estimate the rate of convergence of these series, Yu. I. Mel'nik studied the relation between Leont'ev coefficients and Fourier coefficients, since, for the latter, many results on approximation and rate of convergence of the Fourier series are well-known (see for example [4]). He showed that, under certain conditions, the Leont'ev coefficients of $f \in E^p(D)$ are the Fourier coefficients of some function $F \in L^p([0, 2\pi])$. He estimated the regularity of F with first moduli of continuity. In the following section, we will state his results. Extending his Theorem 1 to moduli of smoothness of arbitrary order, we obtain Theorem 2 in Section 3. The last section contains the respective proof.

2. Yu. I. Mel'nik's results. Yu. I. Mel'nik considered in [5] and [6] the relation of Leont'ev coefficients of $f \in E^p(D)$ to the Fourier coefficients of some suited function $F \in L^p([0, 2\pi])$ for first moduli of continuity. His first step was the reduction of the integral in (2) to a Fourier transform:

Lemma 1 [5].

(i) *Let* $\Phi \in L^p([0, 2\pi])$, $1 < p < \infty$, *and* $\Re(v) > 0$. *Denote*

$$\check{\Phi}(t) := \sum_{m=n(j)}^{\infty} d_m(\Phi) e^{imt},$$

where

$$d_m(\Phi) := \int_0^{2\pi} \Phi(\xi) e^{-mv\xi} d\xi, \qquad m \geq n(j).$$

Then $\check{\Phi} \in L^p([0, 2\pi])$, *and* $\|\check{\Phi}\| \leq \text{const} \cdot \|\Phi\|$ *for some positive constant only depending on* p.

(ii) *Let* $f \in E^p(D)$, $1 < p < \infty$. *For fixed* $1 \leq j \leq N$, *the Leont'ev coefficients* $(\kappa_f(\lambda_m^{(j)}))_{m \geq n(j)}$ *are the Fourier coefficients of some function* $F_j \in L^p([0, 2\pi])$, *and* $\|F_j\|_{L^p} \leq \text{const} \cdot \|f\|_{E^p}$.

This result was extended in [6] using first moduli of continuity. Consider the parametrization $z : \partial D \to [0, T]$ of ∂D

$$z(u) = a_j + \frac{a_{j+1} - a_j}{|a_{j+1} - a_j|}(u - T_{j-1}) \quad \text{for} \quad T_{j-1} \leq u \leq T_j, \quad j = 1, \ldots, N,$$

where $T_0 := 0$, $T_j = \sum_{k=1}^{j} |a_{k+1} - a_k|$ and $T := T_N := \sum_{k=1}^{N} |a_{k+1} - a_k|$. For $f \in E^p(D)$ and $0 < h < 2\pi$, let

$$\delta_1(f, h)_p := \sum_{j=1}^{N} \left\{ \left(\int_0^h \left| f\left(a_j + \frac{a_{j+1} - a_j}{2\pi}\theta\right) \right|^p d\theta \right)^{1/p} + \right.$$

$$\left. + \left(\int_{2\pi - h}^{2\pi} \left| f\left(a_j + \frac{a_{j+1} - a_j}{2\pi}\theta\right) \right|^p d\theta \right)^{1/p} \right\}.$$

The function $\delta_1(f, h)_p$ is continuous, nonincreasing, and vanishing for $h \to 0+$.

Theorem 1 [6]. *Let* $f \in E^p(D)$, $1 < p < \infty$, *and let* $1 \leq j \leq N$ *be fixed.*

Then the Leont'ev coefficients $\kappa_f(\lambda_m^{(j)})$, $m \geq n(j)$, *of* f *are Fourier coefficients of some function* $F_j \in L^p([0, 2\pi])$. *Furthermore,*

$$\omega_1(F_j, h)_p \leq \text{const} \cdot (\omega_1(f \circ z, h)_p + \delta_1(f, h)_p).$$

The proof can be deduced as a special case of Subsection 4.2. Yu. I. Mel'nik used his results in [6] to proof direct approximation theorems for first moduli. As we will see in the following Section 3, Theorem 1 can also be proved for moduli of arbitrary order.

3. Extension to moduli of arbitrary order. For an extension of Theorem 1, we have to define moduli of smoothness of order k for functions $f \in E^p(D)$. This can be done using best-approximation with algebraic polynomials.

Let $f \in E^p(\partial D)$ and let $I \subset \partial D$ be some arc. For $k \in \mathbb{N}_0$, the equation

$$E_k(f, I) = \inf_{P_k} \| f - P_k \|_{L^p(I)}$$

defines the algebraic best-approximation on the arc I. Here, the infimum is taken over all algebraic polynomials P_k of degree at most k. The modulus of order k is defined as follows.

Definition 1. *Let* $f \in E^p(D)$, $1 < p < \infty$. *For* $h > 0$, *consider all partitions* $\partial D = \bigcup_{j=1}^{n} I_j$ *with* $h/2 \leq |I_j| \leq h$. *The k-th metrical modulus of smoothness of the function* f *is defined by*

$$\omega_k(f, h)_p := \omega_{k,\overline{D}}(f, h)_p := \sup\left(\sum_{j=1}^{n} \inf_{P_k} \| f - P_k \|_{L^p(I_j)}\right) = \sup\left(\sum_{j=1}^{n} E_k(f, I_j)\right).$$

Here, the supremum is taken over all such partitions.

One can show that these moduli are equivalent to usual moduli of smoothness defined on finite intervals [7]. We can formulate Theorem 1 for k-th moduli.

Theorem 2. *Let* $f \in E^p(D)$, $1 < p < \infty$, *and let* $1 \leq j \leq N$ *be fixed. Then the Leont'ev coefficients* $\kappa_f(\lambda_m^{(j)})$, $m \geq n(j)$, *are the Fourier coefficients of some function* $F_j \in L^p([0, 2\pi])$:

$$\kappa_f(\lambda_m^{(j)}) = \frac{1}{2\pi} \int_0^{2\pi} F_j(\theta) e^{im\theta} d\theta =: c_m(F_j).$$

The k*-th modulus of* F_j *can be estimated by*

$$\omega_k(F_j, h)_p \leq \text{const} \cdot (\omega_k(f, h)_p + \delta_k(f, h)_p), \tag{3}$$

where

$$\delta_k(f, h)_p := \sum_{j=1}^{N} \sum_{n=1}^{k} \binom{k}{n} \left\{ \left(\int_0^{nh} \left| f\left(a_j - \frac{a_{j+1} - a_j}{2\pi}\theta\right) \right|^p d\theta \right)^{1/p} + \right.$$

$$\left. + \left(\int_{2\pi - nh}^{2\pi} \left| f\left(a_j - \frac{a_{j+1} - a_j}{2\pi}\theta\right) \right|^p d\theta \right)^{1/p} \right\}.$$

The function $\delta_k(f, h)_p$ *is continuous and nonincreasing for* $0 < h < 2\pi/h$ *and satisfies* $\lim_{h \to 0+} \delta_k(f, h)_p = 0$.

This result enables us to transform the Leont'ev coefficients (2) in the Dirichlet series (1) into Fourier coefficients of certain functions F. Since Theorem 2 provides information on the regularity of F, classical Bernstein theorems can be applied on the respective Fourier series. This can be used to proof new results on the rate of approximation of Dirichlet series (1).

The term $\delta_k(f,h)_p$ cannot be omitted from the theorem, as the following example shows. Let $p=2$, $f(z)=1$. Suppose that $L(0)=1$. Then $\omega_k(f,h)_2 \equiv 0$, whereas $\delta_k(f,h)_2 = O(\sqrt{h})$ for $h \to 0$ and all $k \in \mathbb{N}$. For the Leont'ev coefficients, we have

$$\kappa_f(\lambda_m^{(j)}) = -\frac{1}{\lambda_m^{(j)}} = O\left(\frac{1}{m}\right) \quad \text{for} \quad m \to \infty.$$

We know from Lemma 1 that $F_j = \sum_{m \ge n(j)} \kappa_f(\lambda_m^{(j)}) e^{im\cdot} \in L^2([0,2\pi])$. The Bernstein Theorem [4] yields $\omega_k(F_j,h)_2 = O(\sqrt{h})$ since the approximation with partial series $S_n(F_j) = \sum_{m \ge n(j)}^{n} \kappa_f(\lambda_m^{(j)}) e^{im\cdot}$ gives

$$\left\| F_j - S_n(F_j) \right\|_2 = \left\| \sum_{m>n>n(j)}^{\infty} \frac{1}{\lambda_m^{(j)}} e^{im\cdot} \right\| = \left(\sum_{m>n>n(j)}^{\infty} \left| \frac{1}{\lambda_m^{(j)}} \right|^2 \right)^{1/2} = O\left(\frac{1}{\sqrt{n}}\right) \text{ for } n \to \infty.$$

Thus, the term $\delta_k(f,h)_2$ is necessary in (3) (see also [6]).

4. Proof of Theorem 2. *4.1. Preliminaries.* Let us first have a closer look on the quasipolynomial $L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$, where $d_k \in \mathbb{C} \setminus \{0\}$ and a_k are the vertices of D, $k=1,\dots,N$. Let $\Lambda = (\lambda_m)_{m \in \mathbb{N}}$ be its sequence of zeros. The entire function L has the following properties [1] (Chapter 1, § 2):

i) For sufficient large C, the zeros $\lambda_n^{(j)}$ of L with $\left|\lambda_n^{(j)}\right| > C$ are of the form $\lambda_n^{(j)} = \tilde{\lambda}_n^{(j)} + \delta_n^{(j)}$, where $\tilde{\lambda}_n^{(j)} = \dfrac{2\pi n i}{a_{j+1} - a_j} + q_j e^{i\beta_j}$, and $\left|\delta_n^{(j)}\right| \le e^{-an}$. Here, $0 < a =$ $=$ const, $j = 1,\dots,N$, $n > n_0$, and $a_{N+1} := a_1$. The parameters b_j and q_j are defined by $e^{q_j(a_{j+1}-a_j)e^{i\beta_j}} = -d_j / d_{j+1}$, where $d_{N+1} := d_1$. Hence, there zeros $\lambda_n^{(j)}$ are simple. The set of zeros Λ can be represented in the form

$$\Lambda = \{\lambda_n\}_{n=1,\dots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\dots} \right).$$

ii) There are positive constants A_1 and c_1 such that, for all $n \ge n(j)$ and all $\xi \in$ $\in [a_j, a_k]$, we have $\left| e^{-\lambda_n^{(j)}(\xi - a_k)} - e^{-\tilde{\lambda}_n^{(j)}(\xi - a_k)} \right| \le A_1 e^{-c_1 n}$. Here, $[a_j, a_k]$ denotes the line between the vertices a_j and a_k in the complex plane.

For simplicity reasons, we assume that all zeros of L are simple. We will use i) and ii) to treat the zeros of L and to estimate the complex exponentials in the Dirichlet series (1). In addition, we need the following result on multipliers:

Theorem 3 (J. Marcinkiewicz, [8], Theorem 4.14). *Let* $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$ *be some series such that*

$$|a_n| < M \quad \textit{and} \quad \sum_{j=2^n}^{2^{n+1}-1} \left| a_j - a_{j+1} \right| \le M$$

for all $n \in \mathbb{N}_0$ *and some suited positive constant* M. *Let* $f = \sum_{n=0}^{\infty} c_n e^{in\cdot} \in L^p([0, 2\pi])$, $1 < p < \infty$.

Then there exists a function $h \in L^p([0, 2\pi])$ *with*

$$h = \sum_{n=0}^{\infty} c_n a_n e^{in\cdot} \quad \text{and} \quad \|h\| \leq C(p) M \|f\|,$$

where the constant $C(p) > 0$ *only depends on* p.

Now we have all means for the proof of Theorem 2.

4.2. ***Proof.*** The existence of a function $F_j \in L^p([0, 2\pi])$ with the the prementioned properties is shown in Lemma 1 (ii). Thus, we just have to examine the regularity of F_j. Using conditions i) and ii) of Subsection 4.1, we can write

$$\kappa_f(\lambda_n^{(j)}) = \sum_{k=1}^{N} d_k \int_{a_j}^{a_k} f(z) e^{-\lambda_n^{(j)}(z-a_k)} dz =$$

$$= \sum_{k=1}^{N} d_k \int_{a_j}^{a_k} f(z) e^{-\tilde{\lambda}_n^{(j)}(z-a_k)} dz + O(e^{-cn}) =$$

$$= d_{j+1} \frac{a_{j+1} - a_j}{2\pi} \int_0^{2\pi} f\left(a_{j+1} - \frac{a_j - a_{j+1}}{2\pi}\theta\right) e^{q_j e^{i\beta_j} \frac{a_{j+1} - a_j}{2\pi}\theta} e^{in\theta} d\theta +$$

$$+ \sum_{\substack{k=1 \\ k \neq j, j+1}}^{N} d_k \frac{a_k - a_j}{2\pi} \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\theta\right) e^{q_j e^{i\beta_j} \frac{a_k - a_j}{2\pi}\theta} e^{-ni \frac{a_j - a_k}{a_{j+1} - a_j}\theta} d\theta +$$

$$+ O(e^{-cn}).$$

The first term is obviously the n-th Fourier coefficient of some function with modulus of order $\omega_{k,\overline{D}}(f, h)_p$. Using Lemma 1 (i) for the second term, we just have to analyse the regularity of $\check{\Phi}$ with respect to the regularity of some function $\Phi \in L^p([0, 2\pi])$ since

$$\Re\left(i \frac{a_j - a_k}{a_{j+1} - a_j}\right) > 0.$$

Then the assertion follows from $\omega_k(f|_{[a_j, a_{j+1}]}, h)_p \leq \omega_k(f, h)_p$.

Let $h > 0$, $\Re(v) > 0$, $\alpha \in \mathbb{R}$, and $\Phi \in L^p([0, 2\pi])$. We will show that the series of coefficients $(d_m(\Phi))_{m \geq n(j)}$ is the series of Fourier coefficients of some function $\check{\Phi} \in L^p([0, 2\pi])$ with

$$\omega_k(\check{\Phi}, h)_p \leq \text{const} \cdot (\omega_k(\Phi, h)_p + \delta'_k(\Phi, h)_p),$$

where

$$\delta'_k(\Phi, h)_p := \sum_{n=1}^{k} \binom{k}{n} \left\{ \left(\int_0^{nh} |\Phi(u)|^p du \right)^{1/p} + \left(\int_{2\pi - nh}^{2\pi} |\Phi(u)|^p du \right)^{1/p} \right\}.$$

Let $\varphi \in L^p([0, 2\pi])$. Then

$$A := \int_0^{2\pi} (\Delta_{-|\alpha|}^k \varphi)(u) e^{-mvu}\, du - (\Delta_{|\alpha|}^k e^{-mv\cdot})(0) \int_0^{2\pi} \varphi(u) e^{-mvu}\, du =$$

$$= \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \left\{ \int_0^{2\pi} \varphi(u - n|\alpha|) e^{-mvu} du - \int_0^{2\pi} \varphi(u) e^{-mv(u+n|\alpha|)} du \right\} =$$

$$= \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \left\{ \int_0^{2\pi} \varphi(u - n|\alpha|) e^{-mvu} du - \int_{n|\alpha|}^{2\pi+n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du \right\} =$$

$$= \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \left\{ \int_0^{n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du - \int_{2\pi}^{2\pi+n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du \right\} =$$

$$= \left(1 - e^{-2\pi mv}\right) \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_0^{n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du.$$

With notation of Lemma 1 (i), we have

$$\Delta_\alpha^k \check{\varphi}(t) = \sum_{m=n(j)}^{\infty} d_m(\varphi)(\Delta_\alpha^k e^{im\cdot})(t) = \sum_{m=n(j)}^{\infty} d_m(\varphi)(\Delta_\alpha^k e^{im\cdot})(0) e^{imt} =$$

$$= \sum_{m=n(j)}^{\infty} \int_0^{2\pi} \varphi(\xi) e^{-mv\xi} d\xi \,(\Delta_\alpha^k e^{im\cdot})(0) e^{imt} =$$

$$= \sum_{m=n(j)}^{\infty} \left(\int_0^{2\pi} (-\Delta_{-|\alpha|}^k \varphi)(u) e^{-mvu} du + A \right) \frac{(\Delta_\alpha^k e^{im\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-mv\cdot})(0)} e^{imt} =$$

$$= \sum_{m=n(j)}^{\infty} d_m(-\Delta_{-|\alpha|}^k \varphi) \frac{(\Delta_\alpha^k e^{im\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-mv\cdot})(0)} e^{imt} +$$

$$+ \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_0^{n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du \times$$

$$\times \left(1 - e^{-2\pi vm}\right) \frac{(\Delta_\alpha^k e^{im\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-mv\cdot})(0)} e^{imt} =$$

$$= \sum_{m=n(j)}^{\infty} d_m(-\Delta_{-|\alpha|}^k \varphi) \mu_m e^{imt} +$$

$$+ \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_0^{n|\alpha|} \varphi(u - n|\alpha|) e^{-mvu} du\, \tilde{\mu}_m e^{imt}, \tag{4}$$

where

$$\mu_m = \frac{(\Delta_\alpha^k e^{im\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-mv\cdot})(0)} \quad \text{and} \quad \tilde{\mu}_m = \left(1 - e^{-2\pi mv}\right) \mu_m.$$

We will show that μ_m and $\tilde{\mu}_m$ are multipliers in $L^p([0, 2\pi])$. Thus, the Fourier series weighted with $(\mu_m)_{m \geq n(j)}$ and $(\tilde{\mu}_m)_{m \geq n(j)}$ both converge in $L^p([0, 2\pi])$. At first, let us consider μ_m. We have

$$\mu_{m+1} - \mu_m = \int_m^{m+1} \left(\frac{d}{dx} \frac{(\Delta_\alpha^k e^{ix\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-xv\cdot})(0)} \right) dx \tag{5}$$

and, in addition,

$$\frac{d}{dx} \frac{(\Delta_\alpha^k e^{ix\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-xv\cdot})(0)} = -\frac{d}{dx} \frac{(1-e^{ix\alpha})^k}{(1-e^{-xv|\alpha|})^k}.$$

Let $\varepsilon > 0$ be fixed and $|\alpha| < \min\{\Re(v), 1\}$. We split the weighted Fourier series in two parts.

First, let $m|\alpha| < \varepsilon$. For $m \le x \le m+1$, we have

$$\left| \frac{d}{dx} \frac{(\Delta_\alpha^k e^{ix\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-xv\cdot})(0)} \right| = \left| \frac{d}{dx} \frac{(1-e^{ix\alpha})^k}{(1-e^{-xv|\alpha|})^k} \right| =$$

$$= k \left| \frac{(1-e^{ix\alpha})^{k-1}}{(1-e^{-xv|\alpha|})^{k-1}} \right| \left| \frac{d}{dx} \left(\frac{1-e^{ix\alpha}}{1-e^{-xv|\alpha|}} \right) \right|. \tag{6}$$

We investigate how this term behaves for $|\alpha| \to 0$, because for $|\alpha| > \gamma$ with some $\gamma > 0$, the term is bounded in the domain $m|\alpha| < \varepsilon$ for continuity reasons. For $k = 1$, it is easily seen that

$$\left| \frac{d}{dx} \left(\frac{1-e^{ix\alpha}}{1-e^{-xv|\alpha|}} \right) \right| = |\alpha| \left| \frac{-i(1-e^{-xv|\alpha|})e^{ix\alpha} - v\,\mathrm{sign}(\alpha)(1-e^{ix\alpha})e^{-xv|\alpha|}}{(1-e^{-xv|\alpha|})^2} \right|.$$

The second term converges to $\left| \dfrac{1 - iv\,\mathrm{sign}(\alpha)}{2v} \right|$ if $|\alpha| \to 0$. Since $m \le x \le m+1$ and $m|\alpha| < \varepsilon$, we can estimate the whole term by some constant independent of α and x. Hence, $|\mu_m| \le \text{const}$ for $m|\alpha| < \varepsilon$ and some constant independent of α. By induction and equation (6), we get

$$\left| \frac{d}{dx} \frac{(\Delta_\alpha^k e^{ix\cdot})(0)}{(-\Delta_{|\alpha|}^k e^{-xv\cdot})(0)} \right| \le \text{const} \cdot |\alpha|,$$

where the constant does not depend on x and α. Furthermore, $|\mu_m| < \text{const}$ for some constant independent of α. Using equation (5), we deduce

$$|\mu_{m+1} - \mu_m| \le \text{const} \int_m^{m+1} |\alpha| dx = \text{const} \cdot |\alpha|,$$

and thus

$$\sum_{m < \varepsilon/|\alpha|} |\mu_{m+1} - \mu_m| \le \text{const}.$$

For $\Phi \in L^p([0, 2\pi])$ and $\check{\Phi}$ as in Lemma 1, by using Theorem 3, we conclude

$$\left\| \sum_{m < \varepsilon/|\alpha|} c_m(\check{\Phi}) \mu_m e^{im\cdot} \right\| \le \text{const} \left\| \sum_{m < \varepsilon/|\alpha|} c_m(\check{\Phi}) e^{im\cdot} \right\| \le \text{const} \cdot \|\check{\Phi}\|, \tag{7}$$

since $(c_m(\check{\Phi}))_{m \in \mathbb{Z}}$ is the sequence of Fourier coefficients of $\check{\Phi}$.

Second, let $m|\alpha| \ge \varepsilon$. We define

$$\mu'_m := \frac{1}{(1-e^{-m\upsilon|\alpha|})^k}$$

and deduce

$$|\mu'_{m+1} - \mu'_m| \le \left|\frac{(1-e^{-m\upsilon|\alpha|})^k - (1-e^{-(m+1)\upsilon|\alpha|})^k}{(1-e^{-m\upsilon|\alpha|})^k(1-e^{-(m+1)\upsilon|\alpha|})^k}\right| =$$

$$= \left|\frac{\sum_{n=1}^k (-1)^{k-n}\binom{k}{n} e^{-mn\upsilon|\alpha|}(1-e^{-n\upsilon|\alpha|})}{(1-e^{-m\upsilon|\alpha|})^k(1-e^{-(m+1)\upsilon|\alpha|})^k}\right| \le$$

$$\le e^{-m\Re(\upsilon)|\alpha|}\left|\frac{\sum_{n=1}^k (-1)^{k-n}\binom{k}{n}(1-e^{-n\upsilon|\alpha|})}{(1-e^{-m\upsilon|\alpha|})^k(1-e^{-(m+1)\upsilon|\alpha|})^k}\right| \le$$

$$\le e^{-m\Re(\upsilon)|\alpha|}|\alpha|\left|\frac{\sum_{n=1}^k (-1)^{k-n}\binom{k}{n}\frac{1-e^{-n\upsilon|\alpha|}}{|\alpha|}}{(1-e^{-m\upsilon|\alpha|})^k(1-e^{-(m+1)\upsilon|\alpha|})^k}\right| \le$$

$$\le e^{-m\Re(\upsilon)|\alpha|}|\alpha|\left|\frac{C(k,\upsilon)}{(1-e^{-m\upsilon|\alpha|})^k(1-e^{-(m+1)\upsilon|\alpha|})^k}\right| \le$$

$$\le C(k,\upsilon,\varepsilon)\,|\alpha|\,e^{-m\Re(\upsilon)|\alpha|}$$

for positive constants $C(k, \upsilon)$ and $C(k, \upsilon, \varepsilon)$. Thus, since $|\alpha| < \min\{\Re(\upsilon),$

$$\sum_{m \ge \varepsilon/|\alpha|} |\mu'_{m+1} - \mu'_m| \le \sum_{l=0}^{\infty}\left|\mu'_{[\varepsilon/|\alpha|]+l+1} - \mu'_{[\varepsilon/|\alpha|]+l}\right| \le$$

$$\le C(k,\upsilon,\varepsilon)|\alpha|\sum_{l=0}^{\infty} e^{-([\varepsilon/|\alpha|]+l)\Re(\upsilon)|\alpha|} \le$$

$$\le C(k,\upsilon,\varepsilon)|\alpha|\frac{1}{1-e^{-\Re(\upsilon)|\alpha|}}\, e^{-[\varepsilon/|\alpha|]\Re(\upsilon)|\alpha|} \le$$

$$\le C(k,\upsilon,\varepsilon)|\alpha|\frac{1}{1-e^{-\Re(\upsilon)|\alpha|}} \le C(k,\upsilon,\varepsilon)\Re(\upsilon).$$

Obviously, $|\mu'_m| \le C_\varepsilon$ for some positive constant C_ε only depending on ε. for $\Phi \in L^p([0, 2\pi])$ with $\check{\Phi}$ as in Lemma 1, by using Theorem 3, we conclu

$$\left\|\sum_{m\ge\varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu'_m e^{im\cdot}\right\| \le \text{const}\cdot\left\|\sum_{m\ge\varepsilon|\alpha|^{-1}} c_m(\check{\Phi})e^{im\cdot}\right\| \le \text{const}\cdot\|\check{\Phi}\|$$

Denote by

$$\check{\Phi}_\varepsilon(t) := \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu'_m e^{imt},$$

i.e., $c_m(\check{\Phi}_\varepsilon) = c_m(\check{\Phi})\mu'_m$. Hence,

$$\left\| \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu_m e^{im\cdot} \right\| = \left\| \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu'_m (1 - e^{im\alpha})^k e^{im\cdot} \right\| =$$

$$= \left\| \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\check{\Phi}_\varepsilon)(1 - e^{im\alpha})^k e^{im\cdot} \right\| = \left\| \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\Delta^k_{-\alpha}\check{\Phi}_\varepsilon) e^{im\cdot} \right\| \leq$$

$$\leq \left\| \Delta^k_{-\alpha}\check{\Phi}_\varepsilon \right\| \leq (k+1)\|\check{\Phi}_\varepsilon\| \leq \text{const} \cdot \|\check{\Phi}\|, \tag{9}$$

where we have used (8). Using (7) and (9), we can deduce

$$\left\| \sum_{m=n(j)}^{\infty} c_m(\check{\Phi})\mu_m e^{im\cdot} \right\| =$$

$$= \left\| \sum_{n(j) \leq m < \varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu_m e^{im\cdot} \right\| + \left\| \sum_{m \geq \varepsilon|\alpha|^{-1}} c_m(\check{\Phi})\mu_m e^{im\cdot} \right\| \leq$$

$$\leq \text{const} \cdot \|\check{\Phi}\|.$$

Hence, μ_m is a multiplier in $L^p([0, 2\pi])$.

The same can be shown for $\tilde{\mu}_m$. We have

$$|\tilde{\mu}_{m+1} - \tilde{\mu}_m| \leq \left|(1 - e^{-2\pi(m+1)\upsilon})\mu_{m+1} - (1 - e^{-2\pi m\upsilon})\mu_m\right| \leq$$

$$\leq |\mu_{m+1} - \mu_m|(1 + e^{-2\pi m\Re(\upsilon)}) + |\mu_{m+1}|\left|e^{-2\pi m\upsilon} - e^{-2\pi(m+1)\upsilon}\right| \leq$$

$$\leq |\mu_{m+1} - \mu_m|(1 + e^{-2\pi m\Re(\upsilon)}) + |\mu_{m+1}|e^{-2\pi m\Re(\upsilon)}\left|1 - e^{-2\pi\upsilon}\right| \leq$$

$$\leq \text{const} \cdot \left(|\mu_{m+1} - \mu_m| + e^{-am}\right),$$

for some positive constant $a > 0$. Thus, $\tilde{\mu}_m$ is a multiplier in $L^p([0, 2\pi])$, too. We have

$$\left\| \sum_{m=n(j)}^{\infty} c_m(\check{\Phi})\tilde{\mu}_m e^{im\cdot} \right\| \leq \text{const} \cdot \|\check{\Phi}\|.$$

For $|\alpha| < h$, using relations (4) and Lemma 1 we get

$$\left\| \Delta^k_\alpha \check{\Phi} \right\| \leq \left\| \sum_{m=n(j)}^{\infty} d_m(-\Delta^k_{-|\alpha|}\Phi)\mu_m e^{im\cdot} \right\| +$$

$$+ \left\| \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_0^{n|\alpha|} \Phi(u - n|\alpha|) e^{-m\upsilon u} du\, \tilde{\mu}_m e^{im\cdot} \right\| \leq$$

$$\leq \text{const} \cdot \left\| \sum_{m=n(j)}^{\infty} d_m(-\Delta^k_{-|\alpha|}\Phi) e^{im\cdot} \right\| +$$

$$+ \ \text{const} \cdot \left\| \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_{0}^{n|\alpha|} \Phi(u - n|\alpha|) e^{-m\nu u} du \, e^{im\cdot} \right\| \le$$

$$\le \ \text{const} \cdot \left\| \Delta_{-|\alpha|}^{k} \Phi \right\| +$$

$$+ \ \text{const} \cdot \left\| \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \int_{0}^{2\pi} \left(\chi_{[0, n|\alpha|]}(u) \, \Phi(u - n|\alpha|) \right) e^{-m\nu u} du \, e^{im\cdot} \right\| =$$

$$= \ \text{const} \cdot \left(\left\| \Delta_{-|\alpha|}^{k} \Phi \right\| + \left\| \sum_{m=n(j)}^{\infty} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} d_m \left(\chi_{[0, n|\alpha|]} \right) \Phi(\cdot - n|\alpha|) e^{im\cdot} \right\| \right) \le$$

$$\le \ \text{const} \cdot \left(\left\| \Delta_{-|\alpha|}^{k} \Phi \right\| + \left\| \sum_{n=1}^{k} (-1)^{k-n} \binom{k}{n} \chi_{[0, n|\alpha|]} \Phi(\cdot - n|\alpha|) \right\| \right) \le$$

$$\le \ \text{const} \cdot \left(\omega_k(\Phi, h)_p + \sum_{n=1}^{k} \binom{k}{n} \left(\int_{0}^{n|\alpha|} |\Phi(u - n|\alpha|)|^p du \right)^{1/p} \right) \le$$

$$\le \ \text{const} \cdot (\omega_k(\Phi, h)_p + \delta_k'(\Phi, h)_p),$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. Passing to the supremum leads to

$$\omega_k(\check{\Phi}, h)_p = \sup_{0<|\alpha|<h} \left\| \Delta_{\alpha}^{k} \Phi \right\|_p \le \ \text{const} \cdot (\omega_k(\Phi, h)_p + \delta_k'(\Phi, h)_p),$$

and the assertion is proved.

Acknowledgements. The author wants to express her gratitude to V. V. Andrievskii and R. Lasser for many valuable comments on the subject.

1. *Leont'ev A. F.* Exponential series. – Moscow: Nauka, 1976 [in Russian].
2. *Lewin B. Ja., Ljubarskii Ju. I.* Interpolation by means of special classes of entire functions and related expansions in series of exponentials // Math. USSR Izvestija. – 1975. – **9**. – P. 621 – 662.
3. *Sedletckii A. M.* Bases of exponential functions in the space E^p on convex polyygons // Math. USSR Izvestija. – 1979. – **13** (2). – P. 387 – 404.
4. *Achieser N. I.* Theory of approximation. – New York: Dover Publ., 1992.
5. *Mel'nik Yu. I.* Some properties of exponential series representing functions being regular in convex polygons // Some Problems of the Theory of Approximation of Functions. Collect. Sci. Works. – Kiev, 1985. – P. 69 – 81 [in Russian].
6. *Mel'nik Yu. I.* Direct theorems for the approximation of functions, regular in convex polygons, by exponential polynomials in the integral metric // Ukr. Math. J. – 1988. – **40**. – P. 498 – 504.
7. *Brudnyi Yu. A.* Piecewise polynomial approximation, embedding theorem and rational approximation // Lect. Notes Math. – 1976. – **556**. – P. 73 – 98.
8. *Zygmund A.* Trigonometric series: Vol. I, II. – Second ed. – Cambridge Univ. Press, 1988.

Received 21.03.2002