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CONGRUENCES ON TERNARY SEMIGROUPS

КОНГРУЕНЦІЇ ТЕРНАРНИХ ПІВГРУП

We study ternary semigroups as universal algebras with one associative operation. We investigate their algebraic structure and associated representations. We present the results for congruences of ternary semigroups generated by binary relations.

Вивчаються тернарні півгрупи як універсальні алгебри з асоціативною операцією. Досліджуються їх алгебраїчна структура та асоційовані зображення. Наведено результати для конгруенцій тернарних півгруп, породжених бінарними відношеннями.

Ternary semigroups are universal algebras (cf. [1, 2]) with one associative ternary operation. A ternary semigroup is a particular case of an n-ary semigroup (n-semigroup) when $n = 3$ (cf. [3, 4]). The morphisms of two algebraic structures, ordered sets, objects in categories form ternary semigroups with ternary operations defined by means of composition of morphisms [5 – 12].

The results of this paper concern congruences generated by binary relations on the universes of ternary semigroups.

Let X be any set. We establish the following symbols: $B(X)$ — the set of all binary relations on the set X, $E(X)$ — the set of all equivalence relations on the set X, $E_X = \{(x, x) : x \in X\}$.

Assume that $\alpha \in B(X)$. Set

$$\alpha^n = \underbrace{\alpha \circ \alpha \circ \ldots \circ \alpha}_{n}$$

for $n \in \mathcal{N}$. Put

$$\alpha^\infty = \bigcup_{n=1}^{\infty} \alpha^n.$$

The relation α^∞ is called a *transitive closure* of the relation $\alpha \in B(X)$.

Note that

$$\bigcap \{\beta \in E(X) : \alpha \subseteq \beta\} \in E(X).$$

Put

$$\alpha^E = \bigcap \{\beta \in E(X) : \alpha \subseteq \beta\}.$$

We say that the equivalence relation α^E on the set X is *generated* by the relation $\alpha \in B(X)$.

Observe that if $\alpha = \varnothing$, then $\alpha^E = \varnothing^E = E_X$.

If α is a relation on a set X, then

$$\alpha^E = \left(\alpha \cup \alpha^{-1} \cup E_X\right)^\infty.$$

Definition 1. *An algebraic structure* (A, f), *where* A *is a nonempty set, is said to be a ternary semigroup if* $f : A^3 \to A$ *is a ternary operation on* A *such that*

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

for $x_1, \ldots, x_5 \in A$.

Let $(A, \cdot)$ be a semigroup. The ternary semigroup (A, f) with the ternary operation defined by

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3$$

for $x_1, x_2, x_3 \in A$, is called a *ternary semigroup reduct* of the semigroup $(A, \cdot)$.

Definition 2. *Let (A, f) be a ternary semigroup. A relation $\alpha \in B(A)$ is called:*

(a) *left compatible with the ternary operation f if*

$$\forall\, a, b, x_1, x_2 \in A\,[(a, b) \in \alpha \Rightarrow (f(x_1, x_2, a), f(x_1, x_2, b)) \in \alpha];$$

(b) *right compatible with the ternary operation f if*

$$\forall\, a, b, x_1, x_2 \in A\,[(a, b) \in \alpha \Rightarrow (f(a, x_1, x_2), f(b, x_1, x_2)) \in \alpha];$$

(c) *lateral compatible with the ternary operation f if*

$$\forall\, a, b, x_1, x_2 \in A\,[(a, b) \in \alpha \Rightarrow (f(x_1, a, x_2), f(x_1, b, x_2)) \in \alpha];$$

(d) *two-sided compatible with the ternary operation f if*

$$\forall\, a_1, a_2, b_1, b_2, x \in A\,[((a_1, b_1) \in \alpha \wedge (a_2, b_2) \in \alpha) \Rightarrow$$
$$\Rightarrow (f(a_1, x, a_2), f(b_1, x, b_2)) \in \alpha];$$

(e) *compatible with the ternary operation f if*

$$\forall\, a_1, a_2, a_3, b_1, b_2, b_3 \in A\,[((a_1, b_1) \in \alpha \wedge (a_2, b_2) \in \alpha \wedge (a_3, b_3) \in \alpha) \Rightarrow$$
$$\Rightarrow (f(a_1, a_2, a_3), f(b_1, b_2, b_3)) \in \alpha].$$

Definition 3. *Let (A, f) be a ternary semigroup. A relation $\alpha \in E(A)$ compatible [left compatible, right compatible, lateral compatible, two-sided compatible] with the ternary operation f is called a congruence [left congruence, right congruence, lateral congruence, two-sided congruence] on the ternary semigroup (A, f).*

Note that the equality relation E_A and the universal relation U_A are congruences, left congruences, right congruences, lateral congruences, two-sided congruences on the ternary semigroup (A, f).

The symbol $C(A)$ [resp. $C_l(A)$, $C_r(A)$, $C_c(A)$, $C_j(A)$] denotes the set of all congruences [resp. left congruences, right congruences, lateral congruences, two-sided congruences] on the ternary semigroup (A, f).

Proposition 1. *Let (A, f) be a ternary semigroup. If a relation $\alpha \in B(A)$ is reflexive and compatible with the ternary operation f, then the relation α is left, right and lateral compatible with the ternary operation f.*

The proof is evident.

Proposition 2. *Let (A, f) be a ternary semigroup. Assume that a relation $\alpha \in$ $\in B(A)$ is reflexive and transitive. The relation α is compatible with the ternary operation f if and only if the relation α is left, right and lateral compatible with the ternary operation f.*

Proof. The necessary condition follows from Proposition 1. Conversely, assume that the relation α is left, right and lateral compatible with the ternary operation f. Let $(a_i, b_i) \in \alpha$ for $a_i, b_i \in A$, where $i = 1, 2, 3$. Note that $(f(a_1, a_2, a_3), f(b_1, a_2, a_3)) \in \alpha$, $(f(b_1, a_2, a_3), f(b_1, b_2, a_3)) \in \alpha$, $(f(b_1, b_2, a_3), f(b_1, b_2, b_3)) \in \alpha$. Since the relation α is transitive, it follows that $(f(a_1, a_2, a_3), f(b_1, b_2, b_3)) \in \alpha$. Thus, the relation α is compatible with the ternary operation f.

As a consequence of Proposition 2 we get the following corollary.

***Corollary* 1.** *A relation $\alpha \in E(A)$ is a congruence on the ternary semigroup (A,f) if and only if α is a left, right and lateral congruence on the ternary semigroup (A,f).*

Proposition 3. *Let (A,f) be a ternary semigroup. If a relation $\alpha \in B(A)$ is reflexive and two-sided compatible with the ternary operation f, then the relation α is left and right compatible with the ternary operation f.*

The proof is evident.

Proposition 4. *Let (A,f) be a ternary semigroup. Assume that a relation $\alpha \in \in B(A)$ is reflexive and transitive. The relation α is two-sided compatible with the ternary operation f if and only if the relation α is left and right compatible with the ternary operation f.*

The proof of this proposition is similar to the proof of Proposition 2.

By Proposition 4 we obtain the following corollary.

***Corollary* 2.** *A relation $\alpha \in E(A)$ is a two-sided congruence on the ternary semigroup (A,f) if and only if α is a left and right congruence on the ternary semigroup (A,f).*

Now we shall show that the assumptions of reflexivity and transitivity in Propositions $1-3$ and 4 are necessary. Therefore, we consider an example.

Example. Let $A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Define on the set A the following operation:

$$(k_1, l_1) \cdot (k_2, l_2) = (k_1, l_2)$$

for all (k_1, l_1), $(k_2, l_2) \in A$. The algebraic structure $(A, \cdot)$. is a semigroup.

Let (A,f) be the ternary semigroup reduct of the semigroup $(A, \cdot)$. Put $a = (1, 1)$, $b = (1, 2)$, $c = (2, 1)$, $d = (2, 2)$. Consider on the set A the relations: $\alpha = \{(a, b)\}$ and $\beta = E_A \cup \{(a, b), (a, c), (b, d), (c, d)\}$. The relation α is compatible with the ternary operation f, because $(f(a, a, a), f(b, b, b)) = (a, b) \in \alpha$. But the relation α is not left, right, lateral compatible with the ternary operation f. In fact, $f(c, c, a) = c$, $f(c, c, b) = d$, but $(c, d) \notin \alpha$. Next, $f(c, a, c) = c$, $f(c, b, c) = c$, but $(c, c) \notin \alpha$, and also $f(a, c, c) = a$, $f(b, c, c) = a$, but $(a, a) \notin \alpha$.

Note that the relation β is not transitive. It is a routine matter to verify that the relation β is left, right and lateral compatible with the ternary operation f. However, the relation β is not compatible with the ternary operation f. Indeed, $f(a, b, c) = a$ and $f(c, d, d) = d$, but $(a, d) \notin \beta$.

The above example we may apply to Propositions 3 and 4.

Lemma 1. *Let (A,f) be a ternary semigroup. If the relations $\alpha, \beta \in B(A)$ are left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f, then the relation $\alpha \circ \beta$ is left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f.*

Proof. Let the relations $\alpha, \beta \in B(A)$ be left compatible with f. Assume that a, $b, x_1, x_2 \in A$ and $(a, b) \in \alpha \circ \beta$. Therefore, $(a, c) \in \beta$ and $(c, b) \in \alpha$ for some element $c \in A$. Hence, $(f(x_1, x_2, a), f(x_1, x_2, c)) \in \beta$ and $(f(x_1, x_2, c), f(x_1, x_2, b)) \in \in \alpha$. Thus, $(f(x_1, x_2, a), f(x_1, x_2, b)) \in \alpha \circ \beta$, and so the relation $\alpha \circ \beta$ is left compatible with f.

The proofs of the remaining cases are analogous.

In view of Lemma 1, it is easy to prove by induction the following corollary.

***Corollary* 3.** *Let (A,f) be a ternary semigroup. If the relations $\alpha_1, \ldots, \alpha_k \in$*

$\in B(A)$ are left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f, then the relation $\alpha_1 \circ \ldots \ldots \circ \alpha_k$ is left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f.

Proposition 5. *Let (A, f) be a ternary semigroup. Assume that the relations $\alpha_t \in B(A)$ for $t \in T \neq \varnothing$ are left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f.*
Then the relation

$$\bigcup_{t \in T} \alpha_t$$

is left compatible [resp. right compatible, lateral compatible, two-sided compatible, compatible] with the ternary operation f.

The proof is evident.

Proposition 6. *Let (A, f) be a ternary semigroup. Let $\{\alpha_t\}$ for $t \in T$ be a nonempty family of congruences [resp. left congruences, right congruences, lateral congruences, two-sided congruences] on the ternary semigroup (A, f). Then*

$$\bigcap_{t \in T} \alpha_t$$

is a congruence [resp. left congruence, right congruence, lateral congruence, two-sided congruence] on the ternary semigroup (A, f).

The proof is evident.

On the basis of Proposition 6, we can formulate the following definitions.

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. According to Proposition 6

$$\bigcap \{\beta \in C(A) : \alpha \subseteq \beta\} \in C(A).$$

Put

$$\alpha^K = \bigcap \{\beta \in C(A) : \alpha \subseteq \beta\}.$$

We say that the congruence α^K on the ternary semigroup (A, f) is *generated* by the relation $\alpha \in B(A)$.

Similarly, we define a left congruence α^L, right congruence α^R, lateral congruence α^C, two-sided congruence α^J on the ternary semigroup (A, f) *generated* by the relation $\alpha \in B(A)$.

By Proposition 6 we get the following corollary.

***Corollary* 4.** *Let (A, f) be a ternary semigroup. For every relation $\alpha \in B(A)$, there exist unique relations α^L, α^R, α^C, α^J, α^K.*

We shall analyse the form of congruences generated by relations on the universe A of a ternary semigroup (A, f).

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. If $\alpha = \varnothing$, then $\alpha^K = \alpha^L = \alpha^R = \alpha^C = \alpha^J = E_A$.

In the further considerations concerning the generated congruences, we do not distinguish the case of empty relation, treating it as trivial.

Consider the relation

$$\alpha^u = \{(f(x_1, x_2, a), f(x_1, x_2, b)) : (a, b) \in \alpha \wedge x_1, x_2 \in A\}.$$

Lemma 2. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $(\alpha^u)^{-1} = (\alpha^{-1})^u$;

(b) $\alpha \subseteq \beta \Rightarrow \alpha^u \subseteq \beta^u$;

(c) $(\alpha^u)^u \subseteq \alpha^u$;

(d) $(\alpha \cup \beta)^u = \alpha^u \cup \beta^u$.

Proof. (a) Note that

$$(x, y) \in (\alpha^u)^{-1} \Leftrightarrow (y, x) \in \alpha^u \Leftrightarrow$$
$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [y = f(x_1, x_2, b) \wedge x = f(x_1, x_2, a) \wedge (b, a) \in \alpha]) \Leftrightarrow$$
$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [x = f(x_1, x_2, a) \wedge y = f(x_1, x_2, b) \wedge (a, b) \in \alpha^{-1}]) \Leftrightarrow$$
$$\Leftrightarrow (x, y) \in (\alpha^{-1})^u$$

for $(x, y) \in A^2$. Thus, $(\alpha^u)^{-1} = (\alpha^{-1})^u$.

(b) The proof is immediate.

(c) If $(x, y) \in (\alpha^u)^u$, then $x = f(y_1, y_2, c)$, $y = f(y_1, y_2, d)$ for some $(c, d) \in \alpha^u$ and $y_1, y_2 \in A$. Therefore, $c = f(x_1, x_2, a)$, $d = f(x_1, x_2, b)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Thus,

$$x = f(y_1, y_2, f(x_1, x_2, a)) = f(f(y_1, y_2, x_1), x_2, a),$$
$$y = f(y_1, y_2, f(x_1, x_2, b)) = f(f(y_1, y_2, x_1), x_2, b),$$

and so $(x, y) \in \alpha^u$. The condition (c) holds.

(d) We have:

$$(x, y) \in (\alpha \cup \beta)^u \Leftrightarrow$$
$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [x = f(x_1, x_2, a) \wedge y = f(x_1, x_2, b) \wedge (a, b) \in \alpha \cup \beta]) \Leftrightarrow$$
$$\Leftrightarrow ((\exists x_1, x_2, a, b \in A\, [x = f(x_1, x_2, a) \wedge y = f(x_1, x_2, b) \wedge (a, b) \in \alpha]) \vee$$
$$\vee (\exists x_1, x_2, a, b \in A\, [x = f(x_1, x_2, a) \wedge y = f(x_1, x_2, b) \wedge (a, b) \in \beta])) \Leftrightarrow$$
$$\Leftrightarrow (((x, y) \in \alpha^u) \vee ((x, y) \in \beta^u)) \Leftrightarrow$$
$$\Leftrightarrow (x, y) \in \alpha^u \cup \beta^u$$

for $(x, y) \in A^2$. Thus (d) holds.

The lemma is proved.

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. Put

$$\alpha^l = \alpha \cup \alpha^u.$$

Note that $E_A^l = E_A$.

Lemma 3. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $\alpha \subseteq \alpha^l$;

(b) $(\alpha^l)^{-1} = (\alpha^{-1})^l$;

(c) $\alpha \subseteq \beta \Rightarrow \alpha^l \subseteq \beta^l$;

(d) $(\alpha^l)^l = \alpha^l$;

(e) $(\alpha \cup \beta)^l = \alpha^l \cup \beta^l$;

(f) $\alpha = \alpha^l$ *if and only if the relation* α *is left compatible with the ternary operation* f *in the ternary semigroup* (A, f).

Proof. The proofs of the conditions (a) and (c) are routine.

(b) By Lemma 2 (a), we have: $(\alpha^l)^{-1} = (\alpha \cup \alpha^u)^{-1} = \alpha^{-1} \cup (\alpha^u)^{-1} = \alpha^{-1} \cup \cup (\alpha^{-1})^u = (\alpha^{-1})^l$.

(d) In view of Lemma 2 (c) and (d), we obtain: $(\alpha^l)^l = (\alpha \cup \alpha^u)^l = (\alpha \cup \alpha^u) \cup \cup (\alpha \cup \alpha^u)^u = \alpha \cup \alpha^u \cup \alpha^u \cup (\alpha^u)^u = \alpha \cup \alpha^u = \alpha^l$.

(e) Using Lemma 2 (d), we have: $(\alpha \cup \beta)^l = (\alpha \cup \beta) \cup (\alpha \cup \beta)^u = (\alpha \cup \beta) \cup \cup (\alpha^u \cup \beta^u) = (\alpha \cup \alpha^u) \cup (\beta \cup \beta^u) = \alpha^l \cup \beta^l$.

(f) If $\alpha = \alpha^l$, then

$$\forall x_1, x_2, a, b \in A \, [(a, b) \in \alpha \Rightarrow (f(x_1, x_2, a), f(x_1, x_2, b)) \in \alpha^l = \alpha].$$

Therefore, the relation α is left compatible with the ternary operation f in the ternary semigroup (A, f).

Conversely, let the relation α be left compatible with the ternary operation f in the ternary semigroup (A, f). Applying (a), it is enough to show that $\alpha^l \subseteq \alpha$. Assume that $(x, y) \in \alpha^l = \alpha \cup \alpha^u$. If $(x, y) \in \alpha^u$, then $x = f(x_1, x_2, a)$ and $y = f(x_1, x_2, b)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Since $(a, b) \in \alpha$, it follows that $(f(x_1, x_2, a), f(x_1, x_2, b)) \in \alpha$, and so $(x, y) \in \alpha$. Hence $\alpha^l \subseteq \alpha$. Thus (f) holds.

The lemma is proved.

The assertions (d) and (f) in Lemma 3 imply the following corollary.

***Corollary* 5.** *Let* (A, f) *be a ternary semigroup. For any relation* $\alpha \in B(A)$, *the relation* α^l *is left compatible with the ternary operation* f *in the ternary semigroup* (A, f).

Theorem 1. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

$$\alpha^L = (\alpha^l)^E.$$

Proof. Note that $\alpha \subseteq (\alpha^l)^E$. We know that

$$(\alpha^l)^E = \bigcup_{n=1}^{\infty} \left(\alpha^l \cup (\alpha^l)^{-1} \cup E_A\right)^n.$$

Put $\beta = \alpha^l \cup (\alpha^l)^{-1} \cup E_A$. Hence $(\alpha^l)^E = \bigcup_{n=1}^{\infty} \beta^n$. By Lemma 3, $\beta = \alpha^l \cup (\alpha^{-1})^l \cup \cup E_A = (\alpha \cup \alpha^{-1} \cup E_A)^l$. In view of Corollary 5, the relation β is left compatible with the ternary operation f. By Corollary 3 the relation β^n, for every $n \in \mathcal{N}$, is left compatible with the ternary operation f. Note that $(\alpha^l)^E \in C_l(A)$. Indeed, if $(a, b) \in \in (\alpha^l)^E$, then $(a, b) \in \beta^n$ for some $n \in \mathcal{N}$. Thus $(f(x_1, x_2, a), f(x_1, x_2, b)) \in \beta^n$, that is $(f(x_1, x_2, a), f(x_1, x_2, b)) \in (\alpha^l)^E$ for all $x_1, x_2 \in A$. Assume that $\gamma \in C_l(A)$ and $\alpha \subseteq \gamma$. From the conditions (c) and (f) in Lemma 3 it follows that, $\alpha^l \subseteq \gamma^l = \gamma$. Since $\gamma \in E(A)$, it follows that $(\alpha^l)^E \subseteq \gamma$. Thus $\alpha^L = (\alpha^l)^E$.

The theorem is proved.

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. Consider the following relation:

$$\alpha^z = \{(f(a, x_1, x_2), f(b, x_1, x_2)) : (a, b) \in \alpha \wedge x_1, x_2 \in A\}.$$

A similar argument applied to the relations α^r and α^R implies the following statements.

Lemma 4. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $(\alpha^z)^{-1} = (\alpha^{-1})^z$;

(b) $\alpha \subseteq \beta \Rightarrow \alpha^z \subseteq \beta^z$;

(c) $(\alpha^z)^z \subseteq \alpha^z$;

(d) $(\alpha \cup \beta)^z = \alpha^z \cup \beta^z$.

Put

$$\alpha^r = \alpha \cup \alpha^z.$$

Notice that $E_A^r = E_A$.

Lemma 5. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $\alpha \subseteq \alpha^r$;

(b) $(\alpha^r)^{-1} = (\alpha^{-1})^r$;

(c) $\alpha \subseteq \beta \Rightarrow \alpha^r \subseteq \beta^r$;

(d) $(\alpha^r)^r = \alpha^r$;

(e) $(\alpha \cup \beta)^r = \alpha^r \cup \beta^r$;

(f) $\alpha = \alpha^r$ *if and only if the relation* α *is right compatible with the ternary operation* f *in the ternary semigroup* (A, f).

***Corollary* 6.** *Let (A, f) be a ternary semigroup. For any relation $\alpha \in B(A)$ the relation α^r is right compatible with the ternary operation f in the ternary semigroup (A, f).*

Theorem 2. *Let (A, f) be a ternary semigroup. If $\alpha \in B(A)$, then*

$$\alpha^R = (\alpha^r)^E.$$

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. Consider the relations:

$$\alpha^v = \{(f(x_1, a, x_2), f(x_1, b, x_2)) : (a, b) \in \alpha \wedge x_1, x_2 \in A\},$$

$$\alpha^w = \{(f(x_1, a, x_2), f(x_1, b, x_2)) : (a, b) \in \alpha^v \wedge x_1, x_2 \in A\}.$$

Lemma 6. *Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. The following conditions are satisfied:*

(a) $(\alpha^v)^w \subseteq \alpha^v$;

(b) $(\alpha^w)^v \subseteq \alpha^v$.

Proof. (a) Suppose that $(x, y) \in (\alpha^v)^w$. Hence, $x = f(x_1, a, x_2)$, $y = f(x_1, b, x_2)$ for some $(a, b) \in (\alpha^v)^v$ and $x_1, x_2 \in A$. Therefore, $a = f(y_1, a_1, y_2)$, $b = f(y_1, b_1, y_2)$

for some $(a_1, b_1) \in \alpha^{\upsilon}$ and $y_1, y_2 \in A$, and so $a_1 = f(z_1, a_2, z_2)$, $b_1 = f(z_1, b_2, z_2)$ for some $(a_2, b_2) \in \alpha$ and $z_1, z_2 \in A$. Thus,

$$x = f(x_1, f(y_1, f(z_1, a_2, z_2), y_2), x_2) = f(f(x_1, y_1, z_1), a_2, f(z_2, y_2, x_2)),$$

$$y = f(x_1, f(y_1, f(z_1, b_2, z_2), y_2), x_2) = f(f(x_1, y_1, z_1), b_2, f(z_2, y_2, x_2)).$$

Then $(x, y) \in \alpha^{\upsilon}$.

(b) Assume that $(x, y) \in (\alpha^{w})^{\upsilon}$. Hence, $x = f(x_1, a, x_2)$, $y = f(x_1, b, x_2)$ for some $(a, b) \in \alpha^{w}$ and $x_1, x_2 \in A$. Therefore, $a = f(y_1, a_1, y_2)$, $b = f(y_1, b_1, y_2)$ for some $(a_1, b_1) \in \alpha^{\upsilon}$ and $y_1, y_2 \in A$, and so $a_1 = f(z_1, a_2, z_2)$, $b_1 = f(z_1, b_2, z_2)$ for some $(a_2, b_2) \in \alpha$ and $z_1, z_2 \in A$. Thus,

$$x = f(x_1, f(y_1, f(z_1, a_2, z_2), y_2), x_2) = f(f(x_1, y_1, z_1), a_2, f(z_2, y_2, x_2)),$$

$$y = f(x_1, f(y_1, f(z_1, b_2, z_2), y_2), x_2) = f(f(x_1, y_1, z_1), b_2, f(z_2, y_2, x_2)).$$

Then we conclude that $(x, y) \in \alpha^{\upsilon}$.

Lemma 7. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $(\alpha^{\upsilon})^{-1} = (\alpha^{-1})^{\upsilon}$;

(b) $\alpha \subseteq \beta \Rightarrow \alpha^{\upsilon} \subseteq \beta^{\upsilon}$;

(c) $(\alpha^{\upsilon})^{\upsilon} \subseteq \alpha^{w}$;

(d) $(\alpha \cup \beta)^{\upsilon} = \alpha^{\upsilon} \cup \beta^{\upsilon}$.

Proof. (a) Note that

$$(x, y) \in (\alpha^{\upsilon})^{-1} \Leftrightarrow (y, x) \in \alpha^{\upsilon} \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [y = f(x_1, b, x_2) \wedge x = f(x_1, a, x_2) \wedge (b, a) \in \alpha]) \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \alpha^{-1}]) \Leftrightarrow$$

$$\Leftrightarrow (x, y) \in (\alpha^{-1})^{\upsilon}$$

for $(x, y) \in A^2$. Thus $(\alpha^{\upsilon})^{-1} = (\alpha^{-1})^{\upsilon}$.

(b) The proof is immediate.

(c) If $(x, y) \in (\alpha^{\upsilon})^{\upsilon}$, then $x = f(x_1, a, x_2)$, $y = f(x_1, b, x_2)$ for some $(a, b) \in \alpha^{\upsilon}$ and $x_1, x_2 \in A$. Hence $(x, y) \in \alpha^{w}$.

(d) Note that

$$(x, y) \in (\alpha \cup \beta)^{\upsilon} \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\, [x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \alpha \cup \beta]) \Leftrightarrow$$

$$\Leftrightarrow ((\exists x_1, x_2, a, b \in A\, [x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \alpha]) \vee$$

$$\vee (\exists x_1, x_2, a, b \in A\, [x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \beta])) \Leftrightarrow$$

$$\Leftrightarrow (((x, y) \in \alpha^{\upsilon}) \vee ((x, y) \in \beta^{\upsilon})) \Leftrightarrow$$

$$\Leftrightarrow (x, y) \in \alpha^{\upsilon} \cup \beta^{\upsilon}$$

for $(x, y) \in A^2$.

Lemma 8. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $(\alpha^w)^{-1} = (\alpha^{-1})^w$;

(b) $\alpha \subseteq \beta \Rightarrow \alpha^w \subseteq \beta^w$;

(c) $(\alpha^w)^w \subseteq \alpha^w$;

(d) $(\alpha \cup \beta)^w = \alpha^w \cup \beta^w$.

Proof. By Lemma 7 (a), we have:

$$(x, y) \in (\alpha^w)^{-1} \Leftrightarrow (y, x) \in \alpha^w \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\,[y = f(x_1, b, x_2) \wedge x = f(x_1, a, x_2) \wedge (b, a) \in \alpha^v]) \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\,[x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in (\alpha^v)^{-1}]) \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\,[x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in (\alpha^{-1})^v]) \Leftrightarrow$$

$$\Leftrightarrow (x, y) \in (\alpha^{-1})^w$$

for $(x, y) \in A^2$. Thus $(\alpha^w)^{-1} = (\alpha^{-1})^w$.

(b) Applying Lemma 7 (b), we directly get the condition (b).

(c) If $(x, y) \in (\alpha^w)^w$, then $x = f(x_1, a, x_2)$, $y = f(x_1, b, x_2)$ for some $(a, b) \in$ $\in (\alpha^w)^v$ and $x_1, x_2 \in A$. By Lemma 6 (b), $(\alpha^w)^v \subseteq \alpha^v$, and so $(a, b) \in \alpha^v$. Hence $(x, y) \in \alpha^w$. The condition (c) holds.

(d) By Lemma 7 (d), we get:

$$(x, y) \in (\alpha \cup \beta)^w \Leftrightarrow$$

$$\Leftrightarrow (\exists x_1, x_2, a, b \in A\,[x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in (\alpha \cup \beta)^v]) \Leftrightarrow$$

$$\Leftrightarrow ((\exists x_1, x_2, a, b \in A\,[x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \alpha^v]) \vee$$

$$\vee (\exists x_1, x_2, a, b \in A\,[x = f(x_1, a, x_2) \wedge y = f(x_1, b, x_2) \wedge (a, b) \in \beta^v])) \Leftrightarrow$$

$$\Leftrightarrow (((x, y) \in \alpha^w) \vee ((x, y) \in \beta^w)) \Leftrightarrow$$

$$\Leftrightarrow (x, y) \in \alpha^w \cup \beta^w$$

for $(x, y) \in A^2$. The condition (d) holds.

Put

$$\alpha^c = \alpha \cup \alpha^v \cup \alpha^w.$$

Note that $E_A^c = E_A$.

Lemma 9. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $\alpha \subseteq \alpha^c$;

(b) $(\alpha^c)^{-1} = (\alpha^{-1})^c$;

(c) $\alpha \subseteq \beta \Rightarrow \alpha^c \subseteq \beta^c$;

(d) $(\alpha^c)^c = \alpha^c$;

(e) $(\alpha \cup \beta)^c = \alpha^c \cup \beta^c$;

(f) $\alpha = \alpha^c$ *if and only if the relation* α *is lateral compatible with the ternary operation* f *in the ternary semigroup* (A, f).

Proof. The proofs of the conditions (a) and (c) are evident.

(b) According to the Lemmas 7 (a) and 8 (a) we have $(\alpha^c)^{-1} = (\alpha \cup \alpha^v \cup \alpha^w)^{-1} = \alpha^{-1} \cup (\alpha^v)^{-1} \cup (\alpha^w)^{-1} = \alpha^{-1} \cup (\alpha^{-1})^v \cup (\alpha^{-1})^w = (\alpha^{-1})^c$.

(d) By Lemmas 6, 7 (c), (d) and 8 (c), (d) we obtain $(\alpha^c)^c = (\alpha \cup \alpha^v \cup \alpha^w)^c = (\alpha \cup \alpha^v \cup \alpha^w) \cup (\alpha \cup \alpha^v \cup \alpha^w)^v \cup (\alpha \cup \alpha^v \cup \alpha^w)^w = (\alpha \cup \alpha^v \cup \alpha^w) \cup (\alpha^v \cup (\alpha^v)^v \cup (\alpha^w)^v) \cup (\alpha^w \cup (\alpha^v)^w \cup (\alpha^w)^w) = \alpha \cup \alpha^v \cup \alpha^w = \alpha^c$.

(e) In view of Lemmas 7 (d) and 8 (d) we have $(\alpha \cup \beta)^c = (\alpha \cup \beta) \cup (\alpha \cup \beta)^v \cup (\alpha \cup \beta)^w = (\alpha \cup \alpha^v \cup \alpha^w) \cup (\beta \cup \beta^v \cup \beta^w) = \alpha^c \cup \beta^c$.

(f) If $\alpha = \alpha^c$, then

$$\forall\, x_1, x_2, a, b \in A\, [(a, b) \in \alpha \Rightarrow (f(x_1, a, x_2), f(x_1, b, x_2)) \in \alpha^c = \alpha].$$

Therefore, the relation α is lateral compatible with the ternary operation f in the ternary semigroup (A, f).

Conversely, let the relation α be lateral compatible with the ternary operation f in the ternary semigroup (A, f). Assume that $(x, y) \in \alpha^c = \alpha \cup \alpha^v \cup \alpha^w$. If $(x, y) \in \alpha^v$, then $x = f(x_1, a, x_2)$ and $y = f(x_1, b, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence $(x, y) \in \alpha$. If $(x, y) \in \alpha^w$, then $x = f(y_1, c, y_2)$ and $y = f(y_1, d, y_2)$ for some $(c, d) \in \alpha^v$ and $y_1, y_2 \in A$. According to the foregoing $(c, d) \in \alpha$, and so $(x, y) \in \alpha$. Hence, $\alpha^c \subseteq \alpha$, therefore $\alpha^c = \alpha$. The proof of (f) is finished.

Analogously to the relations α^l and α^L, we can prove the following assertions.

Corollary 7. *Let* (A, f) *be a ternary semigroup. For any relation* $\alpha \in B(A)$, *the relation* α^c *is lateral compatible with the ternary operation* f *in the ternary semigroup* (A, f).

Theorem 3. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

$$\alpha^C = (\alpha^c)^E.$$

Lemma 10. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

$$(\alpha^u)^z = (\alpha^z)^u.$$

Proof. Assume that $(x, y) \in (\alpha^u)^z$. Therefore, $x = f(c, y_1, y_2)$, $y = f(d, y_1, y_2)$ for some $(c, d) \in \alpha^u$ and $y_1, y_2 \in A$. Hence, $c = f(x_1, x_2, a)$, $d = f(x_1, x_2, b)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Thus,

$$x = f(f(x_1, x_2, a), y_1, y_2) = f(x_1, x_2, f(a, y_1, y_2)),$$

$$y = f(f(x_1, x_2, b), y_1, y_2) = f(x_1, x_2, f(b, y_1, y_2)).$$

Put $c_1 = f(a, y_1, y_2)$ and $d_1 = f(b, y_1, y_2)$. Hence $(c_1, d_1) \in \alpha^z$. Since $x = f(x_1, x_2, c_1)$ and $y = f(x_1, x_2, d_1)$, it follows that $(x, y) \in (\alpha^z)^u$. Thus $(\alpha^u)^z \subseteq (\alpha^z)^u$. Similarly $(\alpha^z)^u \subseteq (\alpha^u)^z$.

The lemma is proved.

Put $\alpha^{zu} = (\alpha^z)^u$ and $\alpha^{uz} = (\alpha^u)^z$. From Lemma 10 it follows that

$$\alpha^{zu} = \alpha^{uz}.$$

Let (A, f) be a ternary semigroup. Assume that $\alpha \in B(A)$. Consider the relation

$$\alpha^j = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz}.$$

Note that $E_A^j = E_A$.

Lemma 11. *Let (A, f) be a ternary semigroup. Assume that $\alpha, \beta \in B(A)$. The following conditions are satisfied:*

(a) $\alpha \subseteq \alpha^j$;

(b) $(\alpha^j)^{-1} = (\alpha^{-1})^j$;

(c) $\alpha \subseteq \beta \Rightarrow \alpha^j \subseteq \beta^j$;

(d) $(\alpha^j)^j = \alpha^j$;

(e) $(\alpha \cup \beta)^j = \alpha^j \cup \beta^j$;

(f) $\alpha = \alpha^j$ *if and only if the relation* α *is left compatible and right compatible with the ternary operation* f *in the ternary semigroup* (A, f).

Proof. The proof of the condition (a) is evident.

(b) Taking advantage of Lemma 2 (a) and Lemma 4 (a) we get $(\alpha^{uz})^{-1} = ((\alpha^u)^z)^{-1} = ((\alpha^u)^{-1})^z = ((\alpha^{-1})^u)^z = (\alpha^{-1})^{uz}$. Therefore,

$$(\alpha^j)^{-1} = (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz})^{-1} =$$
$$= \alpha^{-1} \cup (\alpha^u)^{-1} \cup (\alpha^z)^{-1} \cup (\alpha^{uz})^{-1} =$$
$$= \alpha^{-1} \cup (\alpha^{-1})^u \cup (\alpha^{-1})^z \cup (\alpha^{-1})^{uz} =$$
$$= (\alpha^{-1})^j.$$

(c) Assume that $\alpha \subseteq \beta$. In view of Lemma 2 (b) and Lemma 4 (b) we obtain $\alpha^u \subseteq \beta^u$, $\alpha^z \subseteq \beta^z$, $\alpha^{uz} \subseteq \beta^{uz}$. Since $\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz} \subseteq \beta \cup \beta^u \cup \beta^z \cup \beta^{uz}$, it follows that $\alpha^j \subseteq \beta^j$.

(d) Taking into account Lemmas 2, 4 and 10, we have $(\alpha^{uz})^u \subseteq \alpha^{uz}$, $(\alpha^{uz})^z \subseteq \alpha^{uz}$, $(\alpha^u)^{uz} \subseteq \alpha^{uz}$, $(\alpha^z)^{uz} \subseteq \alpha^{uz}$, $(\alpha^{uz})^{uz} \subseteq \alpha^{uz}$. Thus,

$$(\alpha^j)^j = (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz})^j =$$
$$= (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz}) \cup (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz})^u \cup$$
$$\cup (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz})^z \cup (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz})^{uz} =$$
$$= (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz}) \cup (\alpha^u \cup \alpha^{uu} \cup \alpha^{zu} \cup (\alpha^{uz})^u) \cup$$
$$\cup (\alpha^z \cup \alpha^{uz} \cup \alpha^{zz} \cup (\alpha^{uz})^z) \cup$$
$$\cup (\alpha^{uz} \cup (\alpha^u)^{uz} \cup (\alpha^z)^{uz} \cup (\alpha^{uz})^{uz}) =$$
$$= \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz} = \alpha^j.$$

(e) According to Lemma 2 (d) and Lemma 4 (d), we get $(\alpha \cup \beta)^u = \alpha^u \cup \beta^u$, $(\alpha \cup \beta)^z = \alpha^z \cup \beta^z$, $(\alpha \cup \beta)^{uz} = \alpha^{uz} \cup \beta^{uz}$. Therefore,

$$(\alpha \cup \beta)^j = (\alpha \cup \beta) \cup (\alpha \cup \beta)^u \cup (\alpha \cup \beta)^z \cup (\alpha \cup \beta)^{uz} =$$

$$= (\alpha \cup \beta) \cup (\alpha^u \cup \beta^u) \cup (\alpha^z \cup \beta^z) \cup (\alpha^{uz} \cup \beta^{uz}) =$$

$$= (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz}) \cup (\beta \cup \beta^u \cup \beta^z \cup \beta^{uz}) =$$

$$= \alpha^j \cup \beta^j.$$

(f) Assume that $\alpha = \alpha^j$. Since $\alpha = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz}$, it follows that $\alpha^u \subseteq \alpha$, $\alpha^z \subseteq \alpha$, $\alpha^{uz} \subseteq \alpha$. Thus $\alpha^l = \alpha \cup \alpha^u \subseteq \alpha$, $\alpha^r = \alpha \cup \alpha^z \subseteq \alpha$, and so $\alpha = \alpha^l$ and $\alpha = \alpha^r$. Therefore, by Lemma 3 (f) and Lemma 5 (f), the relation α is left compatible and right compatible with the ternary operation f in the ternary semigroup (A, f).

Conversely, if the relation α is left compatible and right compatible with the ternary operation f, then $\alpha = \alpha^l$ and $\alpha = \alpha^r$ by Lemma 3 (f) and Lemma 5 (f). Since $\alpha = \alpha \cup \alpha^u$, $\alpha = \alpha \cup \alpha^z$, it follows that $\alpha^u \subseteq \alpha$ and $\alpha^z \subseteq \alpha$. Note that $\alpha^u = (\alpha \cup \alpha^z)^u = \alpha^u \cup \alpha^{zu}$, thus $\alpha^{zu} \subseteq \alpha^u \subseteq \alpha$. Therefore, $\alpha^j = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^{uz} \subseteq \alpha$. Hence $\alpha^j = \alpha$.

The lemma is proved.

An immediate consequence of Lemma 11 (d), (e) is the following corollary.

***Corollary* 8.** *Let (A, f) be a ternary semigroup. For any relation $\alpha \in B(A)$ the relation α^j is left compatible and right compatible with the ternary operation f in the ternary semigroup (A, f).*

Theorem 4. *Let (A, f) be a ternary semigroup. If $\alpha \in E(A)$, then*

$$\alpha^J = (\alpha^j)^E.$$

Proof. Note that $\alpha \subseteq (\alpha^j)^E$. We know that

$$(\alpha^j)^E = \bigcup_{n=1}^{\infty} \left(\alpha^j \cup (\alpha^j)^{-1} \cup E_A\right)^n.$$

Put $\beta = \alpha^j \cup (\alpha^j)^{-1} \cup E_A$. Hence $(\alpha^j)^E = \bigcup_{n=1}^{\infty} \beta^n$. By Lemma 11 (b), (e), $\beta = \alpha^j \cup (\alpha^{-1})^j \cup E_A^j = (\alpha \cup \alpha^{-1} \cup E_A)^j$. In view of Corollary 8, the relation β is left compatible and right compatible with the ternary operation f. By Corollary 3, the relation β^n, for every $n \in \mathcal{N}$, is left compatible and right compatible with the ternary operation f. Note that $(\alpha^j)^E \in C_j(A)$. Indeed, if $(a, b) \in (\alpha^j)^E$, then $(a, b) \in \beta^n$ for some $n \in \mathcal{N}$. Thus, $(f(x_1, x_2, a), f(x_1, x_2, b)) \in \beta^n$, $(f(a, x_1, x_2), f(b, x_1, x_2)) \in \beta^n$, that is $(f(x_1, x_2, a), f(x_1, x_2, b)) \in (\alpha^j)^E$, $(f(a, x_1, x_2), f(b, x_1, x_2)) \in (\alpha^j)^E$ for all $x_1, x_2 \in A$. Therefore, $(\alpha^j)^E \in C_l(A)$ and $(\alpha^j)^E \in C_r(A)$. According to Corollary 2, $(\alpha^j)^E \in C_j(A)$. Assume that $\gamma \in C_j(A)$ and $\alpha \subseteq \gamma$. Thus, $\gamma \in C_l(A)$ and $\gamma \in C_r(A)$. From the conditions (c) and (f) in Lemma 11 it follows that $\alpha^j \subseteq \gamma^j = \gamma$. Since $\gamma \in E(A)$, it follows that $(\alpha^j)^E \subseteq \gamma$. Thus $\alpha^J = (\alpha^j)^E$.

The theorem is proved.

Lemma 12. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in E(A)$, *then*

$$\alpha^{zu} = \alpha^{w}.$$

Proof. Assume that $(x, y) \in \alpha^{zu}$. Therefore, $x = f(y_1, y_2, c)$, $y = f(y_1, y_2, d)$ for some $(c, d) \in \alpha^z$ and $y_1, y_2 \in A$. Thus, $c = f(a, x_1, x_2)$, $d = f(b, x_1, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence, we get $x = f(y_1, y_2, f(a, x_1, x_2)) = f(y_1, f(y_2, a, x_1), x_2)$ and $y = f(y_1, y_2, f(b, x_1, x_2)) = f(y_1, f(y_2, b, x_1), x_2)$. Put $c_1 = f(y_2, a, x_1)$, $d_1 = f(y_2, b, x_1)$. Clearly, $(c_1, d_1) \in \alpha^{v}$. Since $x = f(y_1, c_1, x_2)$ and $y = f(y_1, d_1, x_2)$, it follows that $(x, y) \in \alpha^{w}$. Consequently $\alpha^{zu} \subseteq \alpha^{w}$.

Conversely, assume that $(x, y) \in \alpha^{w}$. Therefore, $x = f(y_1, c, y_2)$, $y = f(y_1, d, y_2)$ for some $(c, d) \in \alpha^{v}$ and $y_1, y_2 \in A$. Thus, $c = f(x_1, a, x_2)$, $d = f(x_1, b, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence, we get $x = f(y_1, f(x_1, a, x_2), y_2) = f(y_1, x_1, f(a, x_2, y_2))$ and $y = f(y_1, f(x_1, b, x_2), y_2) = f(y_1, x_1, f(b, x_2, y_2))$. Put $c_1 = f(a, x_2, y_2)$, $d_1 = f(b, x_2, y_2)$. Clearly, $(c_1, d_1) \in \alpha^{z}$. Since $x = f(y_1, x_1, c_1)$ and $y = f(y_1, x_1, d_1)$, it follows that $(x, y) \in (\alpha^{z})^{u} = \alpha^{zu}$. Consequently $\alpha^{w} \subseteq \alpha^{zu}$.

The lemma is proved.

An immediate consequence of the definition of the relation α^{j} and Lemma 12 is the following corollary.

***Corollary* 9.** *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

$$\alpha^{j} = \alpha \cup \alpha^{u} \cup \alpha^{z} \cup \alpha^{w}.$$

Lemma 13. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

(a) $(\alpha^{v})^{u} \subseteq \alpha^{v}$;

(b) $(\alpha^{v})^{u} = (\alpha^{u})^{v}$.

Proof. (a) Assume that $(x, y) \in (\alpha^{v})^{u}$. Therefore, $x = f(y_1, y_2, c)$, $y = f(y_1, y_2, d)$ for some $(c, d) \in \alpha^{v}$ and $y_1, y_2 \in A$. Thus, $c = f(x_1, a, x_2)$, $d = f(x_1, b, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence $x = f(f(y_1, y_2, x_1), a, x_2)$, $y = f(f(y_1, y_2, x_1), b, x_2)$, and so $(x, y) \in \alpha^{v}$.

(b) Assume that $(x, y) \in (\alpha^{v})^{u}$. Therefore, $x = f(y_1, y_2, c)$, $y = f(y_1, y_2, d)$ for some $(c, d) \in \alpha^{v}$ and $y_1, y_2 \in A$. Thus, $c = f(x_1, a, x_2)$, $d = f(x_1, b, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence, we get $x = f(y_1, y_2, f(x_1, a, x_2)) = f(y_1, f(y_2, x_1, a), x_2)$ and $y = f(y_1, y_2, f(x_1, b, x_2)) = f(y_1, f(y_2, x_1, b), x_2)$. Put $p = f(y_2, x_1, a)$, $q = f(y_2, x_1, b)$. Clearly, $(p, q) \in \alpha^{u}$. Since $x = f(y_1, p, x_2)$ and $y = f(y_1, q, x_2)$, it follows that $(x, y) \in (\alpha^{u})^{v}$. Consequently $(\alpha^{v})^{u} \subseteq (\alpha^{u})^{v}$.

Conversely, assume that $(x, y) \in (\alpha^{u})^{v}$. Therefore, $x = f(y_1, c, y_2)$, $y = f(y_1, d, y_2)$ for some $(c, d) \in \alpha^{u}$ and $y_1, y_2 \in A$. Thus, $c = f(x_1, x_2, a)$, $d = f(x_1, x_2, b)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. Hence, we get $x = f(y_1, f(x_1, x_2, a), y_2) = f(y_1, x_1, f(x_2, a, y_2))$ and $y = f(y_1, f(x_1, x_2, b), y_2) = f(y_1, x_1, f(x_2, b, y_2))$. Put $p = f(x_2, a, y_2)$, $q = f(x_2, b, y_2)$. Clearly, $(p, q) \in \alpha^{v}$. Since $x = f(y_1, x_1, p)$ and $y = f(y_1, x_1, q)$, it follows that $(x, y) \in (\alpha^{v})^{u}$. Consequently $(\alpha^{u})^{v} \subseteq (\alpha^{v})^{u}$.

The lemma is proved.

A closely analogous argument leads to the following lemma.

Lemma 14. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

(a) $(\alpha^{v})^{z} \subseteq \alpha^{v}$;

(b) $(\alpha^{v})^{z} = (\alpha^{z})^{v}$.

Lemma 15. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

(a) $(\alpha^{u})^{w} \subseteq \alpha^{w}$;

(b) $(\alpha^{u})^{w} = (\alpha^{w})^{u}$.

Proof. (a) Assume that $(x, y) \in (\alpha^{u})^{w}$. Therefore, $x = f(z_1, c_1, z_2)$, $y = f(z_1, d_1, z_2)$ for some $(c_1, d_1) \in (\alpha^{u})^{v}$ and $z_1, z_2 \in A$. Thus, $c_1 = f(y_1, c, y_2)$, $d_1 = f(y_1, d, y_2)$ for some $(c, d) \in \alpha^{u}$ and $y_1, y_2 \in A$. Hence, $c = f(x_1, x_2, a)$, $d = f(x_1, x_2, b)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. As an immediate consequence of the above results we get:

$$x = f(z_1, f(y_1, f(x_1, x_2, a), y_2), z_2) = f(f(z_1, y_1, x_1), f(x_2, a, y_2), z_2),$$

$$y = f(z_1, f(y_1, f(x_1, x_2, b), y_2), z_2) = f(f(z_1, y_1, x_1), f(x_2, b, y_2), z_2).$$

Put $c_2 = f(x_2, a, y_2)$, $d_2 = f(x_2, b, y_2)$. Clearly, $(c_2, d_2) \in \alpha^{v}$. Since $x = f(f(z_1, y_1, x_1), c_2, z_2)$, $y = f(f(z_1, y_1, x_1), d_2, z_2)$, it follows that $(x, y) \in \alpha^{w}$. Hence $(\alpha^{u})^{w} \subseteq \subseteq \alpha^{w}$.

(b) Assume that $(x, y) \in (\alpha^{u})^{w}$. From the results obtained in the proof of (a), we get $x = f(z_1, f(y_1, f(x_1, x_2, a), y_2), z_2)$ and $y = f(z_1, f(y_1, f(x_1, x_2, b), y_2), z_2)$. Note that

$$x = f(z_1, y_1, f(x_1, f(x_2, a, y_2), z_2)),$$

$$y = f(z_1, y_1, f(x_1, f(x_2, b, y_2), z_2))$$

for some $(a, b) \in \alpha$ and $x_1, x_2, y_1, y_2, z_1, z_2 \in A$. Put $c_2 = f(x_2, a, y_2)$, $d_2 = f(x_2, b, y_2)$. Hence, $(c_2, d_2) \in \alpha^{v}$. Set $c_3 = f(x_1, c_2, z_2)$, $d_3 = f(x_1, d_2, z_2)$. Hence, $(c_3, d_3) \in \alpha^{w}$. Since $x = f(z_1, y_1, c_3)$, $y = f(z_1, y_1, d_3)$, it follows that $(x, y) \in (\alpha^{w})^{u}$. Then $(\alpha^{u})^{w} \subseteq (\alpha^{w})^{u}$.

Conversely, assume that $(x, y) \in (\alpha^{w})^{u}$. Therefore, $x = f(z_1, z_2, c_1)$, $y = f(z_1, z_2, d_1)$ for some $(c_1, d_1) \in \alpha^{w}$ and $z_1, z_2 \in A$. Thus, $c_1 = f(y_1, c, y_2)$, $d_1 = f(y_1, d, y_2)$ for some $(c, d) \in \alpha^{v}$ and $y_1, y_2 \in A$. Hence, $c = f(x_1, a, x_2)$, $d = f(x_1, b, x_2)$ for some $(a, b) \in \alpha$ and $x_1, x_2 \in A$. As an immediate consequence of the above results, we get:

$$x = f(z_1, z_2, f(y_1, f(x_1, a, x_2), y_2)) = f(z_1, f(z_2, f(y_1, x_1, a), x_2), y_2),$$

$$y = f(z_1, z_2, f(y_1, f(x_1, b, x_2), y_2)) = f(z_1, f(z_2, f(y_1, x_1, b), x_2), y_2).$$

Put $c_2 = f(y_1, x_1, a)$, $d_2 = f(y_1, x_1, b)$. Clearly, $(c_2, d_2) \in \alpha^{u}$. Set $c_3 = f(z_2, c_2, x_2)$, $d_3 = f(z_2, d_2, x_2)$. Hence, $(c_3, d_3) \in (\alpha^{u})^{v}$. Since $x = f(z_1, c_3, y_2)$, $y = f(z_1, d_3, y_2)$, it follows that $(x, y) \in (\alpha^{u})^{w}$. Then $(\alpha^{w})^{u} \subseteq (\alpha^{u})^{w}$.

Lemma 16. *Let* (A, f) *be a ternary semigroup. If* $\alpha \in B(A)$, *then*

(a) $(\alpha^{z})^{w} \subseteq \alpha^{w}$;

(b) $(\alpha^{z})^{w} = (\alpha^{w})^{z}$.

The proof of this lemma is similar to the proof of Lemma 15.

Lemma 17. *Let (A,f) be a ternary semigroup. If $\alpha \in B(A)$, then*

$$(\alpha^c)^j = (\alpha^j)^c = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^v \cup \alpha^w.$$

Proof. In view of Lemmas 2, 4, 8 and Corollary 9 we get:

$$(\alpha^c)^j = (\alpha \cup \alpha^v \cup \alpha^w)^j =$$

$$= (\alpha \cup \alpha^v \cup \alpha^w) \cup (\alpha \cup \alpha^v \cup \alpha^w)^u \cup (\alpha \cup \alpha^v \cup \alpha^w)^z \cup (\alpha \cup \alpha^v \cup \alpha^w)^w =$$

$$= (\alpha \cup \alpha^v \cup \alpha^w) \cup (\alpha^u \cup (\alpha^v)^u \cup (\alpha^w)^u) \cup$$

$$\cup (\alpha^z \cup (\alpha^v)^z \cup (\alpha^w)^z) \cup (\alpha^w \cup (\alpha^v)^w \cup (\alpha^w)^w).$$

From Lemmas 6, 8, 13 – 16 it follows that $(\alpha^c)^j = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^v \cup \alpha^w$.

Lemmas 7 (d), 8 (d) and Corollary 9 imply that

$$(\alpha^j)^c = (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^w)^c =$$

$$= (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^w) \cup (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^w)^v \cup (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^w)^w =$$

$$= (\alpha \cup \alpha^u \cup \alpha^z \cup \alpha^w) \cup (\alpha^v \cup (\alpha^u)^v \cup (\alpha^z)^v) \cup (\alpha^w)^v) \cup$$

$$\cup (\alpha^w \cup (\alpha^u)^w \cup (\alpha^z)^w \cup (\alpha^w)^w).$$

From Lemmas 6, 8, 13 – 16 it follows that $(\alpha^j)^c = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^v \cup \alpha^w$.
Therefore, $(\alpha^c)^j = (\alpha^j)^c$.

The lemma is proved.

Let (A,f) be a ternary semigroup. Assume that $\alpha \in B(A)$.

Put

$$\alpha^k = (\alpha^c)^j.$$

In view of Lemma 17,

$$\alpha^k = \alpha \cup \alpha^u \cup \alpha^z \cup \alpha^v \cup \alpha^w.$$

Note that $E_A^k = E_A$.

Lemma 18. *Let (A,f) be a ternary semigroup. Assume that α, β, $\in B(A)$. The following conditions are satisfied*:

(a) $\alpha \subseteq \alpha^k$;

(b) $(\alpha^k)^{-1} = (\alpha^{-1})^k$;

(c) $\alpha \subseteq \beta \Rightarrow \alpha^k \subseteq \beta^k$;

(d) $(\alpha^k)^k = \alpha^k$;

(e) $(\alpha \cup \beta)^k = \alpha^k \cup \beta^k$;

(f) $\alpha = \alpha^k$ *if and only if the relation α is left compatible, right compatible and lateral compatible with the ternary operation f in the ternary semigroup (A,f).*

Proof. In view of Lemmas 9, 11, and 17 it is easy to prove the assertions (a) – (e).

(f) Suppose that $\alpha = \alpha^k$. Thus $\alpha = (\alpha^c)^j = (\alpha \cup \alpha^v \cup \alpha^w)^j = \alpha^j \cup (\alpha^v)^j \cup (\alpha^w)^j$. Hence, $\alpha^j \subseteq \alpha$, and so $\alpha^j = \alpha$. Moreover, $\alpha = (\alpha^j)^c = \alpha^c$. According to Lemmas 9 (f) and 11 (f), the relation α is left compatible, right compatible and lateral compatible with the ternary operation f.

Conversely, if the relation α is left compatible, right compatible and lateral compatible with the ternary operation f, then $\alpha^j = \alpha$ and $\alpha^c = \alpha$ by Lemmas 9 (f), 11 (f) and so $\alpha^k = (\alpha^c)^j = \alpha$.

In view of Lemma 18 (d), (f) we obtain the following corollary.

***Corollary* 10.** *Let (A, f) be a ternary semigroup. For any relation $\alpha \in B(A)$, the relation α^k is left compatible, right compatible and lateral compatible with the ternary operation f in the ternary semigroup (A, f).*

Theorem 5. *Let (A, f) be a ternary semigroup. If $\alpha \in B(A)$, then*

$$\alpha^K = (\alpha^k)^E.$$

We can prove this theorem similarly to Theorem 4.

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