КОРОТКІ ПОВІДОМЛЕННЯ

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THE ENTIRE SOLUTIONS
OF THE EULER – POISSON EQUATIONS
ЦІЛІ РОЗВ'ЯЗКИ РІВНЯНЬ ЕЙЛЕРА – ПУАССОНА

All entire solutions of Euler – Poisson equations are presented.

Наведено всі цілі розв'язки рівнянь Ейлера – Пуассона.

Introduction. Solutions of the Euler – Poisson equations as analytic functions can be investigated by means of the properties of these solutions in the singular points. But besides, it is necessary to remember that there exist the solutions which have no singular points. In [1], necessary and sufficient conditions for the existence of such solutions were proved.

In this paper, we completely solve the problem of searching of the entire solutions to the Euler – Poisson equations. We prove that the sufficient condition for the existence of the entire solutions formulated in [1] is necessary too. Then we present the full collection of the entire solutions. All these solutions are the Euler [2], Lagrange [2], and Grioli [3] well-known partial solutions.

1. Preliminaries. We analyze the Euler – Poisson equations in the following form:

$$A\dot{p} = Ap \times p + \gamma \times r,$$
$$\dot{\gamma} = \gamma \times p, \tag{1}$$

where $p = (p_1, p_2, p_3) \in \mathbb{C}^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$, A — symmetric operator $A : \mathbb{R}^3 \to \mathbb{R}^3$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$. System (1) has the following first integrals:

$$\mathcal{H}(z) = \frac{1}{2}\langle Ap, p \rangle + \langle \gamma, r \rangle,$$

$$\mathcal{M}(z) = \langle Ap, \gamma \rangle,$$

$$\mathcal{T}(z) = \langle \gamma, \gamma \rangle,$$

where

$$\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i$$

is $\mathbb{C}$-scalar product in $\mathbb{C}^3$.

We use the notations: $z(t) = (p(t), \gamma(t))$, $B_{ij} = A_i - A_j$, $C_{ij} = 2A_i - A_j$, $i, j = 1, 2, 3$.

We use the circle permutation of the indices $\sigma = (1, 2, 3)$ for writing the products or sums (for example, $\sum_\sigma A_1 A_2 = A_1 A_2 + A_2 A_3 + A_3 A_1$, $\prod_\sigma A_1 = A_1 A_2 A_3$) and expressions which differ one from another only in the circle permutation of the indices ($\dot{\gamma} = \gamma \times p$, can be written as $\gamma_1 = p_3 \gamma_2 - p_2 \gamma_3$, σ).

Let $\mathbb{C}$ act as a transformation group on $\mathbb{C}^6$ in the following way:

$$\theta: (p_1, p_2, p_3, \gamma_1, \gamma_2, \gamma_3) \to (\theta p_1, \theta p_2, \theta p_3, \theta^2 \gamma_1, \theta^2 \gamma_2, \theta^2 \gamma_3).$$

It is well known that the factor-space by this action [4] denoted by P_*^5 is a compact holomorphic manifold. The canonical projection $\pi: \mathbb{C}^6 \to P_*^5$ maps the foliation [5] induced by flow (1) onto the foliation $\mathcal{F}$ of the compact holomorphic manifold P_*^5. The trajectories $(p(t), \gamma(t))$ are invariant under the action θ because if $(p(t), \gamma(t))$ is a solution of system (1), then $(\theta p(\theta t), \theta^2 \gamma(\theta t))$ is a solution too. Hence, the image $\pi(z(t))$ of any trajectory is the fiber of the foliation $\mathcal{F}$ on the manifold P_*^5. Besides, if $z(t) \to \infty$, then $\pi(z(t)) \to X^2$, where $X^2 = \{\pi(z): \mathcal{H}(z) = \mathcal{M}(z) = \mathcal{T}(z) = 0\}$. It means that all singular points of the solutions $z(t)$ of system (1) are projected onto the singular points of the foliation $\mathcal{F}\big|_{X^2}$ of the manifold X^2.

Definition 1. *The algebraic system*

$$A \tilde{p}^0 \times \tilde{p}^0 + \tilde{\gamma}^0 \times r + A \tilde{p}^0 = 0,$$

$$\tilde{\gamma}^0 \times \tilde{p}^0 + 2\tilde{\gamma}^0 = 0 \tag{2}$$

is called a characteristic system for the Euler – Poisson equations.

Proposition 1 [6]. *Let* $\prod_\sigma B_{12} r_3 \neq 0$, *then there exist two types of the nonzero solution* $(\tilde{p}^0, \tilde{\gamma}^0)$ *of the characteristic system* (2).

The α-*solutions have a form*

$$\tilde{p}_1^0 = \sqrt{\frac{A_2 A_3}{B_{12} B_{31}}}, \sigma, \quad \tilde{\gamma}_1^0 = 0, \sigma,$$

besides if $((\tilde{p}_1^0, \tilde{p}_2^0, \tilde{p}_3^0), \tilde{\gamma}^0)$ *is a solution of system* (2), *then there exist three more solutions of* (2)

$$\left(-\tilde{p}_1^0, -\tilde{p}_2^0, \tilde{p}_3^0\right), \quad \left(-\tilde{p}_1^0, -\tilde{p}_2^0, -\tilde{p}_3^0\right), \quad \left(\tilde{p}_1^0, -\tilde{p}_2^0, -\tilde{p}_3^0\right).$$

The β-*solutions have a form*

$$\tilde{p}_1^0 = \sqrt{\frac{(2A_2 - \rho)(2A_3 - \rho)}{B_{12} B_{31}}}, \sigma, \quad \tilde{\gamma}^0 = -\frac{A \tilde{p}^0 \times \tilde{p}^0}{\langle p, r \rangle}, \tag{3}$$

where ρ *is a root of the equation*

$$\sum_\sigma r_1 (A_1 - \rho) \sqrt{(2A_2 - \rho)(2A_3 - \rho) B_{23}} = 0 \tag{4}$$

or a root of the polynomial

$$\sum_\sigma \left[r_1^4 B_{23}^2 (A_1 - \rho)^4 (2A_2 - \rho)^2 (2A_3 - \rho)^2 - \right.$$

$$\left. - 2 r_2^2 r_3^2 B_{12} B_{31} (A_2 - \rho)^2 (A_3 - \rho)^2 (2A_1 - \rho) \prod_\sigma (2A_1 - \rho) \right] = 0.$$

Proposition 2 [6]. *Let* $A_1 = A_2$, $r_2 = 0$, $r_1 r_3 (2A_3 - A_1) \neq 0$, *then the characteristic system* (2), *in particular, has the solutions of the following form*:

$$\tilde{p}^0 = (0, \pm 2i, 0), \quad \tilde{\gamma}^0 = \left(\frac{\pm 2A_1 i}{r_3 \pm i r_1}, 0, \frac{\pm 2A_1}{r_3 \pm i r_1} \right), \tag{5}$$

$$\tilde{p}^0 = \left(\mp\frac{2A_3r_3i}{C_{31}r_1}, \frac{2A_3r_3}{C_{31}r_1}, \pm 2i\right), \quad \tilde{\gamma}^0 = \left(\frac{2A_3}{r_1}, \pm\frac{2A_3i}{r_1}, 0\right). \tag{6}$$

Proposition 3 [1]. *All singular points of the foliation* $\mathcal{F}\big|_{x^2}$ *have the form* $\pi(\tilde{p}^0, \tilde{\gamma}^0)$, *where* $(\tilde{p}^0, \tilde{\gamma}^0)$ *is a solution of* (2), *if*

$$\prod_\sigma B_{12} \sum_\sigma r_1\sqrt{B_{23}} \neq 0.$$

Proposition 4 [1]. *The singular points of the foliation* $\mathcal{F}$, *which have the form* $\pi(\tilde{p}^0, \tilde{\gamma}^0)$, *correspond to* α- *or* β-*singular points of the solutions of the Euler – Poisson equations with the following asymptotics.*
The asymptotics of the α-*singular points has the following form*:

$$p(t) = \tilde{p}^0 t^{-1} + \alpha_1 u_1 + \sum_0^2 \psi_i t Ln^i t + \sum_2^4 \alpha_i v_i t + o(t),$$

$$\gamma(t) = \alpha_1 v_1 t^{-2} + k_1\tilde{p}^0 Lnt + k_0 v_1 + \alpha_4\tilde{p}^\sigma + t\sum_0^2 \chi'_i Ln^i t + \alpha_5 v_{-1}t + o(t),$$

$\arg t = \text{const}$; *here,* $\alpha_1, \dots, \alpha_5$ *are the free parameters,* u_1, v_i, ψ_i, χ'_i, k_i *are expressed in terms of* A_i, r_i, *and* $(\tilde{p}^0, \tilde{\gamma}^0)$ *is the* α-*solution of the characteristic system* (2).
The asymptotics of the β-*singular points has the following form*:

$$p(t) = \tilde{p}^0 t^{-1} + \beta_0 u_0 t^{\lambda_0-1} + \beta^0 u^0 t^{\lambda^0-1} + \beta_2 u_2 t + \beta_3 u_3 t^2 +$$

$$+ \beta_4 u_4 t^3 + \dots + \sum_{i+j\geq 2} \beta_0^i(\beta^0)^j \psi_{ij} t^{i\lambda_0+j\lambda^0-1} + \dots,$$

$$\gamma(t) = \tilde{\gamma}^0 t^{-2} + \beta_0 v_0 t^{\lambda_0-2} + \beta^0 v^0 t^{\lambda^0-2} + \beta_2 v_2 + \beta_3 v_3 t +$$

$$+ \beta_4 v_4 t^2 + \dots + \sum_{i+j\geq 2} \beta_0^i(\beta^0)^j \chi_{ij} t^{i\lambda_0+j\lambda^0-2} + \dots$$

where β_i, β_0, β^0 *are free parameters,* u_i, v_i, ψ_{ij}, χ_{ij} *are expressed in terms of* A_i, r_i, $\tilde{p}^0$, $\tilde{\gamma}^0$, $(\tilde{p}^0, \tilde{\gamma}^0)$ *is the* β-*solution of the characteristic system* (2).

Theorem 1 [1]. *For the existence of the solutions of the Euler – Poisson equations* (1) *without singular points, it is necessary that*

$$\prod_\sigma B_{12} \sum_\sigma r_1\sqrt{B_{23}} = 0.$$

For the existence of the solutions of the Euler – Poisson equations (1) *without singular points, it is sufficient that*

$$\sum_\sigma r_1\sqrt{B_{23}} = 0.$$

Remark 1. It follows from the proof of Theorem 1 that the existence of entire solutions of the Euler – Poisson equations is connected with the departure of the roots of the characteristic system $\tilde{z}^0$ to infinity. This takes place under the condition

$$\sum_\sigma r_1\sqrt{B_{23}} = 0.$$

2. The criterion of the existens of the entire solutions to the Euler – Poisson equations.

Proposition 5. *Let some root* $\bar{z}^0$ *of the characteristic system tend to infinity if* $A_1 \to A_2$. *Then*

$$\lim_{A_1 \to A_2} \pi(\bar{z}^0) = (1:\pm i:0:0:0:0).$$

Proof. The condition $A_1 = A_2$ gives the possibility to assume without lost of generality that $r_2 = 0$. Therefore, we suppose that $A_1 \to A_2$ and $r_2 \to 0$.

If a root ρ of the equation (4) tends to infinity, then it follows from (3) that

$$\bar{p}_1^0 \sim \frac{\rho}{\sqrt{B_{12} B_{31}}}, \quad \sigma,$$

$$\bar{\gamma}_1^0 \sim \frac{\rho \sqrt{B_{23}}}{\sum_\sigma r_1 \sqrt{B_{23}}}, \quad \sigma,$$

i. e.,

$$\pi(\bar{z}^0) \to (1:\pm i:0:0:0:.0) = z_\infty.$$

If ρ is bounded, then by (4), ρ tends to A_1 or to $2A_2$ or to $2A_3$.

In the case when $\rho \to A_1$, it follows from (3) that $\pi(\bar{p}^0) \to z_\infty$.

Let $\rho \to 2A_3$, then accordingly to (3) $p_3 \to \pm 2i$, $p_2 \sim \pm ip_1$. Substituting the limiting values of p_3 and $p_2 = \pm ip_1$ into (4), we obtain

$$\bar{p}_1^0 = \mp \frac{2A_3 r_3 i}{C_{31} r_1}$$

and then we find $\bar{\gamma}^0$ from (3). This values $\bar{p}^0$, $\bar{\gamma}^0$ coincide with solutions (6) of the characteristic system. Thus, in the case $\rho \to 2A_3$, the condition of the proposition is not satisfied because $\bar{z}^0$ does not tend to infinity.

Finally, let $\rho \to 2A_2$ if $A_1 \to A_2$, $r_2 \to 0$. We obtain

$$r_1(A_1 - \rho)\sqrt{(2A_3 - \rho)B_{23}} + r_2(A_2 - \rho)\sqrt{\frac{2A_1 - \rho}{2A_2 - \rho}(2A_3 - \rho)B_{23}} +$$

$$+ r_3(A_3 - \rho)\sqrt{(2A_1 - \rho)B_{12}} = 0$$

from (3).

Since the last term is much less than the first one, we have

$$\sqrt{2A_2 - \rho} \sim \pm \frac{r_2}{r_1}\sqrt{2A_1 - \rho}.$$

Thus, $\bar{p}_1^0 = o(\bar{p}_2^0)$ and $\bar{p}_3^0 = o(\bar{p}_2^0)$. According to [6], $\sum_\sigma (\bar{p}_1^0)^2 = -4$, consequently, $\bar{p}_2^0 \to \pm 2i$, $\bar{p}_1^0 \to 0$, $\bar{p}_3^0 \to 0$, and we obtain root (5) of the characteristic system (2). Again, we receive the contradiction with the condition of proposition, which proves it.

Theorem 2. *The Euler – Poisson equations have nontrivial entire solution if and only if the following condition is satisfied:*

$$\sum_\sigma r_1 \sqrt{B_{23}} = 0. \qquad (7)$$

Proof. It follows from Remark 1 and Proposition 3 that the solution $z(t)$ of the system (1) may be entire only if fiber $\pi(z(t))$ enters the singular point z_∞ and does not enter the singular points $\pi(\bar{z}^0)$.

Suppose that condition (7) is not satisfied. Let us consider the foliation $\mathcal{F}$ under conditions $A_1 = A_2$, $r_2 = 0$ and prove that there do not exist nontrivial fibers entering the singular point z_∞ in this case.

Similar to [1], let us write the differential equation of the fibers of the foliation $\mathcal{F}$ in the neighbourhood of the point z_∞:

$$
\begin{aligned}
A_1 \dot{p}_2 &= B_{31} p_1 p_3 + r_1 \gamma_3 - r_3 \gamma_1 - A_1 p_2 f, \\
A_3 \dot{p}_3 &= -r_1 \gamma_2 - A_3 p_3 f, \\
\dot{\gamma} &= \gamma \times p - 2f\gamma,
\end{aligned}
\tag{8}
$$

where $f = (-B_{31} p_2 p_3 + r_3 \gamma_2)(A_1 p_1)^{-1}$, $p_1 = \text{const}$.

Let us consider linear approximation of system (8) in the neighbourhood of the singular point $z_\infty = \pi(1 : i : 0 : 0 : 0 : 0)$. For the simplicity, assume $p_1 = 1$ and make the replacement $p_2 \to i + p_2$.

We obtain the following system:

$$
\begin{aligned}
A_1 \dot{p}_2 &= r_1 \gamma_3 - r_3 \gamma_1 - i r_3 \gamma_2, \\
A_3 \dot{p}_3 &= -r_1 \gamma_2, \\
\dot{\gamma}_1 &= -i \gamma_3, \\
\dot{\gamma}_2 &= \gamma_3, \\
\dot{\gamma}_3 &= i \gamma_1 - \gamma_2.
\end{aligned}
\tag{9}
$$

Since $(i\gamma_1 - \gamma_2) = 0$, we have $i\gamma_1 = \gamma_2$ for the trajectories entering the singular point. Besides, $r_1 \neq 0$, otherwise condition (7) holds. System (9) takes the following form:

$$
\begin{aligned}
A_1 \dot{p}_2 &= r_1 \gamma_3, \\
A_3 \dot{p}_3 &= -r_1 \gamma_2, \\
\dot{\gamma}_1 &= -i \gamma_3, \\
\dot{\gamma}_3 &= 0.
\end{aligned}
\tag{10}
$$

The system (10) of the differential equations has the solution:

$$
\gamma_3 = \gamma_{30}, \quad \gamma_1 = \gamma_{10} - i\gamma_{30} t, \quad p_2 = p_{20} + r_1 \gamma_{30} t,
$$

$$
p_3 = p_{30} - \frac{i r_1}{2 A_3} (2\gamma_{10} t - i\gamma_{30} t^2).
$$

We see that the trajectories of all solutions do not enter the singular point. By means of Picard iterations, one can obtain the asymptotic behavior of the original nonlinear system and see that all trajectories do not enter the singular point z_∞ too.

However, this trajectories do not form the full collection of the trajectories in the neighbourhood of z_∞, because there exists the plane of singular points of system (9): $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

Let us consider the domain U, in which the norm of vectors of speed to the linear system (9) is much more then its nonlinear perturbation by system (8). Then the trajectories entering the singular point are in the complement of the domain U. For this trajectories, the following estimate takes place:

$$\gamma_i = o(|p_2|+|p_3|). \tag{11}$$

The system (8) of differential equations with the equation $\dot{\mu} = -\mu f$ has three integrals

$$A_1 i p_2 + \frac{A_3}{2} p_3^2 + \gamma_1 r_1 + \gamma_3 r_3 = c_1 \mu^2, \tag{12}$$

$$A_1 \gamma_1 + A_1 i \gamma_2 + A_1 p_2 \gamma_2 + A_3 p_3 \gamma_3 = c_2 \mu^3,$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 \mu^4. \tag{13}$$

It follows from relation (13) that

$$\mu^2 = o(|p_2|+|p_3|). \tag{14}$$

Substituting (11) and (14) into (12), we obtain that $p_2 = o(p_3)$. Then we present p_2 as the product p_3 and infinitesimal ε and substitute this presentation into original system of the differential equations (8). We receive $\dot{\varepsilon} \sim$ const but it is impossible.

The theorem is proved.

3. The entire solutions of the Euler – Poisson equations for the cases when $\sum_\sigma r_1 \sqrt{B_{23}} = 0.$ We now find all entire solutions of the Euler – Poisson equations by means of the criterion which was proved in Section 2.

The case $r_1 = r_2 = r_3 = 0$ *(Euler).* One can write the first integrals in the following form:

$$A_1 p_1^2 + A_2 p_2^2 = D v^2 - A_3 p_3^2,$$
$$A_1^2 p_1^2 + A_2^2 p_2^2 = D^2 v^2 - A_3^2 p_3^2. \tag{15}$$

If $A_1 \neq A_2$, then

$$A_3 \dot{p}_3 = B_{23} \sqrt{\frac{Dv^2(D-A_2)+A_3 B_{23} p_3^2}{A_1 B_{12}} \frac{Dv^2(A_1-D)+A_3 B_{31} p_3^2}{A_2 B_{12}}}, \tag{16}$$

hence, the solution $p(t)$, in general, has the singular points in $\mathbb{C}$.

However, the solutions of (16) are expressed by the elliptic function not always. First, the linear system (15) can have the null determinant relatively p_1^2, p_2^2 when $A_1 = A_2$. Second on the right-hand side of (16), the irrationality vanishes under condition $D = A_3$.

Let $A_1 = A_2$, then the solution

$$p_3(t) = p_3(0), \quad p_1(t) = \cos\left(\frac{B_{31}}{A_1} p_3 t + \varphi_0\right),$$

$$p_2(t) = \sin\left(\frac{B_{31}}{A_1} p_3 t + \varphi_0\right) \tag{17}$$

has not the singular points in $\mathbb{C}$. The same is true for the solutions

$$\dot{\gamma} = \gamma \times p.$$

It is well known (see, for example, [2]) that Euler angles have the following presentation:

$$\theta = \arccos\frac{A_3 p_3}{vD}, \quad \varphi = \text{arctg}\frac{A_1 p_1}{A_2 p_2}, \quad \psi = vD \int \frac{A_1 p_1^2 + A_2 p_2^2}{A_1^2 p_1^2 + A_2^2 p_2^2}, \tag{18}$$

and vector $\gamma(t)$ is the linear combination with the constant coefficients of the three independent solutions

$$\gamma = k_1 \begin{pmatrix} \cos(\varphi)\cos(\psi) - \cos(\theta)\sin(\varphi)\sin(\psi) \\ -\sin(\varphi)\cos(\psi) - \cos(\theta)\cos(\varphi)\sin(\psi) \\ \sin(\theta)\sin(\psi) \end{pmatrix} +$$

$$+ k_2 \begin{pmatrix} \cos(\varphi)\sin(\psi) - \cos(\theta)\sin(\varphi)\cos(\psi) \\ -\sin(\varphi)\sin(\psi) + \cos(\theta)\cos(\varphi)\cos(\psi) \\ -\sin(\theta)\cos(\psi) \end{pmatrix} + k_3 \begin{pmatrix} \sin(\varphi)\sin(\theta) \\ \cos(\varphi)\sin(\theta) \\ \cos(\varphi) \end{pmatrix}. \qquad (19)$$

In the case under consideration, the functions $\sin(\theta)$, $\cos(\theta)$ are constant,

$$\sin(\varphi) = \frac{A_1 p_1}{\sqrt{D^2 v^2 - A_3^2 p_3^2}}, \qquad \cos(\varphi) = \frac{A_2 p_2}{\sqrt{D^2 v^2 - A_3^2 p_3^2}},$$

$$\psi = \frac{vD}{A_1} t + \psi_0, \qquad (20)$$

i. e., the solution of system (1) is expressed by the entire functions.

In the case $D = A_3$, we have

$$-A_1 B_{31} p_1^2 + A_2 B_{23} p_2^2 = 0 \Rightarrow p_2^2 = \frac{A_1 B_{31}}{A_2 B_{23}} p_1^2 \Rightarrow p_3 = v \frac{1 + ce^{avt}}{1 - ce^{avt}},$$

where a is expressed by the parameters A_i of the solid body and c is arbitrary constant. Thus, in the case $D = A_i$, the solution has the singular point in the complex plane.

The case $A_1 = A_2$, $r_1 = r_2 = 0$ *(Lagrange)*. It is known [2] that

$$A_1^2 (\dot\gamma_3)^2 = A_1 \left(2\mathcal{H}_0 - A_3 r_3^2 - 2r_3\gamma_3\right)\left(\mathcal{T}_0 - \gamma_3^2\right) -$$

$$- \left(\mathcal{M}_0 - A_3 p_{30}\gamma_3\right)^2 \equiv P_3(\gamma_3). \qquad (21)$$

If roots of the polynomial $P_3(\gamma_3)$ are simple, then the function $\gamma_3(t)$ is expressed by the Jacobi elliptic functions and, hence, has the singular points in $\mathbb{C}$. Therefore, let us consider the cases with the multiple roots (see [2]).

Since $\gamma_3 \in \left[-\sqrt{\mathcal{T}_0}, \sqrt{\mathcal{T}_0}\right]$, there are three cases: the multiple root of the polynomial $P_3(\gamma_3)$ is equal to $-\sqrt{\mathcal{T}_0}$, $\sqrt{\mathcal{T}_0}$, or lies in the interval $\left(-\sqrt{\mathcal{T}_0}, \sqrt{\mathcal{T}_0}\right)$.

Let the multiple root be equal to $\pm\sqrt{\mathcal{T}_0}$, then $\mathcal{M}_0 \mp A_3 p_{30}\sqrt{\mathcal{T}_0} = 0$ and $2\mathcal{H}_0 - A_3 r_3^2 \mp r_3\sqrt{\mathcal{T}_0} = 0$.

Equation (21) takes a form

$$A_1^2 (\dot\gamma_3)^2 = (\pm\mathcal{T}_0 - \gamma_3)\sqrt{(r_3 \pm \mathcal{T}_0 + \gamma_3) - A_3 p_{30}}. \qquad (22)$$

Its solution is expressed by hyperbolic tangent, hence, the differential equation (22) has the singular points in the complex plane.

If the multiple root lies in the interval $\left(-\sqrt{\mathcal{T}_0}, \sqrt{\mathcal{T}_0}\right)$, then, taking into account that $P_3\left(\pm\sqrt{\mathcal{T}_0}\right) < 0$, we get that $P_3(\gamma_3) \le 0$ in the segment $\left[-\sqrt{\mathcal{T}_0}, \sqrt{\mathcal{T}_0}\right]$ and $P_3(\gamma_3) = 0$ at unique point. Since $P_3(\gamma_3) = A_1^2 (\dot\gamma_3)^2$, we have

$$(\dot{\gamma}_3)^2 \equiv 0, \tag{23}$$

because the number $(\dot{\gamma}_3)^2$ cannot be negative. Then we substitute

$$p_1 = \lambda \gamma_1, \qquad p_2 = \lambda \gamma_2 \tag{24}$$

into integral of angular momentum and obtain that

$$\lambda = \text{const.} \tag{25}$$

It means that, in the considered case, the Euler – Poisson equations become linear and the solutions are expressed by the sines and cosines.

The case $r_1\sqrt{B_{23}} + r_2\sqrt{B_{31}} = 0$, $r_3 = 0$ (Grioli). It is suitable to consider this case in the special coordinates [3] in which the inertia operator has a form

$$\tilde{J} = \begin{pmatrix} J_1 & 0 & J_2 \\ 0 & J_1 & 0 \\ J_2 & 0 & J_3 \end{pmatrix}$$

and two coordinates of the center of gravity are equal to zero: $r_1 = r_2 = 0$.

Besides, the eigen-values of the inertia operator are equal to

$$J_1, \quad \frac{J_1 + J_3}{2} \pm \sqrt{\frac{(J_1 - J_3)^2}{4} + J_2^2}.$$

They are real and positive if $J_1 > 0$, $J_1 J_3 > J_2^2$.

The differential equations (1) in this coordinates have a form:

$$J_1\dot{p}_1 + J_2\dot{p}_3 = (J_1 - J_3)p_2 p_3 - J_2 p_1 p_2 + r_3\gamma_2,$$

$$J_1\dot{p}_2 = J_2 p_1^2 - J_2 p_3^2 - (J_1 - J_3)p_1 p_2 - r_3\gamma_1,$$

$$J_2\dot{p}_1 + J_3\dot{p}_3 = J_2 p_2 p_3,$$

$$\dot{\gamma}_1 = p_3\gamma_2 - p_2\gamma_3,$$

$$\dot{\gamma}_2 = p_1\gamma_3 - p_3\gamma_1,$$

$$\dot{\gamma}_3 = p_2\gamma_1 - p_1\gamma_2.$$

The first integrals are equal to

$$\frac{1}{2}\left(J_1 p_1^2 + J_2 p_2^2 + J_3 p_1^3\right) + J_2 p_1 p_3 + r_3\gamma_3 = \mathcal{H},$$

$$J_1 p_1\gamma_1 + J_2(p_1\gamma_3 + p_3\gamma_1) + J_1 p_2\gamma_2 + J_3 p_3\gamma_3 = \mathcal{M},$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \mathcal{T}.$$

If the parameters of a solid body tend to the parameters such as in considered case, then some pair of the conjugate solutions of the characteristic system tend to infinity: $\tilde{z}^0 \to z_\infty$. In order to find the coordinates z_∞, we use the relations

$$\gamma_0 = \nu p_0,$$

$$\langle Ap_0, p_0 \rangle = \langle p_0, p_0 \rangle = \langle p_0, r \rangle = 0, \tag{26}$$

which were obtained in the proof [1] of Proposition 3. We note that they are followed from the conditions $\mathcal{H} = \mathcal{M} = \mathcal{T} = 0$.

We get that $p_3 = 0$. Moreover, without lost of generality, one can suppose that $p_1 = 1$, and according to (26), $p_2 = \pm i$. The case $p_2 = i$ is similar to the case $p_2 = -i$, therefore, we consider only first one. It follows from (26) that $\gamma_1 = \nu$, $\gamma_2 = i\nu$, $\gamma_3 = 0$, where ν is some constant, which will be found below.

Similar to [1], let us write the differential equations which present the foliation $\mathcal{F}$ in the neighbourhood of the point z_∞:

$$J_1\dot{p}_1 + J_2\dot{p}_3 = (J_1-J_3)p_2p_3 - J_2p_1p_2 + r_3\gamma_2 - (J_1p_1 - J_2p_3)f,$$
$$J_1\dot{p}_2 = J_2p_1^2 - J_2p_3^2 - (J_1-J_3)p_1p_2 - r_3\gamma_1 - J_1p_2f,$$
$$J_2\dot{p}_1 + J_3\dot{p}_3 = J_2p_2p_3 - (J_2p_1 + J_3p_3)f, \tag{27}$$
$$\dot{\gamma}_2 = p_1\gamma_3 - p_3\gamma_1 - 2\gamma_2 f,$$
$$\dot{\gamma}_3 = p_2\gamma_1 - p_1\gamma_2 - 2\gamma_3 f;$$

here

$$\gamma_1 = \nu, \quad f = \frac{p_3\gamma_2 - p_2\gamma_3}{2\gamma_1}.$$

It follows from the first equation, that $J_2 = \nu r_3$. Now let us write the linear approximation of this system with the replacement: $p_1 \to 1 + p_1$, $p_2 \to i + p_2$, $\gamma_1 \to \nu$, $\gamma_2 \to \nu i + \gamma_2$:

$$J_1\dot{p}_1 + J_2\dot{p}_3 = -J_2 i p_1 - J_2 p_2 + \frac{J_1 - 2J_3}{2}ip_3 + \frac{J_2\gamma_2}{\nu} + \frac{J_1\gamma_3 i}{2\nu},$$
$$J_1\dot{p}_2 = 2J_2 p_1 - \frac{J_1 - 2J_3}{2}p_3 + \frac{J_1\gamma_3}{2\nu},$$
$$J_2\dot{p}_1 + J_3\dot{p}_3 = \frac{J_2 i}{2}p_3 + \frac{J_2 i\gamma_3}{2\nu}, \tag{28}$$
$$\dot{\gamma}_2 = 0,$$
$$\dot{\gamma}_3 = \nu p_2 + i\nu p_1 + \gamma_2.$$

The existence of the first integral γ_2 is the reason that only 3-parametric family of the trajectories of (27) can enter the singular point. Besides, these trajectories touch on the plane $\gamma_2 = 0$. Now we consider system (28) in this plane:

$$J_1\dot{p}_1 + J_2\dot{p}_3 = -J_2 i p_1 - J_2 p_2 + \frac{J_1 - 2J_3}{2}ip_3 + \frac{J_1\gamma_3 i}{2\nu},$$
$$J_1\dot{p}_2 = 2J_2 p_1 - \frac{J_1 - 2J_3}{2}p_3 + \frac{J_1\gamma_3}{2\nu},$$
$$J_2\dot{p}_1 + J_3\dot{p}_3 = \frac{J_2 i}{2}p_3 + \frac{J_2 i\gamma_3}{2\nu}, \tag{29}$$
$$\dot{\gamma}_3 = \nu p_2 + i\nu p_1.$$

System (29) has also the integral $\nu(J_1p_1 + iJ_1p_2 + J_2p_3) + J_2\gamma_3$, consequently, only 2-parametric family of the trajectories of (27) can enter the singular point. Substituting

$$\gamma_3 = \frac{-\nu}{J_2}(J_1p_1 + iJ_1p_2 + J_2p_3)$$

into system (29), we obtain the system

$$J_1\dot{p}_1 + J_2\dot{p}_3 = -\frac{i}{2J_2}\left(J_1^2 + 2J_2^2\right)p_1 - \frac{1}{2J_2}\left(-J_1^2 + 2J_2^2\right)p_2 - J_3 i p_3,$$
$$J_1\dot{p}_2 = \frac{1}{2J_2}\left(J_1^2 + 4J_2^2\right)p_1 + \frac{J_1^2 i}{2J_2}p_2 + J_3 p_3,$$

$$J_2 \dot{p}_1 + J_3 \dot{p}_3 = -\frac{J_1}{2}(p_1 i - p_2),$$

which also has the integral $\left(J_1^2 + 2 J_2^2\right) p_1 + i J_1^2 p_2 + J_2(J_1 + 2 J_3) p_3$. Thus, only 1-parametric family of the trajectories of (27) can enter the singular point.

Finally, the reduction of system (27) has the following form:

$$\dot{p}_1 = -\frac{J_1^2 i}{2 J_2\left(J_1 + 2 J_3^2\right)} p_1 - \frac{-J_1^3 J_3 + J_1^2 J_2^2 + 4 J_2^2 J_3^2 + 4 J_1 J_2^2 J_3}{2 J_2\left(J_1 J_3 - J_2^2\right)(J_1 + 2 J_3)} p_2,$$

$$\dot{p}_2 = \frac{J_1^3 J_3 + 4 J_2^2 J_3 + 4 J_1 J_2^2}{2 J_1 J_2 (J_1 + 2 J_3)} p_1 + \frac{J_1^2 i}{2 J_2\left(J_1 + 2 J_3^2\right)} p_2.$$

$$(30)$$

System (30) assigns the rotation, because the Jacobian squared of this system is equal to

$$\frac{J_2^2 (J_1 + J_3)}{J_1\left(J_1 J_3 - 2 J_2^2\right)}$$

and is negative always accordingly to the triangle inequality for the eigen-values of the inertia operator A. In order to get the asymptotics of the fiber entering the singular point, it is necessary to consider the solution of system (30) with the imaginary time. In this case, the Jacobian becomes positive and the singular point becomes the hyperbolic. There exist only two trajectories which enter the singular point in general, but, in the considering case, there is one solution which has the both trajectories under conditions $t \to + i_\infty$ and $t \to -i_\infty$.

Now, it is enough to test that the well-known Grioli solution considering in the neighbourhood of the point z_∞ of the foliation $\mathcal{F}$, just assign the fiber, which enter the singular point.

The partial solution of the Grioli case is the following:

$$p_1 = \sin(t), \quad p_2 = \cos(t), \quad p_3 = 1, \tag{31}$$

$$\gamma_1 = J_3 \sin(t) - J_2 \cos^2(t), \quad \gamma_2 = J_3 \cos(t) + J_2 \sin(t) \cos(t),$$

$$\gamma_3 = -J_2 \sin(t). \tag{32}$$

The direct testing of this fact is simple, therefore, we omit it. Thus, the following theorem is true.

Theorem 3. *All entire solutions of the Euler – Poisson equations* (1) *are given by the partial solutions of the Euler* (17) – (20), *Lagrange* (23) – (25) *and Grioli* (31), (32) *cases.*

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