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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A NONLINEAR DIFFERENCE EQUATION WITH A CONTINUOUS ARGUMENT

АСИМПТОТИЧНА ПОВЕДІНКА РОЗВ'ЯЗКІВ НЕЛІНІЙНОГО РІЗНИЦЕВОГО РІВНЯННЯ З НЕПЕРЕРВНИМ АРГУМЕНТОМ

We consider the difference equation with continuous argument

$$x(t+2) - 2\lambda x(t+1) + \lambda^2 x(t) = f(t, x(t)),$$

where $\lambda > 0$, $t \in [0, \infty)$, and $f\colon [0, \infty) \times \mathbf{R} \to \mathbf{R}$. Conditions for the existence and uniqueness of continuous asymptotically periodic solutions of this equation are given. We also prove the following result:

Let $x(t)$ be a real continuous function such that

$$\lim_{t \to \infty} \big(x(t+2) - (1-\alpha)x(t+1) - \alpha x(t)\big) = 0$$

for some $\alpha \in \mathbf{R}$.

Then the boundedness of $x(t)$ always implies

$$\lim_{t \to \infty} \big(x(t+1) - x(t)\big) = 0$$

if and only of $\alpha \in \mathbf{R} \setminus \{1\}$.

Розглянуто різницеве рівняння з неперервним аргументом

$$x(t+2) - 2\lambda x(t+1) + \lambda^2 x(t) = f(t, x(t)),$$

де $\lambda > 0$, $t \in [0, \infty)$ та $f\colon [0, \infty) \times \mathbf{R} \to \mathbf{R}$. Наведено умови існування та єдиності неперервних асимптотично періодичних розв'язків даного рівняння. Доведено також наступне твердження:

Нехай $x(t)$ — дійсна неперервна функція така, що

$$\lim_{t \to \infty} \big(x(t+2) - (1-\alpha)x(t+1) - \alpha x(t)\big) = 0$$

для деякого $\alpha \in \mathbf{R}$.

У цьому випадку з обмеженості $x(t)$ завжди випливає, що

$$\lim_{t \to \infty} \big(x(t+1) - x(t)\big) = 0$$

тоді і тільки тоді, коли $\alpha \in \mathbf{R} \setminus \{1\}$.

1. Introduction. In [1], G. P. Pelyukh investigated the problem of existence and uniqueness of continuous asymptotically periodic solutions of the following nonlinear difference equation with a continuous argument:

$$x(t+1) = x(t) + f(t, x(t)), \tag{1}$$

where $f\colon [0, \infty) \times \mathbf{R} \to \mathbf{R}$ is a continuous function such that

(a) $f(t, 0) \equiv 0$, $|f(t, x) - f(t, y)| \le \phi(t)|x - y|$, $x, y \in \mathbf{R}$, where $\phi(t)$ is nonnegative on $[0, \infty)$,

(b) $\Phi(t) = \sum_{i=0}^{\infty} \phi(t+i)$ is a uniformly convergent series on $[0, \infty)$ such that $\Phi(t) \le \theta < 1$.

It is natural to investigate the same problem for the difference equation

$$\Delta^2 x(t) = f(t, x(t)), \tag{2}$$

where $\Delta x(t) = x(t+1) - x(t)$.

In Section 2, we consider the problem of the existence and uniqueness of continuous solution of Eq. (2) for $t \geq 0$, which satisfies the condition

$$\lim_{t\to\infty} \left(x(t) - \omega(t)\right) = 0, \tag{3}$$

where $\omega(t)$ is a continuous function periodic with period one. We assume that the function f satisfies (a) and the following condition:

$(\mathrm{b})_1$ $\Phi_1(t) = \sum_{i=0}^{\infty} (i+1)\phi(t+i)$ is a uniformly convergent series on $[0, \infty)$ such that $\Phi_1(t) \leq \theta < 1$.

Note that, from condition (a), we have

$$\left|f(t, x(t))\right| = \left|f(t, x(t)) - f(t, 0)\right| \leq \phi(t)\left|x(t)\right|.$$

Since it follows from (b) that $\phi(t) \to 0$ as $t \to \infty$, we obtain that $\left|f(t, x(t))\right| = o(x(t))$ as $t \to \infty$.

It is easy to see that if $x(t)$ is a continuous solution for the problem (2), (3), then

$$x(t+1) - x(t) = -\sum_{i=0}^{\infty} f\left(t+i, x(t+i)\right)$$

and, consequently,

$$x(t) = \omega(t) + \sum_{i=0}^{\infty} (i+1) f\left(t+i, x(t+i)\right). \tag{4}$$

The convergence of the last sum follows from the boundedness of $x(t)$ and from conditions (a) and $(\mathrm{b})_1$. Conversely, if $x(t)$ is a bounded and continuous solution of Eq. (4) on $[0, \infty)$, then it is a solution of the problem (2), (3).

In Section 3, we give conditions under which every solution of a difference equation satisfies the condition

$$\lim_{t\to\infty} \left(x(t+1) - x(t)\right) = 0.$$

2. Existence and uniqueness results. In this section, we prove the following result on the existence and uniqueness of solution of Eq. (4). In the proof of the result, we follow the lines of the proof of Theorem 1 in [1].

Theorem 1. *Assume that conditions* (a) *and* $(\mathrm{b})_1$ *are satisfied. Then Eq.* (4) *has one and only one continuous and bounded solution on* $[0, \infty)$.

Proof. Let $x_0(t) = \omega(t)$ and

$$x_n(t) = \omega(t) + \sum_{i=0}^{\infty} (i+1) f\left(t+i, x_{n-1}(t+i)\right), \qquad n = 1, 2, \ldots. \tag{5}$$

For every $n \geq 1$ and $t \in [0, \infty)$, we show that

$$\left|x_n(t) - x_{n-1}(t)\right| \leq M\theta^n, \tag{6}$$

where $M = \max_{t\in[0,1]}\left|\omega(t)\right|$. We have

$$\left|x_1(t) - x_0(t)\right| \leq \sum_{i=0}^{\infty} (i+1)\left|f\left(t+i, x_0(t+i)\right)\right| =$$

$$= \sum_{i=0}^{\infty} (i+1)\left| f\big(t+i, x_0(t)\big) - f(t+i, 0)\right| =$$

$$= \sum_{i=0}^{\infty} (i+1)\phi(t+i)\left| x_0(t)\right| = \Phi_1(t)\left| x_0(t)\right| \leq M\theta.$$

Suppose that we have already proved (6) for some $m \in \mathbf{N}$. Then

$$\left| x_{m+1}(t) - x_m(t)\right| \leq \sum_{i=0}^{\infty} (i+1)\left| f\big(t+i, x_m(t+i)\big) - f\big(t+i, x_{m-1}(t+i)\big)\right| =$$

$$= \sum_{i=0}^{\infty} (i+1)\phi(t+i)\left| x_m(t+i) - x_{m-1}(t+i)\right| \leq$$

$$\leq M\theta^m \Phi_1(t) \leq M\theta^{m+1}.$$

Hence, (6) follows by induction. From (6) we have

$$\left| x_n(t)\right| \leq M(1+\theta+\ldots+\theta^n) < \frac{M}{1-\theta}.$$

Thus, the sequence $x_n(t)$ uniformly converges to a bounded continuous function $x(t)$. Letting $n \to \infty$ in (5), we obtain that $x(t)$ is a solution of Eq. (4).

We now prove that this solution is unique. Let $y(t)$ be another bounded continuous solution of (4). Then the function $x(t)-y(t)$ is bounded and continuous on $[0, \infty)$. By conditions (a) and $(\mathrm{b})_1$, we have

$$|x(t) - y(t)| = \sum_{i=0}^{\infty} (i+1)\,| f(t+i, x(t+i)) - f(t+i, y(t+i))| \leq$$

$$\leq \sum_{i=0}^{\infty} (i+1)\phi(t+i)|\, x(t+i) - y(t+i)| \leq$$

$$\leq \Phi_1(t)\| x(t)-y(t)\|_{C[0,\infty)} \leq \theta\| x(t)-y(t)\|_{C[0,\infty)}$$

and, consequently,

$$\| x(t)-y(t)\|_{C[0,\infty)} \leq \theta\| x(t)-y(t)\|_{C[0,\infty)},$$

whence the required result follows.

Similarly we can prove the following theorem:

Theorem 2. *Consider the difference equation*

$$\Delta^n x(t) = f(t, x(t)). \tag{7}$$

Assume that condition (a) *and the following condition are satisfied*:

$(\mathrm{b})_2$ $\Phi_n(t) = \sum_{i=0}^{\infty} (i+1)^{n-1}\phi(t+i)$ *is a uniformly convergent series on* $[0, \infty)$ *such that* $\Phi_1(t) \leq \theta < 1$.

Then Eq. (7) *has one and only one continuous solution on* $[0, \infty)$ *which satisfies* (3).

Using the change $x(t) = \lambda^t y(t)$, we obtain the corollary.

***Corollary* 1.** *Consider the difference equation*

$$\Delta_\lambda^n x(t) = f(t, x(t)), \tag{8}$$

where $\lambda > 0$, $\Delta_\lambda x(t) = x(t+1) - \lambda x(t)$. *Assume that the function*

$$f_1(t, y) = \lambda^{-(t+n)} f\left(t, \lambda^t y\right)$$

satisfies conditions (a) *and* $(b)_2$.

Then Eq. (8) *has one and only one continuous solution* $x(t)$ *on* $[0, \infty)$ *such that*

$$\lim_{t\to\infty}\left(\lambda^{-t}x(t) - \omega(t)\right) = 0.$$

3. Variation of solutions. By Theorem 1, we have found a solution of problem (2), (3). It is clear that the solution satisfies

$$\lim_{t\to\infty}(x(t+1) - x(t)) = 0. \tag{9}$$

On the other hand, we see that every bounded continuous solution of Eq. (2) satisfies

$$\lim_{t\to\infty}\Delta^2 x(t) = 0 \tag{10}$$

if (a) and $(b)_1$ are satisfied.

The following natural question arises: Is property (9) characteristic only for the solution of problem (2), (3)? The answer is negative. Moreover, difference equation which satisfy (10) have the property that every bounded solution of these ones satisfies (9).

We consider a somewhat generalized version of Eq. (2), namely, we consider second order nonhomogeneous difference equations with constant coefficients such that the sum of the coefficients of the corresponding homogeneous difference equations are equal to zero. There are many papers concerning discrete difference equations of this type. These equations appear in a large class of Mathematical Biology models, for example: Discrete delay logistic difference equation [2, 3], Generalized Beddington – Holt stock recruitment model [4 – 7], Mosquito population equations [8], Perennial grass model [5, 9], Flour beetle population model [10, 11] (see also [12]). The following result is the main in this section:

Theorem 3. *Let* $x(t)$ *be a real continuous function such that*

$$\lim_{t\to\infty}(x(t+2) - (1-\alpha)x(t+1) - \alpha x(t)) = 0 \tag{11}$$

for some $\alpha \in \mathbf{R}$.

Then the boundedness of $x(t)$ *always implies*

$$\lim_{t\to\infty}(x(t+1) - x(t)) = 0$$

if and only if $\alpha \in \mathbf{R}\setminus\{1\}$.

For the proof of Theorem 3, we need an auxiliary result which is contained in the lemma which follows.

Lemma 1. *Let* $x(t)$ *be a bounded continuous complex-valued function such that*

$$\lim_{t\to\infty}(x(t+1) + \alpha x(t)) = 0.$$

Then the following statements are true:

(a) *if* $\alpha \in \mathbf{C}\setminus\{z \mid |z| = 1\}$, *then* $x(t) \to 0$ *as* $t \to +\infty$;

(b) *for each* α, $|\alpha| = 1$, $x(t)$ *need not be convergent as* $t \to +\infty$.

Proof. a) Suppose that $|\alpha| < 1$. Let $b(t) = x(t+1) + \alpha x(t)$, $n \in \mathbf{N}$. By standard procedure, we obtain

$$x(t) = (-\alpha)^{[t]} x(\{t\}) + \sum_{k=0}^{[t]-1} b(\{t\}+k)(-\alpha)^{[t]-(k+1)}, \tag{12}$$

where $\{t\}$ is so-called rational part of number t, that is $\{t\} = t - [t]$, where $[t]$ is the integer part of t.

Since $\lim_{t\to+\infty} b(t) = 0$ and $|\alpha| < 1$, it is easy to prove that the sum

$$\sum_{k=0}^{n-1} b(\{t\}+k)(-\alpha)^{[t]-(k+1)}$$

tends to zero as $t \to +\infty$. It follows from (12) that $\lim_{t\to+\infty} x(t) = 0$.

Let $|\alpha| > 1$. Since $x(t)$ is bounded, there exists $M > 0$ such that $|x(t)| \le M$ for each $t \in [0, \infty)$. Hence,

$$\left| x(\{t\}) + \sum_{k=0}^{[t]-1} \frac{b(\{t\}+k)}{(-\alpha)^{k+1}} \right| \le \frac{M}{|\alpha|^{[t]}}, \quad \text{for all} \quad t \in [0, \infty). \tag{13}$$

Replacing t in (13) by $t_n = n + t_0$, $n \in \mathbf{N}$, where t_0 is a fixed number in $[0, 1)$, and then letting $n \to \infty$, we obtain

$$x(t_0) = -\sum_{k=0}^{+\infty} \frac{b(t_0+k)}{(-\alpha)^{k+1}}. \tag{14}$$

Taking $\{t\} = t_0$ and then substituting (14) in (12), we obtain

$$x(t) = -\frac{1}{\alpha} \sum_{k=[t]}^{+\infty} \frac{b(\{t\}+k)}{(-\alpha)^{k-[t]}}.$$

Since $\lim_{t\to+\infty} b(t) = 0$ and $|\alpha| > 1$, we obtain

$$\lim_{t\to+\infty} \sum_{k=[t]}^{+\infty} \frac{b(\{t\}+k)}{(-\alpha)^{k-[t]}},$$

whence the required result follows.

b) If $|\alpha| = 1$, i.e., $\alpha = e^{i\theta}$, $\theta \in [-\pi, \pi]$, then the function $x(t) = e^{it(\theta - \pi/2)}$ is bounded and satisfies the condition $x(t+1) + \alpha x(t) = 0$, $t \in [0, \infty)$, and the limit $\lim_{t\to\infty} x(t)$ does not exist.

Remark. The last example shows that $x(t)$ in Lemma 1 need not be convergent.

***Proof of Theorem* 3.** The case $\alpha = 0$ is trivial. Suppose $\alpha \notin \{-1, 0, 1\}$. Put $b(t) = x(t+1) - x(t)$. Then, from (11) it follows that $b(t+1) + \alpha b(t) \to 0$ as $t \to \infty$. The boundedness of $x(t)$ implies the boundedness of $b(t)$. Since $\alpha \notin \{-1, 1\}$, by Lemma 1 we obtain $b(t) \to 0$ as $t \to \infty$, as desired.

Let $\alpha = -1$. Hence,

$$\lim_{t\to\infty} \left(x(t+2) - 2x(t+1) + x(t) \right) = 0.$$

Now assume that

$$\limsup_{t\to\infty} \left(x(t+1) - x(t) \right) = a > 0.$$

Hence, for every $\varepsilon \in (0, a)$, there exists $t_0 > 0$ such that

$$x(t_0+1) - x(t_0) > a - \varepsilon \tag{15}$$

and

$$x(t+2) - x(t+1) - \big(x(t+1) - x(t)\big) > -\varepsilon \quad \text{for} \quad t \geq t_0. \tag{16}$$

For a fixed $k_0 \in \mathbf{N}$, we choose $\varepsilon = a/2(k_0+1)$ and, for such ε, find $t_0 = t_0(\varepsilon)$ such that (15) and (16) hold. Then we have

$$x(t_0+k+1) - x(t_0+k) > a - (k+1)\varepsilon = a - \frac{k+1}{2(k_0+1)}a \geq \frac{a}{2}$$

for every $k \in \{1, 2, \ldots, k_0\}$. Hence,

$$x(t_0+k_0+1) > x(t_0+1) + k_0\frac{a}{2},$$

which contradicts the boundedness of $x(t)$.

The assumption

$$\liminf_{t\to\infty} \big(x(t+1) - x(t)\big) = a < 0$$

leads to a contradiction by a similar argument.

Let $\alpha = 1$. Then the function $x(t) = \sin \pi t$ is bounded, $x(t+2) - x(t) = 0$, and the limit

$$\lim_{t\to\infty} \big(x(t+1) - x(t)\big)$$

does not exist.

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