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STABILITY OF SOLUTIONS OF QUASILINEAR INDEX-2 TRACTABLE DIFFERENTIAL ALGEBRAIC EQUATION BY LIAPUNOV'S SECOND METHOD

СТІЙКІСТЬ РОЗВ'ЯЗКІВ КВАЗІЛІНІЙНОГО РОЗВ'ЯЗУВАНОГО ЗА ІНДЕКСОМ 2 ДИФЕРЕНЦІАЛЬНОГО АЛГЕБРАЇЧНОГО РІВНЯННЯ ЗА ДРУГИМ МЕТОДОМ ЛЯПУНОВА

Liapunov's second method is an important tool in the qualitative theory of ordinary differential equations. In this paper, we consider behavior of solutions of quasilinear index-2 tractable differential algebraic equations (DAEs). By Liapunov's second method, some sufficient conditions for the stability of zero solution of such equations are proved.

Другий метод Ляпунова є важливим інструментом в якісній теорії звичайних диференціальних рівнянь. У даній статті розглянуто поведінку розв'язків квазілінійних розв'язуваних за індексом 2 диференціальних алгебраїчних рівнянь. За допомогою другого методу Ляпунова доведено достатні умови стійкості нульового розв'язку таких рівнянь.

1. Preliminaries. ***The index-2 tractable DAE.*** Consider the following index-2 tractable DAEs:

$$A(t)x' + B(t)x = f(t, x), \tag{1.1}$$

$$A(t)x' + B(t)x = 0, \tag{1.2}$$

where

$$A \in C^1\big(I, L(\mathbb{R}^m)\big), \tag{1.3}$$

$B \in C\big(I, L(\mathbb{R}^m)\big)$, $I = \{t\colon a \le x < \infty\}$, $x \in \mathbb{R}^m$, $f(t, 0) \equiv 0$, $f(t, x) \in C^{01}_{tx}\big(I \times \mathbb{R}^m\big)$, $\det A(t) = 0$ for all $t \in I$, $N(t) = \ker A(t)$ is smooth on I [1], and

$$\| A(t) \| \le M \quad \text{for all } t \in I \quad (M > 0 \text{ is a constant}). \tag{1.4}$$

In the case of unboundedness of $A(t)$, multiplying (1.1), (1.2) by suitable scalar function $k(t)$, $k \in C^1(I)$, $k(t) \ne 0$ for all $t \in I$, we obtain the equivalent equations in which the coefficient matrix of x' is bounded.

Choose $P(t) = A^+(t)A(t)$, $Q(t) = I - P(t)$, then $P(t)$ is the orthoprojector onto $\operatorname{im}\big(A^T(t)\big)$ along $N(t)$ [1].

Denote

$$A_1(t) = A(t) + \big(B(t) - A(t)P'(t)\big)Q(t), \qquad N_1(t) = \ker A_1(t),$$

$$S(t) = \big\{z \in \mathbb{R}^m\colon B(t)z \in \operatorname{im} A(t)\big\}, \qquad S_1(t) = \big\{z \in \mathbb{R}^m\colon B(t)P(t)z \in \operatorname{im} A_1(t)\big\},$$

$$P_1(t) = I - Q_1(t), \quad t \in I,$$

where $Q_1(t)$ is the projector onto $N_1(t)$ along $S_1(t)$.

The DAE (1.2) is said to be *index-2 tractable* (shortly *index-2*) on I if the following conditions are valid [1]:

$$\dim N_1(t) = \text{const} > 0, \qquad N_1(t) \oplus S_1(t) = \mathbb{R}^m \quad \text{for all } t \in I. \tag{1.5}$$

Suppose that

$$P,\ Q_1 \in C^1\big(I, L(\mathbb{R}^m)\big), \tag{1.6}$$

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and the range of $(A(t)P(t)P_1(t))$ is fixed, i.e.,

$$R(A(t)P(t)P_1(t)) \equiv R(A(a)P(a)P_1(a)) \quad \text{for all} \quad t \in I. \tag{1.7}$$

Denote

$$R(A(a)P(a)P_1(a)) = \mathbb{E}^k \subset \mathbb{R}^m, \quad 1 \le k < m.$$

Now putting

$$A_2(t) = A_1(t) + B_1(t)Q_1(t), \tag{1.8}$$

where $B_1(t) \doteq \left(B - A_1(PP_1)'\right)P(t)$, then the relation (1.5) is equivalent to $\det A_2(t) \neq 0$ for all $t \in I$ [2, 3]. Furthermore, we have

$$Q_1 = Q_1 A_2^{-1} BP, \quad Q_1 Q = 0.$$

Denote

$$\pi_{\text{can}} = \left(I - \left(QQ_1'\right) - QP_1A_2^{-1}B\right)PP_1. \tag{1.9}$$

In that case, $\pi_{\text{can}}^2 = \pi_{\text{can}}$ and $\operatorname{im}\pi_{\text{can}}(t)$ is the subspace of $\mathbb{R}^m$.

Let $\tilde{x}(t)$ be a solution of (1.2), then we have [3, 4]

$$\tilde{x}(t) \in \operatorname{im}\pi_{\text{can}}(t), \quad t \in I. \tag{1.10}$$

Further, let the equation (1.1) have *index*-2 on $I \times \mathbb{R}^m$. This means that the following conditions are satisfied [4]:

$$\dim \overline{N_1}(t, x) = \text{const} > 0,$$

$$\overline{N_1}(t, x) \oplus \overline{S_1}(t, x) = \mathbb{R}^m \quad \text{for all} \quad (t, x) \in I \times \mathbb{R}^m,$$

where

$$\overline{N_1}(t, x) = \ker \overline{A_1}(t, x), \quad \overline{S_1}(t, x) = \left\{z \in \mathbb{R}^m : \left(B - f_x'(t, x)\right)P(t)z \in \operatorname{im}\overline{A_1}(t, x)\right\},$$

$$\overline{A_1}(t, x) = A + \left(B - f_x'(t, x) - AP'\right)Q(t).$$

Suppose that

$$Q_1 A_2^{-1}(t) f(t, x(t)) \equiv 0 \tag{1.11}$$

and there exists a matrix $D \in C\left(I, L\left(\mathbb{R}^m\right)\right)$ such that

$$Q(t)A_2^{-1}(t)f(t, x(t)) \equiv Q(t)D(t)P(t)P_1(t)x(t) \quad \text{for all solutions} \quad x(t) \quad \text{of (1.1).} \tag{1.12}$$

We will use the following decomposition:

$$I = PP_1 + QP_1 + Q_1. \tag{1.13}$$

Since $x(t)$ is a solution of (1.1), we have

$$A(t)x'(t) + B(t)x(t) = f(t, x(t)), \quad t \in I.$$

Multiplying all members of the above identity by $PP_1A_2^{-1}(t)$, $QP_1A_2^{-1}(t)$, $Q_1A_2^{-1}(t)$, respectively, and due to (1.11), (1.12), (1.13), we obtain

$$x(t) = \left(I - (QQ_1)' - QP_1A_2^{-1}B + QD\right)PP_1(t)x(t), \tag{1.14}$$

where $P(t)P_1(t)x(t) = u(t)$ is a solution of the following equation [4]:

$$u' = (PP_1)'(t)u - PP_1A_2^{-1}B(t)u + f\left(t, \left(I - (QQ_1)' - QP_1A_2^{-1}B + QD\right)(t)u\right).$$

Putting

$$\pi = \Big(I - (QQ_1)' - QP_1A_2^{-1}B + QD\Big)PP_1, \tag{1.15}$$

we have $\pi^2 = \pi$, which means that π is a projector. Moreover, we can see that π and any solution $x(t)$ of (1.1) is invariant for all matrices D satisfying the condition (1.12) and

$$x(t) \in \operatorname{im}\pi(t), \quad t \in I. \tag{1.16}$$

Lemma 1.1. *The following conclusions are valid:*

(i) $PP_1 = A_2^{-1}APP_1$; (1.17)

(ii) if $x \in \operatorname{im}\pi(t)$ *or* $x \in \operatorname{im}\pi_{\text{can}}(t)$, *then,* $PP_1(t)x = A_2^{-1}(t)A(t)x$; (1.18)

(iii) $A\pi = APP_1$, $A\pi_{\text{can}} = APP_1$. (1.19)

Now, by (1.3) we can rewrite (1.1), (1.2) as the following equivalent equations:

$$(A(t)x)' = \big(A'(t) - B(t)\big)x + f(t, x), \tag{1.20}$$

$$(A(t)x)' = \big(A'(t) - B(t)\big)x. \tag{1.21}$$

Due to $x \in \operatorname{im}\pi(t)$, $x = \pi(t)x$, thus, from (1.3), (1.15), (1.16) and (1.18), we can see that, on $\operatorname{im}\pi(t)$, the equation (1.20) is equivalent to

$$(A(t)x)' = \big(A'(t) - B(t)\big)T(t)A(t)x + f(t, T(t)A(t)x),$$

where

$$T(t) = \Big(I - (QQ_1)' - QP_1A_2^{-1}B + QD\Big)A_2^{-1}(t). \tag{1.22}$$

Similarly, on $\operatorname{im}\pi_{\text{can}}(t)$ (1.21) is equivalent to

$$(A(t)x)' = \big(A'(t) - B(t)\big)T_1(t)A(t)x,$$

where

$$T_1(t) = \big(T - QDA_2^{-1}\big)(t). \tag{1.23}$$

Hence, together with (1.20), (1.21) we consider two following ordinary differential equations:

$$y' = \big(A'(t) - B(t)\big)T(t)y + f(t, T(t)y), \tag{1.24}$$

$$y' = \big(A'(t) - B(t)\big)T_1(t)y, \qquad y \in \mathbb{R}^m, \quad t \in I, \tag{1.25}$$

and for solutions of (1.1), (1.2) we ask, respectively,

$$PP_1(t_0)\big(x(t_0) - x^0\big) = 0, \tag{1.26}$$

$$PP_1(t_0)\big(\bar{x}(t_0) - x^0\big) = 0, \qquad t_0 \in I, \quad x^0 \in \mathbb{R}^m. \tag{1.27}$$

Obviously, because of (1.16), (1.26) and (1.10), (1.27) we have

$$x(t_0) = \pi(t_0)x^0, \tag{1.28}$$

$$\bar{x}(t_0) = \pi_{\text{can}}(t_0)x^0. \tag{1.29}$$

Further, we always assume that (1.1), (1.2) have index-2, the conditions (1.3), (1.4), (1.6), (1.7), (1.11), (1.12) are satisfied, and, for each $\big(t_0, z^0\big) \in I \times \mathbb{R}^m$, the equations (1.1), (1.2), (1.24), (1.25) have the unique solution which is defined on $[t_0, \infty)$ and satisfies the given initial condition.

We have the following lemma:

Lemma 1.2. *If* $x(t) = x\big(t, t_0, x^0\big)$ *is a solution of* (1.1) *satisfying* (1.26), *then*

$y(t)=A(t)x(t,t_0,x^0)$ *is a solution of* (1.24) *satisfying* $y(t_0)=A(t_0)P(t_0)P_1(t_0)x^0$.

Conversely, if $y(t)=y(t,t_0,y^*)$, *where* $y^*\in\mathbb{E}^k$, *is a solution of* (1.24), *then there exists* $x^*\in\mathbb{R}^m$ *such that* $y^*=A(t_0)P(t_0)P_1(t_0)x^*$ *and*

$$x(t,t_0,x^*) = \left(I-(QQ_1)' - QP_1A_2^{-1}B+QD\right)A_2^{-1}(t)y(t)$$

is a solution of (1.1) *satisfying* $P(t_0)P_1(t_0)\left(x(t_0)-x^*\right)=0$.

We also have the similar conclusions for solutions of the equations (1.2) and (1.25).

Proof. First, if $x(t)=x(t,t_0,x^0)$ is a solution of (1.1) satisfying (1.26), then $x(t)\in\operatorname{im}\pi(t)$ for all $t\ge t_0$. Hence, in case $y=A(t)x(t)$, the right-hand side of (1.24) coincides with the right-hand side of (1.20). This implies that $y(t)=A(t)x(t)$ is a solution of (1.24) and, because of (1.19), we have

$$y(t_0) = A(t_0)x(t_0) = APP_1(t_0)x(t_0) = APP_1(t_0)x^0.$$

Further, let $y(t)=y(t,t_0,y^*)$, $y^*\in\mathbb{E}^k$, be a solution of (1.24). Since (1.7), there exists $x^*\in\mathbb{R}^m$ such that $y^*=A(t_0)P(t_0)P_1(t_0)x^*$. Assume that $x(t)=x(t,t_0,x^*)$ is a solution of (1.1) satisfying $P(t_0)P_1(t_0)x(t_0)=P(t_0)P_1(t_0)x^*$. According to the beginning of this proof, we have that $\bar{y}(t)=A(t)x(t,t_0,x^*)$ is a solution of (1.24) satisfying

$$\bar{y}(t_0)=A(t_0)P(t_0)P_1(t_0)x^*=y^*=y(t_0).$$

Due to the unique existence of solution of (1.24), we have $\bar{y}(t_0)\equiv y(t)$, whence $y(t)\equiv A(t)x(t,t_0,x^*)$ for all $t\ge t_0$. Multiplying all members of this identity by $A_2^{-1}(t)$ and due to (1.18), we have $P(t)P_1(t)x(t,t_0,x^*)=A_2^{-1}(t)y(t)$. From this and (1.11), (1.12), (1.14), we obtain

$$x(t,t_0,x^*) = \left(I-(QQ_1)' - QP_1A_2^{-1}B+QD\right)A_2^{-1}(t)y(t) \quad \text{for all} \quad t\ge t_0.$$

The lemma is proved.

Lemma 1.3. *The condition* (1.7) *implies*

$$R(A(t)P(t)P_1(t)) = R\left(A(t)|_{\operatorname{im}\pi(t)}\right) = R\left(A(a)|_{\operatorname{im}\pi(a)}\right) = \mathbb{E}^k \quad \textit{for all} \quad t\in I.$$

***Remark* 1.1.** (a) Let the following conditions hold:

(i) the equation $v=Q_1A_2^{-1}(t)f(t,w+u+Pv)$ has the unique solution $v=0$ for all $t\in I$;

(ii) there exists a matrix $D\in C\left(I,L(\mathbb{R}^m)\right)$ such that the equation

$$w + \left((QQ_1)'+QP_1A_2^{-1}B\right)(t)u = QA_2^{-1}(t)f(t,w+u)$$

has the unique solution

$$w = -\left((QQ_1)'+QP_1A_2^{-1}B-QD\right)(t)u$$

for all $t\in I$, where $u=PP_1x$, $v=Q_1x$, $w=Qx$.

Then, the conditions (1.11), (1.12) are satisfied.

(b) Due to (1.7) and the unique existence of solution of (1.11), we have

$$\left(\left(A'(t)-B(t)\right)x+f(t,x)\right)\in\mathbb{E}^k \quad \text{for all} \quad x\in\operatorname{im}\pi(t), \quad t\in I.$$

Definitions of stability.

Definition 1.1 [1]. *The solution* $x(t)\equiv 0$ *of* (1.1) *is stable (in the sense of*

Liapunov) if for any $\varepsilon > 0$ *and any* $t_0 \in I$, *there exists a* $\delta = \delta(t_0, \varepsilon) > 0$ *such that if* $x^0 \in \mathbb{R}^m$, $\| P(t_0) P_1(t_0) x^0 \| < \delta$, $\| x(t, t_0, x^0) \| < \varepsilon$ *for all* $t \geq t_0$.

Definition 1.2 [1]. *The zero solution of* (1.1) *is asymptotically stable, if* $x(t) \equiv 0$ *is stable and if there exists a* $\delta_0(t_0) > 0$ *such that, if* $\| P(t_0) P_1(t_0) x^0 \| < \delta_0(t_0)$, $\| x(t, t_0, x^0) \| \to 0$ *as* $t \to \infty$.

We assume that $\varphi(t)$ is a continuous positive scalar function on I and $\varphi(t) \to \infty$ as $t \to \infty$.

Definition 1.3. *The zero solution of* (1.1) *is* φ*-asymptotically stable, if* $x(t) \equiv 0$ *is stable and there exists a* $C(t_0, \alpha) > 0$ *such that if* $\| P(t_0) P_1(t_0) x^0 \| < \alpha$, $\| x(t, t_0, x^0) \| \leq C(t_0, \alpha) \dfrac{\varphi(t_0)}{\varphi(t)}$ *for all* $t \geq t_0$.

Definition 1.4 [5, 6]. *The solution* $x(t) \equiv 0$ *of* (1.1) *is exponential-asymptotically stable if there exists* $c > 0$ *and, given any* $\varepsilon > 0$ *the exists* $\delta(\varepsilon) > 0$ *such that if* $\| P(t_0) P_1(t_0) x^0 \| < \delta(\varepsilon)$, $\| x(t, t_0, x^0) \| \leq \varepsilon \exp(-c(t - t_0))$ *for all* $t \geq t_0$.

Definition 1.5 [5]. *Let* $\mathbb{E}$ *denote a subspace of* $\mathbb{R}^m$. *The zero solution of* (1.25) *is stable on* $\mathbb{E}$ *if* $(t_0, y^0) \in (I \times \mathbb{E})$, *a solution* $y(t, t_0, y^0)$ *of* (1.25) *satisfies* $y(t, t_0, y^0) \in \mathbb{E}$ *for all* $t \geq t_0$, *and given any* $\varepsilon > 0$ *there exists* $\delta(t_0, \varepsilon) > 0$ *such that if* $\| y^0 \| < \delta(t_0, \varepsilon)$, $\| y(t, t_0, y^0) \| < \varepsilon$ *for all* $t \geq t_0$.

The zero solution of (1.25) is *asymptotically stable*, φ*-asymptotically stable*, *exponential-asymptotically stable* on $\mathbb{E}$ defined similarly.

Denote $\Delta = \{ y \in \mathbb{E}^k : \| y \| < MH \}$, where $H > 0$, M is determined by (1.4), and $\mathbb{E}^k = R(A(a)|_{\operatorname{im} \pi(a)})$.

Let $V(t, y)$ be a Liapunov function defined on $I \times \Delta$ and $y(t)$ be a solution of (1.24) which stays on $I \times \Delta$. Denote by $V'(t, y(t))$ the upper right-hand derivative of $V(t, y(t))$ and

$$V'_{(1.24)}(t, y) = \overline{\lim_{h \to 0+}} \frac{1}{h} \left(V(t + h, y + hG(t, y)) - V(t, y) \right),$$

where

$$G(t, y) = \left(A'(t) - B(t) \right) T(t) y + f(t, T(t) y).$$

We have [6]

$$V'(t, y(t)) = V'_{(1.24)}(t, y). \tag{1.30}$$

In case $y = A(t) x$, $x \in \operatorname{im} \pi(t)$ $(\| x \| < H)$, the right-hand side of (1.24) coincides with the right-hand side of (1.20), therefore by (1.30), Lemma 1.2, Lemma 1.3 and (b) of Remark 1.1 we obtain

$$V'(t, A(t) x(t)) = V'_{(1.20)}(t, A(t) x) \quad \text{for all} \quad x \in \operatorname{im} \pi(t), \quad \| x \| < H, \quad t \in I,$$

where $x(t)$ is a solution of (1.1).

Similarly, we also have $V'(t, A(t) x(t)) = V'_{(1.21)}(t, A(t) x)$, where $x(t)$ is a solution of (1.2) and $x \in \operatorname{im} \pi_{\text{can}}(t)$, $t \in I$.

In case $V(t, y)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1.20)}(t, A(t) x) = \frac{\partial V}{\partial t} + \left(\operatorname{grad} V, \left(\left(A' - B \right)(t) x + f(t, x) \right) \right)$$

for all $x \in \operatorname{im} \pi(t)$, $\| x \| < H$, $t \in I$.

2. Sufficient conditions.

Theorem 2.1. *Suppose that there exists a Liapunov function* $V(t, y)$ *defined on* $I \times \Delta$, *which satisfies the following conditions for all* $x \in \operatorname{im} \pi(t)$, $\|x\| < H$, $t \in I$,

(i) $V(t, 0) \equiv 0$;

(ii) $a(\|x\|) \le V(t, A(t)x)$, *where* $a(r) \in CIP$ [6];

(iii) $V'_{(1.20)}(t, A(t)x) \le 0$.

Then the solution $x(t) \equiv 0$ *of* (1.1) *is stable.*

Proof. Since $V(t, y)$ is continuous, for any $\varepsilon > 0$ $(\varepsilon < H)$, $t_0 \in I$, there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that $\|y^0\| < \delta_1$ implies

$$V(t_0, y^0) < a(\varepsilon). \tag{2.1}$$

Let $x(t) = x(t, t_0, x^0)$ be a solution of (1.1) which satisfies $\|P(t_0) P_1(t_0) x^0\| < \frac{\delta_1}{M}$, M be the number determined by (1.4). Then, according to Lemma 1.2, $y(t) = A(t)x(t)$ is a solution of (1.24) satisfying $y(t_0) = A(t_0)x(t_0) = A(t_0)P(t_0)P_1(t_0)x^0$ and

$$\|y(t_0)\| \le \|A(t_0)\| \|P(t_0) P_1(t_0) x^0\| \le M \frac{\delta_1}{M} = \delta_1. \tag{2.2}$$

Thereby, from (ii), (iii), (2.1) and (2.2) we have

$$a(\|x(t, t_0, x^0)\|) \le V(t, A(t)x(t)) \le$$

$$\le V(t_0, A(t_0)P(t_0)P_1(t_0)x^0) = V(t_0, y(t_0)) < a(\varepsilon)$$

for all $t \ge t_0$. This implies $\|x(t, t_0, x^0)\| < \varepsilon$ for all $t \ge t_0$ if $\|P(t_0)P_1(t_0)x^0\| < \delta = \frac{\delta_1}{M}$. That is, the zero solution of (1.1) is stable.

The theorem is proved.

By the same argument used in the proof of Theorem 8.3 in [6, p. 32], we can prove the following theorem.

Theorem 2.2. *Suppose that there exists a Liapunov function* $V(t, y)$ *defined on* $I \times \Delta$, *which satisfies the following conditions for all* $x \in \operatorname{im} \pi(t)$, $\|x\| < H$, $t \in I$:

(i) $a(\|x\|) \le V(t, A(t)x) \le b(\|x\|)$, *where* $a(r) \in CIP$, $b(r) \in CIP$;

(ii) $V'_{(1.20)}(t, A(t)x) \le -c(\|x\|)$, *where* $c(r)$ *is continuous on* $[0, H]$ *and positive definite.*

Then the solution $x(t) \equiv 0$ *of* (1.1) *is asymptotically stable.*

Theorem 2.3. *Suppose that there exists a Liapunov function* $V(t, y)$ *defined on* $I \times \Delta$, *which satisfies the following conditions for all* $x \in \operatorname{im} \pi(t)$, $\|x\| < H$, $t \in I$:

(i) $V(t, 0) \equiv 0$;

(ii) $\|x\| \le V(t, A(t)x)$;

(iii) $V'_{(1.20)}(t, A(t)x) \le -\lambda(t) V(t, A(t)x)$, *where* $\lambda(t)$ *is a continuous positive scalar function on* I *and* $\int_a^\infty \lambda(t)\,dt = \infty$.

Then the solution $x(t) \equiv 0$ *of* (1.1) *is* φ-*asymptotically stable.*

Proof. Due to (iii) and Theorem 4.1 in [6], we have

$$V(t, A(t)x(t, t_0, x^0)) \le V(t, A(t_0)x(t_0)) \exp\left(-\int_{t_0}^{t} \lambda(s)\,ds\right) =$$

$$= V(t_0, A(t_0)P(t_0)P_1(t_0)x^0) \exp\left(-\int_{t_0}^{t} \lambda(s)\,ds\right). \tag{2.3}$$

Denote $\varphi(t) = \exp\left(\int_a^t \lambda(s)\,ds\right)$. Because of the feature of function λ, we can see that φ is continuous positive function on I and $\varphi(t) \to \infty$ as $t \to \infty$. Then the estimation (2.3) leads to

$$V\left(t, A(t)x\left(t, t_0, x^0\right)\right) \le V\left(t_0, A(t_0)P(t_0)P_1(t_0)x^0\right)\frac{\varphi(t_0)}{\varphi(t)} \quad \text{for all} \quad t \ge t_0. \tag{2.4}$$

We define

$$C(t_0, \alpha) = \max_{\|y\| \le \alpha M} V(t_0, y), \tag{2.5}$$

where $0 < \alpha < H$, M is determined by (1.4).

Let $\left\| P(t_0)P_1(t_0)x^0 \right\| \le \alpha$, then

$$\left\| A(t_0)P(t_0)P_1(t_0)x^0 \right\| \le \alpha M. \tag{2.6}$$

Therefore, together with (2.4), (2.5) and (2.6), we have

$$V\left(t, A(t)x\left(t, t_0, x^0\right)\right) \le C(t_0, \alpha)\frac{\varphi(t_0)}{\varphi(t)}. \tag{2.7}$$

Finally, from (ii) and (2.7), we obtain $\left\| x\left(t, t_0, x^0\right) \right\| \le C(t_0, \alpha)\dfrac{\varphi(t_0)}{\varphi(t)}$ for all $t \ge t_0$, if $\left\| P(t_0)P_1(t_0)x^0 \right\| < \alpha$. Thus, the zero solution of (1.1) is φ-*asymptotically stable*.

The theorem is proved.

***Example* 2.1.** Consider the following equation:

$$A(t)x' + B(t)x = f(t, x), \tag{2.8}$$

where

$$A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} -t^{-1} & -1 & 2t^{-1} \\ 2t^3 & 1 & -t^{-3} \\ 0 & 1 & 0 \end{pmatrix}$$

and $f(t, x) = (f_1, f_2, f_3)^T$, $f_1 = -x_1(x_3 - 2x_1)^2$, $f_2 = -t^{-2}x_1\cos x_2$, $f_3 = -x_2 x_3^2$, $I = [1, \infty)$, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$.

We can see that the equation (2.8) has index-2 and $f(t, x)$ satisfies the conditions (1.11), (1.12) with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^{-5} & 0 & 0 \end{pmatrix}.$$

Moreover, $\operatorname{im}\pi(t) = \left\{\left(\alpha, 0, \left(2 + t^{-5}\right)\alpha\right)^T;\ \alpha \in \mathbb{R}\right\}$. The equation (2.8) is equivalent to

$$\begin{pmatrix} x_1' \\ \left(t^{-1}x_2\right)' \\ 0 \end{pmatrix} = \begin{pmatrix} t^{-1}x_1 + x_2 - 2t^{-1}x_3 - x_1(x_3 - 2x_1)^2 \\ -2t^3x_1 - \left(1 + t^{-2}\right)x_2 + t^2x_3 - t^{-2}x_1\cos x_2 \\ -x_2 - x_2x_3^2 \end{pmatrix}. \tag{2.9}$$

Since $R\left(A(t)|_{\operatorname{im}\pi(t)}\right) = \left\{(\alpha, 0, 0)^T;\ \alpha \in \mathbb{R}\right\} := \mathbb{E}^1$, we choose $V(t, y) = 10|y_1|$, where $y = (y_1, 0, 0)^T \in \mathbb{E}^1$. In that case, we can see that $V(t, y)$ is a Liapunov function defined on $I \times \mathbb{E}^1$, and together with $y = A(t)x$, $x \in \operatorname{im}\pi(t)$ we have:

$$V(t, A(t)x) = 10|x_1| \geq \left(1+\left(2+t^{-5}\right)^2\right)^{\frac{1}{2}}|x_1| = \|x\|,$$

$$V'_{(2.9)}(t, A(t)x) \leq -3t^{-1} V(t, A(t)x) \quad \text{for all} \quad x \in \operatorname{im}\pi(t), \quad t \in I.$$

According to Theorem 2.3, the zero solution of (2.8) is t^3-asymptotically stable and

$$\left\|x\left(t, t_0, x^0\right)\right\| \leq 10 t_0^3 t^{-3}$$

for all $t \geq t_0 \geq 1$, if $\left\|P(t_0) P_1(t_0) x^0\right\| = \left|x_1^0 + 2t_0^{-5} x_2^0\right| < 2^{-1}$, $x^0 = \left(x_1^0, x_2^0, x_3^0\right)^T$.

Theorem 2.4. *Suppose that:*

I. There exists a Liapunov function $V(t, y)$ defined on $I \times \Delta$, which satisfies the following conditions for all $x \in \operatorname{im}\pi(t)$, $\|x\| < H$, $t \in I$:

(i) $\|x\| \leq V(t, A(t)x) \leq K\|x\|$;

(ii) $V'_{(1.20)}(t, A(t)x) \leq -cV(t, A(t)x)$, $K \geq 1$, $c > 0$ are constants.

II. There exists $L_0 > 0$ such that $\left\|\left(I - (QQ_1)' - QP_1A_2^{-1}B + QD\right)(t)\right\| \leq L_0$ for all $t \in I$.

Then the solution $x(t) \equiv 0$ of (1.1) is exponential-asymptotically stable.

Proof. First of all, since $x(t)$ is a solution of (1.1), $x(t) \in \operatorname{im}\pi(t)$, this implies $x(t_0) \in \operatorname{im}\pi(t_0)$. Therefore, due to the condition II of this theorem, we have

$$\|x(t_0)\| = \left\|\left(I - (QQ_1)' - QP_1A_2^{-1}B + QD\right)PP_1(t_0)x^0\right\| \leq L_0\left\|P(t_0)P_1(t_0)x^0\right\|.$$

Further, this theorem can be proved by the same argument used in the proof of Theorem 2.3.

Theorem 2.5. *Suppose that there exists a Liapunov function $V(t, y)$ defined on $I \times \Delta$, which satisfies the following conditions for all $x \in \operatorname{im}\pi(t)$, $\|x\| < H$, $t \in I$:*

(i) $\|x\| \leq V(t, A(t)x) \leq K\|A(t)x\|$;

(ii) $V'_{(1.20)}(t, A(t)x) \leq -cV(t, A(t)x)$.

Then the solution $x(t) \equiv 0$ of (1.1) is exponential-asymptotically stable.

Lemma 2.1. *Let the conditions (1.3), (1.4), (1.5), (1.7) hold and there exists a $L > 0$ such that*

$$\left\|A_2^{-1}(t)\right\| \leq L, \quad \left\|\left(I - (QQ_1)' - QP_1A_2^{-1}B\right)A_2^{-1}(t)\right\| \leq L \quad \textit{for all} \quad t \in I, \quad (2.10)$$

where $A_2(t)$ is determined by (1.8).

Then the zero solution of (1.2) is stable if and only if the zero solution of (1.25) is stable on $\mathbb{E}^k$.

We have the similar conclusions for asymptotically, φ-asymptotically, exponential-asymptotically stability of the zero solution of (1.2) and (1.25).

Proof. First of all, we assume that the zero solution of (1.2) is stable. In that case, for all $t_0 \in I$, given $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $\left\|P(t_0)P_1(t_0)x^0\right\| < \delta$, $\left\|x\left(t, t_0, x^0\right)\right\| < \dfrac{\varepsilon}{M}$ for all $t \geq t_0$, M is determined by (1.4).

Let $y(t) = y\left(t, t_0, y^0\right)$ be a solution of (1.25), where $y^0 \in \mathbb{E}^k$. Then there exists $x^0 \in \mathbb{R}^m$ such that $y^0 = A(t_0)P(t_0)P_1(t_0)x^0$. Due to Lemma 1.2, $y(t) \equiv A(t)x\left(t, t_0, x^0\right)$, where $x\left(t, t_0, x^0\right)$ is the solution of (1.2) satisfying $P(t_0)P_1(t_0)x(t_0) = P(t_0)P_1(t_0)x^0$.

Because of (1.17) and (2.10), we have

$$\left\| P(t_0) P_1(t_0) x^0 \right\| = \left\| A_2^{-1}(t_0) A(t_0) P(t_0) P_1(t_0) x^0 \right\| \leq$$
$$\leq \left\| A_2^{-1}(t_0) \right\| \left\| A(t_0) P(t_0) P_1(t_0) x^0 \right\| \leq L \left\| y^0 \right\|.$$

Therefore, if $\left\| y^0 \right\| < \dfrac{\delta}{L}$, then $\left\| P(t_0) P_1(t_0) x^0 \right\| < \delta$, and hence, it holds that

$$y\left(t, t_0, y^0\right) = \left\| A(t) x\left(t, t_0, x^0\right) \right\| \leq \| A(t) \| \left\| x\left(t, t_0, x^0\right) \right\| < M \frac{\varepsilon}{M} = \varepsilon.$$

Moreover, we can see that, by the unique existence of solution of (1.25), $y(t) \in \mathbb{E}^k$ for all $t \geq t_0$. Hence, according to Definition 1.5, the zero solution of (1.25) is stable of $\mathbb{E}^k$.

Conversely, let the zero solution of (1.25) be stable on $\mathbb{E}^k$. Then, for any $t_0 \in I$ and given $\varepsilon > 0$, there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that

$$\left\| y\left(t, t_0, y^0\right) \right\| < \frac{\varepsilon}{L} \tag{2.11}$$

for all $t \geq t_0$ if $\left\| y^0 \right\| < \delta_1$ $\left(y^0 \in \mathbb{E}^k\right)$.

Now, let $x\left(t, t_0, x^0\right)$ be a solution of (1.2).

Denote

$$y^* = A(t_0) P(t_0) P_1(t_0) x^0. \tag{2.12}$$

By (1.7), $y^* \in \mathbb{E}^k$. Moreover, due to Lemma 1.2, we see that $A(t) x\left(t, t_0, x^0\right)$ is a solution of (1.25) and $y\left(t, t_0, y^*\right) = A(t) x\left(t, t_0, x^0\right)$. Since $x\left(t, t_0, x^0\right) \in \operatorname{im} \pi_{\mathrm{can}}(t)$ for all $t \geq t_0$, it holds that

$$x\left(t, t_0, x^0\right) = \pi_{\mathrm{can}}(t) x\left(t, t_0, x^0\right) =$$
$$= \left(I - (QQ_1)' - QP_1 A_2^{-1} B\right) A_2^{-1}(t) A(t) x\left(t, t_0, x^0\right) =$$
$$= \left(I - (QQ_1)' - QP_1 A_2^{-1} B\right) A_2^{-1}(t) y\left(t, t_0, y^*\right).$$

Thereby, together with (2.10), we obtain

$$\left\| x\left(t, t_0, x^0\right) \right\| < L \left\| y\left(t, t_0, y^*\right) \right\|. \tag{2.13}$$

On the other hand, if $\left\| P(t_0) P_1(t_0) x^0 \right\| < \dfrac{\delta_1}{M}$, (2.12) implies

$$\left\| y^* \right\| \leq M \left\| P(t_0) P_1(t_0) x^0 \right\| < \delta_1. \tag{2.14}$$

Thus, because of (2.14), (2.11) and (2.13), we have $\left\| x\left(t, t_0, x^0\right) \right\| \leq L \dfrac{\varepsilon}{L} = \varepsilon$. That is, the zero solution of (1.2) is stable.

The lemma is proved.

Theorem 2.6. *Suppose that:*

(i) there exists $N > 0$ *such that*

$$\left\| A_2^{-1}(t) \right\| \leq N, \quad \| T(t) \| \leq N, \quad \| T_1(t) \| \leq N \tag{2.15}$$

for all $t \in I$, *where* T, T_1 *are defined by* (1.22) *and* (1.23), A_2 *is defined by* (1.8);

(ii) there exists $K_1 > 0$ *such that*

$$\left\| x\left(t, t_0, x^0\right) \right\| \leq K_1 \left\| P(t_0) P_1(t_0) x^0 \right\| \quad \textit{for all} \quad t \geq t_0 \tag{2.16}$$

where $x(t, t_0, x^0)$ *is a solution of* (1.2);

(*iii*) $\|(A' - B)QDP(t)x + f(t, x)\| \le \alpha(t)\|x\|$, *or* $\|(A' - B)QDP(t)x + f(t, x)\| \le \le \alpha(t)\|\pi_{\text{can}}(t)x\|$ *for all* $x \in \operatorname{im}\pi(t)$, $\|x\| < H$, $t \in I$, *where the matrix* D *is determined by* (1.12) *and* $\alpha(t)$ *is positive continuous on* $[a, \infty)$ *satisfying*

$$\int_a^\infty \alpha(t)\,dt = \alpha_0 < \infty. \tag{2.17}$$

Then the solution $x(t) \equiv 0$ *of* (1.1) *is stable.*

Proof. First, if $x \in \operatorname{im}\pi_{\text{can}}(t)$, then because of (1.18) in Lemma 1.1 and (1.9), we have:

$$x = \pi_{\text{can}}(t)x = \left(I - (QQ_1)' - QP_1A_2^{-1}B\right)PP_1(t)x =$$

$$= \left(I - (QQ_1)' - QP_1A_2^{-1}B\right)A_2^{-1}A(t)x = T_1(t)A(t)x.$$

Because of (2.15), this implies

$$\|x\| \le N\|A(t)x\|. \tag{2.18}$$

Due to (i), (ii) and Lemma 2.1, we can see that the zero solution of (1.25) is stable on $\mathbb{E}^k$. By (1.4), (2.15), (2.16), (1.17) and Lemma 1.2, we have

$$\left\|y(t, t_0, y^0)\right\| \le K\left\|y^0\right\|, \tag{2.19}$$

where $K = MNK_1$ for all $t \ge t_0$, $y(t, t_0, y^0)$ is a solution of (1.25), $y^0 \in \mathbb{E}^k$.

Denote

$$V(t, y) = N\sup_{\tau\ge 0}\|y(t+\tau, t, y)\|, \quad y \in \mathbb{E}^k.$$

Since the equation (1.25) is linear, then, basing on (2.18), (2.19), Lemma 1.2 and Lemma 1.3, we can prove that $V(t, y)$ is a Liapunov function defined on $I \times \mathbb{E}^k$, which satisfies

$$|V(t, y_1) - V(t, y_2)| \le KN\|y_1 - y_2\| \quad \text{for all} \quad y_1 \in \mathbb{E}^k, \quad y_2 \in \mathbb{E}^k, \tag{2.20}$$

$$V(t, 0) \equiv 0 \quad \text{and if} \quad x \in \operatorname{im}\pi_{\text{can}}(t), \quad t \in I, \quad \text{then} \quad \|x\| \le V(t, A(t)x),$$

$$V'_{(1.21)}(t, A(t)x) \le 0. \tag{2.21}$$

Thus, the zero solution of (1.2) is stable.

Denote

$$W(t, y) = V(t, y)e^{KN\alpha_0}\exp\left(-KN\int_a^t \alpha(s)\,ds\right).$$

We see that $W(t, y)$ is also Liapunov function defined on $I \times \mathbb{E}^k$.

Moreover, $W(t, 0) \equiv 0$ and in case $x \in \operatorname{im}\pi(t)$, $x = \pi(t)x = T(t)A(t)x$, this implies $\|x\| \le \|T(t)\|\|A(t)x\| \le N\|A(t)x\|$, and hence, $\|x\| \le W(t, A(t)x)$, simultaneously, we also have

$$\|\pi_{\text{can}}(t)x\| \le V(t, A(t)x), \quad \|x\| \le V(t, A(t)x). \tag{2.22}$$

On the other hand, because of (iii), (2.20), (2.22) and (2.21), $x \in \operatorname{im}\pi(t)$ we have

$$e^{-KN\alpha_0}W'_{(1.20)}(t, A(t)x) \le -KN\alpha(t)V(t, A(t)x)\exp\left(-KN\int_a^t \alpha(s)\,ds\right) +$$

$$+ \left\{V'(t, A(t)\pi_{\text{can}}(t)x) + KN\left\|\left(A' - B\right)QDP(t)x + f(t, x)\right\|\right\} \exp\left(-KN \int_a^t \alpha(s)\,ds\right) \le$$

$$\le -KN\alpha(t) V(t, A(t)x) \exp\left(-KN \int_a^t \alpha(s)\,ds\right) +$$

$$+ \left\{V'(t, A(t)\pi_{\text{can}}(t)x) + KN\,\alpha(t) V(t, A(t)x)\right\} \exp\left(-KN \int_a^t \alpha(s)\,ds\right) =$$

$$= V'(t, A(t)\pi_{\text{can}}(t)x) \exp\left(-KN \int_a^t \alpha(s)\,ds\right) \le 0.$$

This implies $W'_{(1.20)}(t, A(t)x) \le 0$ for all $x \in \operatorname{im}\pi(t)$, $t \in I$. So, $W(t, A(t)x)$ satisfies all assumptions of Theorem 2.1, and this means that the zero solution of (1.1) is stable.

The theorem is proved.

Lemma 2.2. *Consider the following equation:*

$$A(t)x + B(t)x = A(t)F(t, x), \tag{2.23}$$

where

$$F(t, 0) \equiv 0, \quad F(t, x) \in C^{01}_{tx}\left(I \times \mathbb{R}^m\right).$$

Let (1.2) *have index-2 on* I *and*

$$A(t)Q_1(t)F(t, x) \equiv 0 \quad \textit{for all} \quad (t, x) \in I \times \mathbb{R}^m. \tag{2.24}$$

Then (2.23) *has index-2 on* $I \times \mathbb{R}^m$. *Moreover, the conditions* (1.11), (1.12) *are fulfilled and any solution* $x(t)$ *of* (2.23) *satisfies* $x(t) \in \operatorname{im}\pi_{\text{can}}(t)$, $t \in I$.

Proof. Denote

$$\overline{S}(t, x) = \left\{z \in \mathbb{R}^m : (B(t) - AF'_x(t, x))z \in \operatorname{im} A(t)\right\},$$

$$\overline{S_1}(t, x) = \left\{z \in \mathbb{R}^m : (B(t) - A(t)F'_x(t, x))P(t)z \in \operatorname{im}\overline{A_1}(t, x)\right\},$$

$$\overline{N_1}(t, x) = \ker \overline{A_1}(t, x),$$

where

$$\overline{A_1}(t, x) = A(t) + \left(B(t) - A(t)F'_x(t, x) - AP'\right)Q(t) = A_1(t) - A(t)F'_x(t, x)Q(t). \tag{2.25}$$

Let arbitrary $\eta \in \mathbb{R}^m$. Because of (2.24) and (2.25), for all $t \in I$, we have

$$\overline{A_1}(t, \eta)P(t)P_1(t) = A_1(t)P(t)P_1(t) = A(t)P(t)P_1(t), \tag{2.26}$$

$$A(t)F'_x(t, \eta) = A(t)PP_1(t)F'_x(t, \eta). \tag{2.27}$$

Due to (2.25), (2.26) and (2.27), we obtain

$$\operatorname{im}\overline{A_1}(t, \eta) = \operatorname{im} A_1(t). \tag{2.28}$$

It follows from (2.28) that $\dim\overline{N_1}(t, \eta) = \dim N_1(t) = \text{const} > 0$ for any $t \in I$. Since (2.28) and

$$\left(B(t) - A(t)F'_x(t, \eta)\right)P(t)z = \left(B(t) - \overline{A_1}(t, \eta)PP_1(t)F'_x(t, \eta)\right)P(t)z \in \operatorname{im}\overline{A_1}(t, \eta)$$

if and only if $B(t)P(t)z \in \operatorname{im} A_1(t)$, we have

$$\overline{S_1}(t, \eta) \equiv S_1(t), \quad (t, \eta) \in I \times \mathbb{R}^m.$$

Further, let $y \in \overline{N_1}(t, \eta) \cap S_1(t)$. By $y \in \overline{N_1}(t, \eta)$, $\overline{A_1}(t, \eta) y = 0$, this implies

$$A_1(t) y - A_1(t) P P_1(t) F_x'(t, \eta) y = A_1(t)\big(y - P P_1(t) F_x'(t, \eta) y\big) = 0,$$

i.e.,

$$\xi = \big(y - P P_1(t) F_x'(t, \eta) y\big) \in N_1(t).$$

Thus, due to $y \in S_1(t)$, we have $\xi = Q_1(t)\xi = Q_1(t) y = 0$, and hence,

$$y = P P_1(t) F_x'(t, \eta) y.$$

From this we obtain

$$A_1(t) y = A_1(t) P(t) P_1(t) F_x'(t, \eta) y = \overline{A_1}(t, \eta) P(t) P_1(t) F_x'(t, \eta) y = \overline{A_1}(t, \eta) y = 0,$$

i.e., $y \in N_1(t)$. Since (1.2) has index-2, $y = 0$. This implies $\overline{N_1}(t, \eta) \oplus S_1(t) = \mathbb{R}^m$, for all $t \in I$. Because η is arbitrary, we conclude

$$\dim \overline{N_1}(t, x) = \text{const} > 0 \quad \text{and} \quad \overline{N_1}(t, x) \oplus S_1(t) = \mathbb{R}^m \quad \text{for all} \quad (t, x) \in I \times \mathbb{R}^m,$$

this means, the equation (2.23) has index-2 on $I \times \mathbb{R}^m$.

The second assertion of this lemma is proved by $A_1^{-1} A(t) = P(t)$ and $D \equiv 0$.

Lemma 2.2 and Theorem 2.6 yield the following corollary.

***Corollary* 2.1.** *Under the same assumptions as in Theorem* 2.6, *if the conditions (i) and (iii) are replaced by the following:*

There exists $N > 0$ *such that* $\|A_2^{-1}(t)\| \le N$, $\|T_1(t)\| \le N$ *for all* $t \in I$ *and* $\|A(t) F(t, x)\| \le \alpha(t) \|x\|$ *for all* $x \in \operatorname{im} \pi_{\text{can}}(t)$, $\|x\| < H$, $t \in I$, *then the zero solution of* (2.23) *is stable.*

Theorem 2.7. *Under the same assumptions as in Theorem* 2.6, *where the condition (ii) is replaced by the following:*

(ii)′ there exists a $K_1 > 0$ *such that* $\left\|x\big(t, t_0, x^0\big)\right\| \le K_1 \left\| P(t_0) P_1(t_0) x^0 \right\| \dfrac{\varphi(t_0)}{\varphi(t)}$ *for all* $t \ge t_0$, *where* $x\big(t, t_0, x^0\big)$ *is a solution of* (1.2) *and* $\varphi \in C^1(I)$, $\varphi(t) > 0$, $\varphi'(t) > 0$ *for all* $t \in I$, $\varphi(t) \to \infty$ *as* $t \to \infty$, *the zero solution of* (1.1) *is* φ-*asymptotically stable.*

Proof. Due to the condition (ii)′, we have

$$\left\|y\big(t, t_0, y^0\big)\right\| \le K \left\|y^0\right\| \frac{\varphi(t_0)}{\varphi(t)} \quad \text{for all} \quad t \ge t_0,$$

where $K = MNK_1$, $y^0 \in \mathbb{E}^k$ and $y\big(t, t_0, y^0\big)$ is a solution of (1.25).

Denote

$$V(t, y) = N \sup_{\tau \ge 0} \| y(t + \tau, t, y) \| \frac{\varphi(t + \tau)}{\varphi(t)}, \quad y^0 \in \mathbb{E}^k.$$

We can see that $V(t, y)$ is a Liapunov function defined on $I \times \mathbb{E}^k$ satisfying the following properties:

$$V(t, 0) \equiv 0, \quad \|x\| \le V(t, A(t) x), \quad V'_{(1.21)}(t, A(t) x) \le -\lambda(t) V(t, A(t) x)$$

for all $x \in \operatorname{im} \pi_{\text{can}}(t)$, $t \in I$, where $\lambda(t) = \dfrac{\varphi'(t)}{\varphi(t)}$.

Let

$$W(t, y) = V(t, y) e^{KN\alpha_0} \exp\left(-KN \int_a^t \alpha(s)\, ds\right).$$

We can prove that $W(t, y)$ is also a Liapunov function which satisfies all assumptions of Theorem 2.3.

In fact, $W'_{(1.20)}(t, A(t)x) \leq -\lambda(t) W(t, A(t)x)$ for all $x \in \operatorname{im}\pi(t)$, $t \in I$.

Thereby, the solution $x(t) \equiv 0$ of (1.1) is φ-asymptotically stable.

Theorem 2.8. *Suppose that:*

(i) there exists $N > 0$ *such that* $\left\| A_2^{-1}(t) \right\| \leq N$, $\| T(t) \| \leq N$, $\| T_1(t) \| \leq N$ *for all* $t \in I$, *where* T_1, T *are defined by* (1.23) *and* (1.22);

(ii) there exist $K_1 > 0$ *and* $c > 0$ *such that*

$$\left\| x\left(t, t_0, x^0\right) \right\| \leq K_1 \left\| P(t_0) P_1(t_0) x^0 \right\| \exp(-c(t - t_0))$$

for all $t \geq t_0$, *where* $x\left(t, t_0, x^0\right)$ *is a solution of* (1.2);

(iii) $f(t, x) = g(t, x) + h(t, x)$, *where* $g(t, x)$, $h(t, x)$ *satisfy the following conditions for all* $x \in \operatorname{im}\pi(t)$, $\| x \| < H$, $t \in I$,

+) $\left\| \left(A' - B\right) QDP(t) x + g(t, x) \right\| = o(\| x \|)$ $(\| x \| \to 0)$ *and* $\| h(t, x) \| \leq \alpha(t) \| x \|$

or

+) $\| g(t, x) \| = o(\| x \|)$ $(\| x \| \to 0)$ *and* $\left\| \left(A' - B\right) QDP(t) x + h(t, x) \right\| \leq \alpha(t) \| x \|$,

$\alpha(t)$ *satisfies* (2.17) *and the matrix* D *is determined by* (1.12).

Then the solution $x(t) \equiv 0$ *of* (1.1) *is exponential-asymptotically stable.*

This theorem can be proved by the same method used in the proof of Theorem 2.6 with

$$V(t, y) = N \sup_{\tau \geq 0} \| y(t + \tau, t, y) \| e^{c\tau} \quad \text{and} \quad W(t, y) = V(t, y) e^{KN\alpha_0} \exp\left(-KN \int_a^t \alpha(s)\, ds\right),$$

where $y \in \mathbb{E}^k$, $K = MNK_1$.

***Remark* 2.2.** (a). Under the same assumptions as in Theorem 2.8, where $o(\| x \|)$ $(\| x \| \to 0)$ and $\alpha(t) \| x \|$ in the condition (iii) are replaced by $o(\| \pi_{\text{can}}(t) x \|)$ $(\| \pi_{\text{can}}(t) x \| \to 0)$ and $\alpha(t) \| \pi_{\text{can}}(t) x \|$, the zero solution of (1.1) is exponential-asymptotically stable.

(b). Similarly, from Theorems 2.7 and 2.8, we also have corresponding corollaries for the zero solution of the equation (2.23).

(c). If the equation (1.1) has index-2 on $I \times \{x \colon \| x \| < H, \text{ small } H\}$, then, the above theorems are true.

***Example* 2.2.** Consider the following equations:

$$A(t) x' + B(t) x = f(t, x), \tag{2.29}$$

$$A(t) x' + B(t) x = 0, \tag{2.30}$$

where

$$A(t) = \begin{pmatrix} 1 & t^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} -1 & -t^{-2} & 3 \\ 1 & t^{-1} & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $f(t, x) = (f_1, f_2, f_3)^T$, $f_1 = (x_1 - x_3)^2$, $f_2 = -t^{-2} x_1 \left(1 + x_2^2\right)^{\frac{1}{2}}$, $f_3 = -x_2 \sin^2 x_3$, $I = [1, \infty)$, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$.

We see that the equations (2.29), (2.30) have index-2 and $f(t, x)$ satisfies the conditions (1.11) and (1.12) with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^{-2} & 0 & 0 \end{pmatrix}.$$

We compute,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & (3t+1)t^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix} 1 & 3 & (3t-2)t^{-1} \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

and

$$\operatorname{im}\pi(t) = \left\{\left(\alpha, 0, \left(1+t^{-2}\right)\alpha\right)^T;\ \alpha\in\mathbb{R}\right\}.$$

Next, we have $R\left(A(t)\,P(t)\,P_1(t)\right) = R\left(A(1)\,P(1)\,P_1(1)\right)$ for all $t\in I$. Let $\tilde{x}\left(t, t_0, x^0\right)$ be a solution of (2.30), which satisfies $P(t_0)\,P_1(t_0)\left(\tilde{x}(t_0)-x^0\right)=0$, where $x^0 = \left(x_1^0, x_2^0, x_3^0\right)^T$, then,

$$\tilde{x}\left(t, t_0, x^0\right) =$$

$$= \left(\left(x_1^0 + \left(3+t^{-1}\right)x_2^0\right)\exp(-2(t-t_0)),\ 0,\ \left(x_1^0 + \left(3+t^{-1}\right)x_2^0\right)\exp(-2(t-t_0))\right)^T.$$

Hence, we have

$$\left\|x\left(t, t_0, x^0\right)\right\| \le 2\left|x_1^0 + \left(3+t^{-1}\right)x_2^0\right|e^{-2(t-t_0)} = 2\left\|P(t_0)\,P_1(t_0)\,x^0\right\|e^{-2(t-t_0)} \quad (2.31)$$

for all $t \ge t_0 \ge 1$.

On the other hand, we have

$$\|A(t)\| \le 3, \quad \left\|A_2^{-1}(t)\right\| \le 11, \quad \|T_1(t)\| \le 11, \quad \|T(t)\| \le 11 \quad \text{for all} \quad t\in I, \quad (2.32)$$

$$\left\|\left(A'-B\right)QDPx + f(t,x)\right\| \le 3t^{-2}\|x\| \quad \text{for all} \quad x\in\operatorname{im}\pi(t), \quad \|x\|<1. \quad (2.33)$$

Thus, from (2.31), (2.32) and (2.33), we see that the assumptions of Theorem 2.8 are satisfied, hence, the zero solution of (2.29) is a exponential-asymptotically stable. Moreover,

$$\left\|x\left(t, t_0, x^0\right)\right\| \le 22\exp(-2(t-t_0)) \quad \text{for all} \quad t\ge t_0\ge 1$$

if

$$\left\|P(t_0)\,P_1(t_0)\,x^0\right\| = \left|x_1^0 + \left(3+t_0^{-1}\right)x_2^0\right| < e^{-1452}.$$

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