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HOPFICITY AND CO-HOPFICITY IN SOLUBLE GROUPS

ХОПФОВІСТЬ І КОХОПФОВІСТЬ У РОЗВ'ЯЗУВАНИХ ГРУПАХ

We show that a soluble group satisfying the minimal condition for its normal subgroups is co-hopfian and that a torsion-free finitely generated soluble group of finite rank is hopfian. The latter property is a consequence of a stronger result: in a minimax soluble group, the kernel of an endomorphism is finite if and only if its image is of finite index in the group.

Показано, що розв'язувана група, яка задовольняє умову мінімальності для її нормальних підгруп, є кохопфовою і скінченнопороджена розв'язувана група скінченного рангу без скруту є хопфовою. Остання властивість є наслідком сильнішого результату: в мінімаксній розв'язуваній групі ядро ендоморфізму скінченне тоді і тільки тоді, коли його образ має скінченний індекс у групі.

1. Introduction and main results. Recall that a group G is said to be *hopfian* (respectively, *co-hopfian*) if every surjective (respectively, injective) endomorphism $\varphi: G \to G$ is an automorphism. In other words, a group is hopfian if for every normal subgroup $H \trianglelefteq G$, the fact that G/H and G are isomorphic implies that $H = \{1\}$. In the same way, a group is co-hopfian if for every subgroup $H \leq G$, the fact that H and G are isomorphic implies that $H = G$. As usual, we shall say that a group G satisfies Min (respectively, Min-n) to mean that each nonempty subset of subgroups of G (respectively, normal subgroups of G) contains a minimal element. Groups satisfying Max or Max-n are likewise defined, by substituting "maximal" for "minimal".

It is every easy to see that a group satisfying Min is co-hopfian and a group satisfying Max-n is hopfian. In Section 3, we shall give a more general form of these results. On the other hand, a group satisfying Min-n is not necessarily co-hopfian. For example, let $A(\mathbb{N})$ be the finitary alternating group on the set of natural integers. Hence $A(\mathbb{N})$ is the set of products of an even (finite) number of transpositions of $\mathbb{N}$. Consider the "1-right-shift mapping" $\varphi: A(\mathbb{N}) \to A(\mathbb{N})$, where for each element $f \in A(\mathbb{N})$, the permutation $\varphi(f)$ is defined by

$$\varphi(f)(0) = 0, \quad \text{and} \quad \varphi(f)(n) = 1 + f(n-1) \quad \text{when} \quad n > 0.$$

It is easy to verify that φ is an injective endomorphism. But φ is not surjective since $\varphi(A(\mathbb{N})) = \{f \in A(\mathbb{N}) \mid f(0) = 0\}$. Thus the group $A(\mathbb{N})$ is not co-hopfian; nevertheless it satisfies Min-n for it is simple.

Our first result shows that the situation is different in the class of soluble groups.

Theorem 1. *A soluble group satisfying* Min-n *is co-hopfian.*

It is worth to point out that a soluble group which satisfies Min-n is locally finite [1] (Theorem 5. 25). Also note that in the class of soluble groups, the property Min-n does not imply the property Min [1, p. 152 – 153] (Part 1).

The property of co-hopficity is rather strong. For instance, even the infinite cyclic group is not co-hopfian; however, in this case, the image of each injective endomorphism is of finite index in the group. It is not difficult to show that this result remains true in any finitely generated abelian group. But in a finitely generated metabelian group, the image of an injective endomorphism can be of infinite index in the group. For example, consider the subring $\mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$ of $\mathbb{Q}(X, Y)$ generated by X, Y, X^{-1}, Y^{-1} and denote by G the group of matrices of the form

$$\begin{pmatrix} X^iY^j & f \\ 0 & 1 \end{pmatrix}, \quad \text{with} \quad i, j \in \mathbb{Z},\ f \in \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$$

(in other words, G is the restricted wreath product of $\mathbb{Z}$ and $\mathbb{Z}^2$). This group is metabelian, generated by $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. But the endomorphism φ of G defined by

$$\varphi: \begin{pmatrix} X^iY^j & f \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} X^iY^j & (X+Y)f \\ 0 & 1 \end{pmatrix}$$

is injective with an image of infinite index in G. It follows from our next result that this cannot happen in a soluble group of finite rank. Furthermore, it is not necessary to suppose that the endomorphism is injective but only that its kernel is finite. Recall that a group is said to be of finite rank (in the sense of Prüfer) if there is a positive integer n such that every finitely generated subgroup can be generated by at most n elements; in this case, the least integer n with this property is the rank of the group.

Theorem 2. *Let G be a soluble group of finite rank and let φ be an endomorphism. Then, if* $\ker\varphi$ *is finite, so is the index* $|G:\varphi(G)|$.

In this theorem, if we suppose in addition that G is torsion-free, we can substitute "locally soluble" for "soluble", since a torsion-free locally soluble group of finite rank is soluble [1] (Corollary of Lemma 10.39).

We remark that in Theorem 2, the converse property of φ fails, namely the fact that $|G:\varphi(G)|$ is finite does not imply that $\ker\varphi$ is finite. Indeed, consider for example the direct product G of all quasicyclic p-groups when p ranges the set of primes, and the endomorphism φ of G defined on each p-component by $\varphi(x) = x^p$. Then G is an abelian group of finite rank 1 and φ is surjective, but $\ker\varphi$ is infinite. Nevertheless, the converse property holds when G is a minimax group (that is, G has a series of finite length in which each factor satisfies either Max or Min).

Theorem 3. *Let G be a minimax soluble group and let φ be an endomorphism. Then, if the index* $|G:\varphi(G)|$ *is finite, so is* $\ker\varphi$.

A similar result was proven by Hirshon in the case where G is a finitely generated residually finite group [2].

***Corollary* 1.** *Let G be a torsion-free minimax soluble group and let φ be an endomorphism. Then, if $|G:\varphi(G)|$ is finite, φ is injective. In particular, G is hopfian.*

The quasicyclic p-group with the endomorphism $x \to x^p$ shows that a minimax soluble group can be non-hopfian.

In Corollary 1, when G is metabelian, the property of hopficity is a consequence of a more general result. Indeed, a finitely generated metabelian group is residually finite [3] (Theorem 15.4.1) and so is hopfian [3] (Theorem 6.1.11). That follows as well from the fact that a finitely generated metabelian group satisfies Max-n [1] (Theorem 5.34). But on the other hand, a finitely generated soluble group of derived length $d \geq 3$ is not necessarily hopfian [4].

Since a soluble minimax group has finite rank [1, p. 166] (Part 2), Theorems 2 and 3 imply the following result.

***Corollary* 2.** *In a minimax soluble group, the kernel of an endomorphism is finite if and only if its image is of finite index in the group.*

Note that a finitely generated soluble group of finite rank is a minimax group [1] (Theorem 10.38).

2. Proofs of the theorems. We shall always employ the multiplicative notation, even when the group is abelian. Let $\mathcal{X}$ be a class of groups in the usual sense (a trivial group belongs to $\mathcal{X}$, and for any group $G \in \mathcal{X}$, all groups isomorphic to G are in $\mathcal{X}$). We say that $\mathcal{X}$ is *inductive* if in any group G, for any chain of subgroups $(K_i)_{i \in I}$ with $K_i \in \mathcal{X}$, the union $\bigcup_{i \in I} K_i$ belongs to $\mathcal{X}$.

Lemma 1. *Let G be a group and let φ be an endomorphism. Consider an inductive class of groups $\mathcal{X}$, closed under taking subgroups, and such that for each normal subgroup $H \trianglelefteq G$ containing $\ker\varphi$, if $H/\ker\varphi \in \mathcal{X}$, then $H \in \mathcal{X}$. Under these conditions, for any normal subgroup $K_0 \trianglelefteq G$, with $K_0 \in \mathcal{X}$ and $\varphi(K_0) \le K_0$, there exists a normal subgroup $K_1 \trianglelefteq G$ containing K_0, with $K_1 \in \mathcal{X}$ and $\varphi(K_1) \le K_1$, and such that the endomorphism induced by φ on G/K_1 is injective.*

Proof. Let $\overline{\varphi}$ be the endomorphism of G/K_0 induced by φ. Define inductively the sequence $(K_{0,n})_{n \ge 0}$ of normal subgroups of G like this: $K_{0,0} = K_0$ and for $n > 0$, $K_{0,n}$ is the normal subgroup of G containing K_0 and such $K_{0,n}/K_0 = \ker\overline{\varphi}^n$. Since $\varphi(K_{0,n}) \le K_{0,n-1}$, φ induces a homomorphism from $K_{0,n}$ to $K_{0,n-1}$ and so the quotient $K_{0,n}/\ker\varphi$ is isomorphic to a subgroup of $K_{0,n-1}$ (note that $K_{0,n}$ contains $\ker\varphi$ for $n > 0$). It follows by an immediate induction on n that $K_{0,n}$ belongs to $\mathcal{X}$. This class being assumed inductive, the union $K_1 = \bigcup_{n \ge 0} K_{0,n}$ also is in $\mathcal{X}$. Then clearly K_1 satisfies all the desired properties of Lemma 1.

***Proof of Theorem* 1.** Let G be a soluble group satisfying Min-n and let φ be an injective endomorphism. We must prove that φ is surjective. For that, we argue by induction on the derived length d of G. If $d = 1$, then G satisfies Min and so the result is clear.

Now suppose that $d > 1$ and apply Lemma 1, by choosing for $\mathcal{X}$ the class of abelian groups, and for K_0 the $(d-1)$-th derived subgroup $G^{(d-1)}$. Consequently, there exists a normal abelian subgroup $K_1 \trianglelefteq G$ containing $G^{(d-1)}$, with $\varphi(K_1) \le K_1$, and such that the endomorphism $\overline{\varphi}$ of G/K_1 induced by φ is injective. By the inductive hypothesis, G/K_1 is co-hopfian and so $\overline{\varphi}$ is surjective. It follows that $G = K_1\varphi(G)$, and more generally that $G = K_1\varphi^n(G)$ for any positive integer n. Now consider the descending sequence of subgroups $\left(\varphi^n(K_1)\right)_{n>0}$. These subgroups are normal in G. Indeed, for any $x \in G$ and $\varphi^n(a) \in \varphi^n(K_1)$ $(a \in K_1)$, we may express x in the form $x = b\varphi^n(y)$ $(b \in K_1,\ y \in G)$ for $G = K_1\varphi^n(G)$; thus, using the fact that K_1 is abelian, we can write

$$x^{-1}\varphi^n(a)x = \varphi^n(y)^{-1}b^{-1}\varphi^n(a)b\varphi^n(y) = \varphi^n(y)^{-1}\varphi^n(a)\varphi^n(y) = \varphi^n(y^{-1}ay)$$

and so $x^{-1}\varphi^n(a)x$ belongs to $\varphi^n(K_1)$. Therefore, since G satisfies the condition Min-n, we have $\varphi^m(K_1) = \varphi^{m+1}(K_1)$ for some positive integer m. It follows that $K_1 = \varphi(K_1)$ for φ is injective, hence

$$G = K_1\varphi(G) = \varphi(K_1)\varphi(G) = \varphi(G),$$

as required.

The theorem is proved.

Lemma 2. *Let G be a group, φ an endomorphism and H a normal subgroup of G such that $\varphi(H) \le H$. Denote by $\overline{\varphi}$ (respectively, φ_0) the* *endomorphism of* $\overline{G} = G / H$ *(respectively,* H*) induced by* φ. *Then:*

(i) *If the indices* $|\overline{G} : \overline{\varphi}(\overline{G})|$ *and* $|H : \varphi_0(H)|$ *are finite, so is the index* $|G : \varphi(G)|$.

(ii) *If* $\ker \overline{\varphi}$ *and* $\ker \varphi_0$ *are finite, so is* $\ker \varphi$.

Proof. (i) We have $\overline{\varphi}(\overline{G}) = H\varphi(G)/H$, thus $|\overline{G} : \overline{\varphi}(\overline{G})| = |G : H\varphi(G)|$. If $r = |G : H\varphi(G)|$ and $s = |H : \varphi_0(H)|$, there are elements $a_1, \ldots, a_r \in G$ and $b_1, \ldots, b_s \in H$ such that

$$G = \bigcup_{i=1,\ldots,r} a_i H\varphi(G), \quad H = \bigcup_{j=1,\ldots,s} b_j \varphi(H).$$

Therefore, we can express any element $x \in G$ in the form $x = a_i h \varphi(x')$ (with $h = b_j \varphi(h') \in H$), and so $x = a_i b_j \varphi(h') \varphi(x')$ belongs to $a_i b_j \varphi(G)$ for some integers i, j (with $1 \le i \le r$ and $1 \le j \le s$). It follows that $|G : \varphi(G)| \le rs$.

(ii) Since the quotient $H. \ker \varphi / H$ is a subgroup of $\ker \overline{\varphi}$, it is finite. But this quotient is isomorphic to $\ker \varphi / H \cap \ker \varphi$ and $H \cap \ker \varphi = \ker \varphi_0$ is finite, so $\ker \varphi$ is finite as well.

In the following, we shall use the (obvious) fact that the class of groups of finite rank is closed under taking subgroups and quotients. The next lemma is a particular case of Theorem 2.

Lemma 3. *Let* G *be a soluble group of finite rank and let* φ *be an endomorphism. The, if* φ *is injective, the index* $|G : \varphi(G)|$ *is finite.*

Proof. We argue by induction on the derived length d of G. First suppose that $d = 1$. For each prime p, denote by T_p the p-primary component of G and by $T = \Pi_p T_p$ its torsion-subgroup. Since T_p is of finite rank, it satisfies Min [3] (Theorem 4. 3. 13) and so is co-hopfian. It follows that $\varphi(T_p) = T_p$, therefore we have $\varphi(T) = T$. If we denote by $\overline{G}$ the quotient G/T, it is easy to see that the endomorphism $\overline{\varphi} : \overline{G} \to \overline{G}$ induced by φ is injective. Consequently, since $\overline{G}$ is a torsion-free abelian group of finite rank, $|\overline{G} : \overline{\varphi}(\overline{G})|$ is finite by a result of Fuchs [3] (Theorem 15. 2. 3). We can then apply Lemma 2(i) and it follows that the index $|G : \varphi(G)|$ is finite.

Now suppose that $d > 1$ and apply Lemma 1 by taking for χ the class of abelian groups, and for K_0 the $(d-1)$-th derived subgroup $G^{(d-1)}$. Hence there exists a normal abelian subgroup $K_1 \trianglelefteq G$ containing $G^{(d-1)}$, with $\varphi(K_1) \le K_1$, and such that the endomorphism $\overline{\varphi}$ of $\overline{G} = G/K_1$ induced by φ is injective. By the inductive hypothesis, $|\overline{G} : \overline{\varphi}(\overline{G})|$ is finite. In the same way, since K_1 is abelian of finite rank, $|K_1 : \varphi(K_1)|$ is finite. The result now follows from Lemma 2(i).

Let Π be a set of primes. As usual, if for each element of a torsion group G, all prime divisors of its order belong to Π, we say that G is a Π-group.

Lemma 4. *Let* Π *be a finite set of primes and let* G *be a soluble* Π*-group of finite rank. Then* G *satisfies* Min.

Proof. According to a forementioned result [3] (Theorem 4.3.13), an abelian p-group of finite rank satisfies Min for any prime p. Therefore, if G is abelian, it is the direct product of finitely many p-groups and so satisfies Min. Since the class of groups satisfying Min is closed with respect to forming extensions, the result follows from an induction on the derived length of G.

Lemma 5. *Let* G *be a group and let* φ *be an endomorphism. Then:*

(i) *If* $\ker\varphi$ *is finite, so is* $\ker\varphi^n$ *for any positive integer* n.

(ii) *If* $\varphi^m(G) = \varphi^{m+1}(G)$ *for some positive integer* m, *we have the equality* $G = \varphi(G)\ker\varphi^m$.

(iii) *If* $\ker\varphi$ *is finite and if* G *satisfies* Min, *the index* $|G : \varphi(G)|$ *is finite.*

Proof. (i) Since $\varphi(\ker\varphi^n) \le \ker\varphi^{n-1}$ for any $n > 0$, φ induces a homomorphism from $\ker\varphi^n$ to $\ker\varphi^{n-1}$, hence the quotient $\ker\varphi^n/\ker\varphi$ is isomorphic to a subgroup of $\ker\varphi^{n-1}$. It is then easy to prove by induction on n that $\ker\varphi^n$ is finite.

(ii) Let x be an element of G. Since $\varphi^m(G) = \varphi^{m+1}(G)$, there is an element $x' \in G$ such that $\varphi^m(x) = \varphi^{m+1}(x')$. It follows that $\varphi^m\left(\varphi(x')^{-1}x\right) = 1$ and so $x = \varphi(x')y$, where y belongs to $\ker\varphi^m$.

(iii) We have $\varphi^m(G) = \varphi^{m+1}(G)$ for some integer $m > 0$ because G satisfies Min. By (ii), $G = \varphi(G)$. $\ker\varphi^m$ and by (i), $\ker\varphi^n$ is finite. The result follows.

***Proof of Theorem* 2.** Suppose that $\ker\varphi$ is finite and denote by Π the set of primes dividing the order of $\ker\varphi$. Clearly, we may apply Lemma 1 by taking for χ the class of (torsion) Π-groups, with $K_0 = \ker\varphi$. It follows that there exists a normal Π-subgroup $K_1 \trianglelefteq G$ containing $\ker\varphi$, with $\varphi(K_1) \le K_1$, and such that the endomorphism $\overline{\varphi}$ induced by φ on $\overline{G} = G/K_1$ is injective. By Lemma 3, the index $|\overline{G} : \overline{\varphi}(\overline{G})|$ is finite. Furthermore, by Lemma 4, K_1 satisfies Min. Hence we may apply Lemma 5 (iii) to the endomorphism induced by φ in K_1 and so $|K_1 : \varphi(K_1)|$ is finite. The result now follows from Lemma 2 (i).

Lemma 6. *Let* G *be a group,* φ *an endomorphism and* H *a normal subgroup of* G *such that* $\varphi(H) \le H$. *Denote by* $\overline{\varphi}$ *the endomorphism of* $\overline{G} = G/H$ *induced by* φ *and suppose that the index* $|G : \varphi(G)|$ *and the kernel* $\ker\overline{\varphi}$ *are finite. Then* $|H : \varphi(H)|$ *is finite.*

Proof. We have $\varphi(H) \le (H \cap \varphi(G)) \le H$. Clearly, since $|G : \varphi(G)|$ is finite, so is the index $|H : H \cap \varphi(G)|$. Hence it suffices to prove that $|H \cap \varphi(G) : \varphi(H)|$ is finite. To show this, consider the subgroup $K \trianglelefteq G$ containing H such that $\ker\overline{\varphi} = K/H$. If $r = |K : H|$, there are r elements $a_1, \ldots, a_r \in K$ such that $K = \bigcup_{i=1,\ldots,r} a_i H$. Let x be an element of $H \cap \varphi(G)$. Thus we may write $x = \varphi(x') \in H$ for some $x' \in G$. In fact, x' belongs to K since $\varphi(x') \in H$. Consequently, we can express x' in the form $x' = a_j h$, with $j \in \{1, \ldots, r\}$ and $h \in H$. It follows that $x = \varphi(x') = \varphi(a_j)\varphi(h)$ and that implies that $|H \cap \varphi(G) : \varphi(H)| \le r$.

Before to prove Theorem 3, observe that the class of minimax groups is closed under taking subgroups and quotients. Moreover, we shall use the fact that a soluble minimax group has finite rank [1, p. 166] (Part 2).

***Proof of Theorem* 3.** First suppose that G is abelian and denote by T_p the p-primary component of G and by $T = \Pi_p T_p$ its torsion-subgroup. Obviously, a torsion abelian minimax group satisfies Min. Hence T is a direct product of finitely many quasicyclic groups and cyclic groups of prime-power order. In particular, there exists a positive integer q such T^q is a product of finitely many quasicyclic groups (with possibly $T^q = \{1\}$). Note that T/T^q is finite. According to the result of Fuchs already used [3] (Theorem 15.2.3), the endomorphism of G/T induced by φ is injective. Thus φ induces on $\overline{G} = G/T^q$ an endomorphism $\overline{\varphi}$ whose the kernel is included in T/T^q and so is finite. We can then apply Lemma 6 and we obtain that $|T^q : \varphi(T^q)|$ is finite. But T^q is a product of finitely many quasicyclic groups and so $\varphi(T^q) = T^q$. Now, in order to obtain a contradiction, suppose that $\ker\varphi$ is infinite. Since φ induces on G/T an injective endomorphism, $\ker\varphi$ is included in T. Furthermore, since T/T^q is finite, the intersection $T^q \cap \ker\varphi$ is necessarily infinite. It follows that in at least one p-primary component of T^q, say T_p^q, φ induces an endomorphism with an infinite kernel. Hence this kernel contains a quasicyclic p-subgroup A. Since A is divisible, T_p^q is the direct product of A and a subgroup B. But

$$\varphi\left(T_p^q\right) = \varphi(B) = T_p^q \quad (\text{for } \varphi(T^q) = T^q),$$

so the rank of T_p^q would be equal to the rank of a homomorphic image of B, a contradiction. Hence our theorem is proved when G is abelian.

Now suppose that G is soluble of derived length $d > 1$ and consider the endomorphism $\overline{\varphi} : G/G^{(d-1)} \to G/G^{(d-1)}$ induced by φ. By induction, we can say that $\ker\overline{\varphi}$ is finite. Hence, by Lemma 6, $|G^{(d-1)} : \varphi(G^{(d-1)})|$ is finite. Since our theorem is proved in the case of abelian groups, we may apply it to $G^{(d-1)}$ with the endomorphism $\varphi_0 : G^{(d-1)} \to G^{(d-1)}$ induced by φ; thus $\ker\varphi_0$ is finite. Lemma 2(ii) can now be used to give the result.

3. Groups satisfying Min or Max-n. Let Π be a set of primes. We say that an integer $m > 0$ is a Π-number if each prime divisor of m belongs to Π. In particular, if Π is empty, we have $m = 1$.

The two following results generalize the well-known facts mentioned in the introduction: a group satisfying Min is co-hopfian and a group satisfying Max-n is hopfian.

Proposition 1. *Let G be a group satisfying* Min *and let φ be an endomorphism such that* $\ker\varphi$ *is finite. Then* $|G : \varphi(G)|$ *is a (finite)* Π-*number, where Π is the set of primes dividing the order of* $\ker\varphi$.

Proof. Arguing as in the proof of Lemma 5 (i), it is easy to see that $\ker\varphi^n$ is a finite Π-group for any integer $n > 0$. Lemma 5 shows that we have $G = \varphi(G)\ker\varphi^m$ for some positive integer m. This implies the relation $|G : \varphi(G)| = |\ker\varphi^m : \varphi(G) \cap \cap \ker\varphi^m|$ and the result follows.

Proposition 2. *Let G be a group satisfying* Max-n *and let φ be an endomorphism such that the index $|G : \varphi(G)|$ is finite. Then* $\ker\varphi$ *is a finite* Π-*group, where Π is the set of primes dividing* $|G : \varphi(G)|$.

To prove this result, we need two further lemmas.

Lemma 7. *Let G be a group and let φ be an endomorphism. Then, if* $\ker\varphi^m = \ker\varphi^{m+1}$ *for some positive integer m, we have* $\varphi^m(G) \cap \ker\varphi^m = \{1\}$.

Proof. For any element $x \in \varphi^m(G) \cap \ker\varphi^m$, we have $x = \varphi^m(x')$ (for some $x' \in G$) and $\varphi^m(x) = 1$. It follows that $\varphi^{2m}(x') = 1$ and so x' belongs to $\ker\varphi^{2m}$. But $\ker\varphi^{2m} = \ker\varphi^m$, thus $\varphi^m(x') = x = 1$, as required.

If φ is an endomorphism of a group G such that the index $|G : \varphi(G)|$ is finite, it is easy to see that $|G : \varphi^n(G)|$ is finite as well, and that we have the inequality

$|G: \varphi^n(G)| \le |G: \varphi(G)|^n$. But to prove Proposition 2, a more precise result is necessary.

Lemma 8. *Let G be a group and let φ be an endomorphism such that the index $|G: \varphi(G)|$ is finite. Then, for each integer $n > 0$:*

(i) $|G: \varphi^{n+1}(G)| = |G: \varphi^n(G)| \times |G: \varphi(G)\ker\varphi^n|$;

(ii) $|G: \varphi^n(G)|$ *divides* $|G: \varphi(G)|^n$.

Proof. (i) Denote by r and s the respective indices $|G: \varphi^n(G)|$ and $|G: \varphi(G)\ker\varphi^n|$. Thus there exist elements $a_1, \ldots, a_r, b_1, \ldots, b_s \in G$ such that

$$G = \bigcup_{i=1,\ldots,r} a_i\varphi^n(G) = \bigcup_{j=1,\ldots,s} b_j\varphi(G)\ker\varphi^n.$$

Consider an element $x \in G$. It is of the form $x = a_i\varphi^n(x')$, with $x' \in G$. Since $G = \bigcup_{j=1,\ldots,s} b_j\,\varphi(G)\ker\varphi^n$, the element x' can be written in the form

$$x' = b_j\varphi(x'')y, \quad x'' \in G, \quad y \in \ker\varphi^n.$$

It follows the expression $x = a_i\varphi^n(b_j)\varphi^{n+1}(x'')$, and so G is the union of (at most) rs left cosets $a_i\varphi^n(b_j)\varphi^{n+1}(G)$, where i and j range $\{1, \ldots, r\}$ and $\{1, \ldots, s\}$ respectively. It remains to prove that these cosets are distinct, namely that a relation of the form

$$a_i\varphi^n(b_j) = a_{i'}\varphi^n\left(b_{j'}\right)\varphi^{n+1}(z), \quad z \in G,$$

implies the equalities $i = i'$ and $j = j'$. First observe that the relation implies that $a_{i'}^{-1}a_i$ belongs to $\varphi^n(G)$, so $i = i'$. It follows that $\varphi^n(b_j) = \varphi^n\left(b_{j'}\right)\varphi^{n+1}(z)$ and hence we obtain $\varphi^n\left(\varphi(z)^{-1}b_{j'}^{-1}b_j\right) = 1$. This shows that $b_{j'}^{-1}b_j$ belongs to $\varphi(G)\ker\varphi^n$. But $b_1, \ldots, b_s$ is a left transversal to $\varphi(G)\ker\varphi^n$ in G, so $j = j'$, as required.

(ii) Since $|G: \varphi(G)\ker\varphi^n(G)|$ divides $|G: \varphi(G)|$, the proof follows from (i) by induction on n.

***Proof of Proposition* 2.** There exists an integer $m > 0$ such that $\ker\varphi^m = \ker\varphi^{m+1}$ since G satisfies Max-n. If Π is the set of primes dividing $|G: \varphi(G)|$, it follows from Lemma 8 (ii) that $|G: \varphi^m(G)|$ is a Π-number. The index $|\varphi^m(G)\ker\varphi^m : \varphi^m(G)|$ is a Π-number as well for it divides $|G: \varphi^m(G)|$. But $\varphi^m(G) \cap \ker\varphi^m = \{1\}$ (Lemma 7), thus $|\varphi^m(G)\ker\varphi^m : \varphi^m(G)|$ coincide with the order of $\ker\varphi^m$, which is so a Π-group. Since $\ker\varphi$ is a subgroup of $\ker\varphi^m$, the proof is complete.

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