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A HIGH DIMENSIONAL VERSION OF THE BRODY REPARAMETRIZATION LEMMA

БАГАТОВИМІРНИЙ ВАРІАНТ ЛЕМИ БРОДІ ПРО РЕПАРАМЕТРИЗАЦІЮ

We prove the generalization of the Brody reparametrization lemma.

Доведено узагальнення леми Броді про репараметризацію.

1. Introduction. In [1, 2], S. Kobayashi introduced, for every complex manifold M with tangent bundle TM, a pseudodistance

$$d_M\colon M\times M\to[0,+\infty)$$

and a pseudometric

$$F_M\colon TM\to[0,\infty)$$

now called the Kobayashi pseudodistance and Kobayashi differential metric of M, respectively (see [3]). If d_M is a distance, M is called hyperbolic in the sense of Kobayashi.

In [4], R. Brody considered the hyperbolicity on compact complex space and obtained the following theorem:

Let M be a compact complex space. Then M is Kobayashi hyperbolic iff every $f\in\operatorname{Hol}(\mathbb{C},M)$ is constant.

The key of the Brody demonstration is the reparametrization lemma. Generalizing this lemma, we prove the following theorem:

Theorem. *Let M be a complex manifold of dimension n and $\langle\cdot,\cdot\rangle$ be an Hermitian metric on Λ^kTM. Suppose $f\colon B^k(r)\to M$ is a holomorphic mapping such that $H(f'(0))\ge c>0$, where $0=(0,0,\dots,0)\in B^k(r)$. Then there exists a holomorphic mapping $g: B^k(r)\to M$ satisfying the following conditions*:

1) $H(g'(0)) = H\left(g_*\dfrac{\partial}{\partial z^1}\wedge\ldots\wedge g_*\dfrac{\partial}{\partial z^k}(0)\right) = \dfrac{c}{2}$;

2) $\dfrac{H(g'(z))}{\eta_r(z)} \le \dfrac{c}{2}$ *for all* $z\in B^k(r)$, *where* $\eta_r(z) = \left(\dfrac{r^2}{r^2-\|z\|^2}\right)^{(k+1)/2}$, $\|z\|$ *is Euclidean norm on* $\mathbb{C}^k$;

3) $g(B^k(r))\subset f(B^k(r))$.

2. Basic definitions. In this section, we recall the basic definitions needed for the succeeding sections.

2. 1. Let M be a complex manifold of dimension n and $p\in M$. We denote by T_pM (resp. TM) the holomorphic space tangent to M at p (resp. the holomorphic tangent bundle). Let Λ^kT_pM be the k^{th} exterior power of T_pM. We denote by D_p^kM the decomposable elements of Λ^kT_pM, i. e., the elements of type $\alpha=v_1\wedge\ldots$ $\ldots\wedge v_k\in\Lambda^kT_pM$, where $\dim\operatorname{span}_{\mathbb{C}}\{v_1,\dots,v_k\}=k$.

If $\langle\cdot,\cdot\rangle$ is an Hermitian metric in TM, it can be extended to an Hermitian metric on Λ^kTM as follows: for $\alpha,\beta\in D_p^kM$ with $\alpha=v_1\wedge\ldots\wedge v_k$, $\beta=w_1\wedge\ldots\wedge w_k$, set

$$\langle \alpha, \beta \rangle = \det \left\{ \left\langle v_i, w_j \right\rangle \right\},$$

$i, j = 1, \ldots, k$, and extend this definition to arbitrary elements of $\Lambda^k T_p M$ by linearity (see [5]). Write $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. Denote $\|\alpha\|$ also by $H(\alpha)$ or $H(p, \alpha)$ for $\alpha \in M$.

2. 2. Let $B^k(r) = \left\{ z \in \mathbb{C}^k : \|z\| < r \right\}$, where $\|\cdot\|$ is the Euclidean norm. The Bergman metric on $B^k(r)$ is defined by

$$ds^2 = \sum_{i,j=1}^{k} \frac{\overline{z}^i \overline{z}^j + \left(r^2 - \|z\|^2\right)\delta_{ij}}{\left(r^2 - \|z\|^2\right)^2} dz^i \, d\overline{z}^j.$$

This is an Hermitian metric on $B^k(r)$. For $u, v \in T_z B^k(r)$, the Hermitian product of u with v will be written $\langle u, v \rangle_z$.

We denote by $\operatorname{Aut}(B^k(r))$ the group of automorphisms (biholomorphic) of $B^k(r)$.

2. 3. For $a \in B^k(r)$, we define the $k \times k$ matrix, $\Gamma_r(a)$, by

$$\Gamma_r(a) = \frac{a\, {}^t\overline{a}}{r - v_r(a)} - v_r(a) I,$$

where a is regarded as a column matrix, $v_r(a) = \sqrt{r^2 - \|a\|^2}$, I is unit matrix. In the case where $r = 1$, we write $v(a)$ instead of $v_1(a)$ for simplicity.

2. 4. **Definition.** *For* $a \in B^k(r)$, *we define* $g_a^r : B^k(r) \to B^k(r)$ *by*

$$g_a^r(z) = r \Gamma_r(a) \frac{z - a}{r^2 - {}^t\overline{a} z}, \quad z \in B^k(r).$$

When $r = 1$, we write $\Gamma(a)$ instead of $\Gamma_1(a)$, $g_a(z)$ instead of $g_a^1(z)$ for simplicity.

3. Some results about the group $\operatorname{Aut}(B^k(r))$. In [6], some facts about the group of automorphisms of $B^k = \left\{ z \in \mathbb{C}^k : \|z\| < 1 \right\}$ are given. In this section, we prove some facts important for us about the group $\operatorname{Aut}(B^k(r))$.

3. 1. We have $\Gamma_r(a) = r\Gamma(a/r)$. It is obvious by Definition 2.3.

3. 2. For $a, z \in B^k(r)$, the following equality holds:

$$g_a^r(z) = r g_{a/r}\left(\frac{z}{r}\right).$$

Indeed,

$$r g_{a/r}\left(\frac{z}{r}\right) = r^2 \Gamma\left(\frac{a}{r}\right) \frac{z - a}{r^2 - {}^t\overline{a} z} = r \Gamma_r(a) \frac{z - a}{r^2 - {}^t\overline{a} z} = g_a^r(z).$$

We have $g_a^r \in \operatorname{Aut}(B^k(r))$ and $g_a^r(a) = 0$, $g_a^r(0) = r g_{a/r}(0) = r(-a/r) = -a$ (by [6, p. 6]).

This implies that the group $\operatorname{Aut}(B^k(r))$ acts transitively on $B^k(r)$.

3. 3. We have

$$\operatorname{Aut} B^k(r) = \left\{ A g_a^r : A \in \mathcal{U}(k),\ a \in B^k(r) \right\},$$

where $\mathcal{U}_k$ is the unitary group.

Indeed, it is obvious that $\mathrm{Aut}(B^k(r))$ and $\mathrm{Aut}(B^k)$ are isomorphic, furthermore by [6, p. 6],

$$\mathrm{Aut}(B^k) = \{A \cdot g_a : A \in \mathcal{U}_k,\ g_a \text{ defined in 2.4}\},$$

whence the required assertion follows.

3. 4. We have

$$\Gamma_r(a)a = ra \quad \text{for} \quad a \in B^k(r).$$

In fact,

$$\Gamma_r(a)a = r\Gamma\left(\frac{a}{r}\right)a = r^2\Gamma\left(\frac{a}{r}\right)\frac{a}{r} = r^2\frac{a}{r} = ra \quad \text{(see [6, p. 6])}.$$

3. 5. We have

$${}^t\overline{\Gamma_r(a)} = \Gamma_r(a),$$

hence ${}^t\overline{a}\Gamma_r(a) = r\,{}^t\overline{a}$. In fact,

$${}^t\overline{\Gamma_r(a)} = \overline{{}^t r\Gamma\left(\frac{a}{r}\right)} = r\,\overline{{}^t\Gamma\left(\frac{a}{r}\right)} = r\Gamma\left(\frac{a}{r}\right) \text{ (by [6, p. 6])} = \Gamma_r(a) \text{ (by 3. 1)}.$$

Furthermore,

$${}^t\overline{a}\Gamma_r(a) = {}^t\overline{a}\, r\Gamma\left(\frac{a}{r}\right) = r^2\frac{{}^t\overline{a}}{r}\Gamma\left(\frac{a}{r}\right) = r^2\frac{{}^t\overline{a}}{r} = r\,{}^t\overline{a}$$

(from [6, p. 6]).

3. 6. We have

$$\Gamma_r(a)^2 = (r - \upsilon_r(a))\Gamma_r(a) + r\upsilon_r(a)I = a\,{}^t\overline{a} + \upsilon_r(a)^2 I.$$

In fact,

$$\Gamma_r(a)^2 = r^2\Gamma\left(\frac{a}{r}\right)^2 = r^2\left[\left(1 - \upsilon\left(\frac{a}{r}\right)\right)\Gamma\left(\frac{a}{r}\right) + \upsilon\left(\frac{a}{r}\right)I\right] \text{ [6, p. 6]} =$$

$$= (r - \upsilon_r(a))\, r\Gamma\left(\frac{a}{r}\right) + r\upsilon_r(a)I = (r - \upsilon_r(a))\Gamma_r(a) + r\upsilon_r(a)I$$

and

$$\Gamma_r(a)^2 = r^2\Gamma\left(\frac{a}{r}\right)^2 = r^2\left(\frac{a}{r}\frac{{}^t\overline{a}}{r} + \upsilon\left(\frac{a}{r}\right)^2 I\right) = a\,{}^t\overline{a} + r^2\upsilon\left(\frac{a}{r}\right)^2 I = a\,{}^t\overline{a} + \upsilon_r(a)^2 I$$

(by [6, p. 6]).

3. 7. We have

$$\Gamma_r(a)^{-1} = \frac{1}{r\upsilon_r(a)}(\Gamma_r(a) + (\upsilon_r(a) - r)I) = \frac{1}{r\upsilon_r(a)}\left(\frac{a\,{}^t\overline{a}}{r - \upsilon_r(a)} - rI\right).$$

Proof. To compute $\Gamma_r(a)^{-1}$, we have

$$\Gamma_r(a)^{-1} = \left(r\Gamma\left(\frac{a}{r}\right)\right)^{-1} = \frac{1}{r}\Gamma\left(\frac{a}{r}\right)^{-1} =$$

$$= \frac{1}{r}\frac{1}{\upsilon(a/r)}\left(\Gamma\left(\frac{a}{r}\right) + \left(\upsilon\left(\frac{a}{r}\right) - 1\right)I\right) = \frac{1}{\upsilon_r(a)}\left(\frac{1}{r}\Gamma_r(a) + \frac{1}{r}(\upsilon_r(a) - r)I\right) =$$

$$= \frac{1}{r\upsilon_r(a)}(\Gamma_r(a) + (\upsilon_r(a) - r)I).$$

Furthermore, again from [6, p. 6], we have

$$\Gamma_r(a)^{-1} = \frac{1}{r}\Gamma\left(\frac{a}{r}\right)^{-1} = \frac{1}{rv(a/r)}\left(\frac{a\,{}^t\overline{a}}{r^2(1-v(a/r))} - I\right) =$$

$$= \frac{1}{v_r(a)}\left(\frac{a\,{}^t\overline{a}}{r(r-v_r(a))} - I\right) = \frac{1}{rv_r(a)}\left(\frac{a\,{}^t\overline{a}}{r-v_r(a)} - rI\right).$$

3. 8. We have

$$\Gamma_r(a)^{-2} = \frac{1}{r^3 v_r(a)}\left(-a\,{}^t\overline{a} + r^2 I\right).$$

In fact,

$$\Gamma_r(a)^{-2} = \frac{1}{r^2}\Gamma\left(\frac{a}{r}\right)^{-2} = \frac{1}{r^2}\frac{1}{v(a/r)}\left(-\frac{a\,{}^t\overline{a}}{r^2} + I\right) = \frac{1}{r^3 v_r(a)}\left(-a\,{}^t\overline{a} + r^2 I\right).$$

3. 9. We have

$$\det\Gamma_r(a) = r(-v_r(a))^{k-1} = r\left(-\sqrt{r^2 - \|a\|^2}\right)^{k-1}.$$

Indeed,

$$\det\Gamma_r(a) = \det\left(r\Gamma\left(\frac{a}{r}\right)\right) = r^k \det\Gamma\left(\frac{a}{r}\right) = r^k\left(-v\left(\frac{a}{r}\right)\right)^{k-1} =$$

$$= r^k\left(-\sqrt{1-\left\|\frac{a}{r}\right\|^2}\right)^{k-1} = r\left(-\sqrt{r^2 - \|a\|^2}\right)^{k-1}.$$

3. 10. We have

$$g^r_{Aa} = A\, g^r_a A^{-1} \quad \text{for} \quad A \in \mathcal{U}_k.$$

Proof. To compute g^r_{Aa}, we have

$$g^r_{Aa}(z) = r\, g_{Aa/r}\left(\frac{z}{r}\right) = r\, g_{A(a/r)}\left(\frac{z}{r}\right) = r A\, g_{a/r} A^{-1}\left(\frac{z}{r}\right) =$$

$$= A\left(r\, g_{a/r}\left(\frac{1}{r}A^{-1}(z)\right)\right) = A\, g^r_a A^{-1}(z),$$

hence $g^r_{Aa} = A\, g^r_a A^{-1}$.

3. 11. We have

$$d\left(g^r_a\right)_a = \frac{r\Gamma_r(a)}{v_r(a)^2} = \frac{r}{r^2 - \|a\|^2}\Gamma_r(a),$$

where $d\left(g^r_a\right)_a$ is the Jacobian matrix of the g^r_a at a.

In fact,

$$g^r_a(z) = r\, g_{a/r}\left(\frac{z}{r}\right) = r\, g_{a/r}\, h(z),$$

where $h = \frac{1}{r}\mathrm{id}\left(B^k(R)\right)$, hence

$$d\left(g^r_a\right)_a = r d\left(g_{a/r}\, h\right)_{(a)} = r\left(d\, g_{a/r}\right)_{h(a)} dh_{(a)} =$$

$$= \left(d g_{a/r}\right)_{h(a)} = \frac{\Gamma(a/r)}{1-\|a/r\|^2} = \frac{1}{r^2-\|a\|^2} r^2 \Gamma(a/r) =$$

$$= \frac{r}{r^2-\|a\|^2}\Gamma_r(a) \quad \text{(by 3.1 and [6, p. 7])}.$$

***3. 12.* Proposition.** *Let*

$$u = \sum_{j=1}^{k} a^j \frac{\partial}{\partial z^j}, \qquad v = \sum_{j=1}^{k} b^j \frac{\partial}{\partial z^j}$$

be tangent vectors of $B^k(r)$ *at the point* z; $u, v \in T_z B^k(r)$. *Then*

$$\langle u, v\rangle_z = \frac{1}{r^2}\left[a\left(dg_z^r\right)_z\right]\left[\overline{{}^t b\left(dg_z^r\right)_z}\right] = \left\langle g_{z*}^r(u), g_{z*}^r(v)\right\rangle_0,$$

where $\langle u, v\rangle_z$ *is the Hermitian product of* u *with* v *with respect to* ds^2, $a = (a^j)$, $b=(b^j)$ *are* $1\times k$ *matrix,* $z=(z^j)$ *is the column vector.*

Proof. On the one hand,

$$\langle u, v\rangle_z = ds^2(u, v) = \sum_{i,j=1}^{k} \frac{\bar z^i z^j + \left(r^2-\|z\|^2\right)\delta_{ij}}{\left(r^2-\|z\|^2\right)^2} a^i \bar b^j =$$

$$= \frac{1}{\left(r^2-\|z\|^2\right)^2}\left(\sum_{i,j=1}^{k} \bar z^i z^j a^i \bar b^j + \left(r^2-\|z\|^2\right) a\, {}^t\bar b\right), \tag{1}$$

on the other hand,

$$a\left(dg_z^r\right)_z \overline{{}^t b\left(dg_z^r\right)_z} = a\left(dg_z^r\right)_z \overline{{}^t\left(dg_z^r\right)_z}\, {}^t\bar b =$$

$$= a\left(\frac{r}{r^2-\|z\|^2}\right)^2 \Gamma_r(z)\overline{{}^t\Gamma_r(z)}\, {}^t\bar b \;\text{(by 3. 11)} = \frac{r^2}{\left(r^2-\|z\|^2\right)^2} a\Gamma_r^2(z)\, {}^t\bar b \;\text{(by 3. 5)} =$$

$$= \frac{r^2}{\left(r^2-\|z\|^2\right)^2} a\left(z\, {}^t\bar z + \left(r^2-\|z\|^2\right)I\right) {}^t\bar b \;\text{(by 3. 6)} =$$

$$= \frac{r^2}{\left(r^2-\|z\|^2\right)^2}\left[a z\, {}^t\bar z\, {}^t\bar b + \left(r^2-\|z\|^2\right) a\, {}^t\bar b\right] =$$

$$= \frac{r^2}{\left(r^2-\|z\|^2\right)^2}\left[\sum_{i,j=1}^{k} a^i z^i \bar z^j \bar b^j + \left(r^2-\|z\|^2\right) a\, {}^t\bar b\right]. \tag{2}$$

From (1) and (2), we have

$$\langle u, v\rangle_z = \frac{1}{r^2}\left[a\left(dg_z^r\right)_z\right]\left[\overline{{}^t b\left(dg_z^r\right)_z}\right],$$

furthermore,

$$g_{z*}^r(u) = \sum_{i,j=1}^{k} a^j g_{z*}^r\left(\frac{\partial}{\partial z^j}\right) = a\left(dg_z^r\right)_z, \qquad g_{z*}^r(v) = \sum_{j=1}^{k} b^j g_{z*}^r\left(\frac{\partial}{\partial z^j}\right) = b\left(dg_z^r\right)_z,$$

hence

$$\left\langle g_{z*}^r(u), g_{z*}^r(v)\right\rangle_0 = ds^2\left\langle g_{z*}^r(u), g_{z*}^r(v)\right\rangle_0 =$$

$$= \frac{1}{r^2}\left(\sum_{i,j=1}^{k} dz^i d\bar z^j\right)\left(g_{z*}^r(u), g_{z*}^r(v)\right)_0 = \frac{1}{r^2}\left(a\left(dg_z^r\right)_z\right)\overline{{}^t b\left(dg_z^r\right)_z} = \langle u, v\rangle_z.$$

3. 13. **Proposition.** *For every* $h \in \operatorname{Aut}(B^k(r))$, *we have that* h *is isometric with respect to the Begman metric* ds^2 *on* $B^k(r)$.

Proof. We prove that $h^* ds^2 = ds^2$, i.e., for $u, v \in T_z(B^k(r))$, we have

$$\langle h_*(u), h_*(v) \rangle_{h(z)} = \langle u, v \rangle_z.$$

In fact, let $w = h(z)$. Then $g_w^r(w) = 0$ by definition, so $g_w^r h = A g_z^r$ for some $A \in \mathcal{U}_k$ (since $g_w^r h \in \operatorname{Aut}(B^k(r))$) and by 3.3.

Thus, we have

$$\langle h_*(u), h_*(v) \rangle_w = \left\langle g_{w*}^r h_*(u), g_{w*}^r h_*(v) \right\rangle_0 = \left\langle A_* g_{z*}^r(u), A_* g_{z*}^r(v) \right\rangle_0 =$$

$$= \left\langle g_{z*}^r(u), g_{z*}^r(v) \right\rangle_0 = \langle u, v \rangle_z \quad \text{(by 3. 12)}.$$

The proof of the following proposition is omitted since it is evident.

3. 14. **Proposition.** *Let*

$$u_j = \sum_{i=1}^{k} a_j^i \frac{\partial}{\partial z^i}$$

be a tangent vector of $B^k(r)$ *at the point* O, $u_j \in T_0(B^k(r))$, $j = 1, \ldots, k$. *Then*

$$\det\left(\left\langle u_i, u_j \right\rangle_0\right) = \det\left(A\, {}^t\overline{A}\right),$$

where $A = \left(a_j^i\right) \in \operatorname{Mat}(k, \mathbb{C})$.

4. A high dimensional version of the Brody reparametrization lemma. Recall that, for an Hermitian metric $\langle u, v \rangle$ in the holomorphic tangent bundle TM of a complex manifold M, the Hermitian metric extended on $\Lambda^k TM$ is denoted also by $\langle \cdot, \cdot \rangle$. For $\alpha \in \Lambda^k TM$, denote $\| \alpha \| = \sqrt{\langle \alpha, \alpha \rangle}$ by $H(p, \alpha)$ or $H(\alpha)$ for simplicity. For $f \in \operatorname{Hol}(B^k(r), M)$ and $z \in B^k(r)$, we put

$$H(f'(z)) = H\left(f_*\left(\frac{\partial}{\partial z^1} \wedge \ldots \wedge \frac{\partial}{\partial z^k}(z) \right)\right) = H\left(f_*\left(\frac{\partial}{\partial z^1} \right) \wedge \ldots \wedge f_*\left(\frac{\partial}{\partial z^k} \right) z \right).$$

We have the following theorem:

Theorem. *Let* M *be a complex manifold of dimension* n *and* $\langle \cdot, \cdot \rangle$ *be an Hermitian metric on* $\Lambda^k TM$. *Suppose* $f \colon B^k(r) \to M$ *is a holomorphic mapping such that* $H(f'(0)) \geq c > 0$, *where* $0 = (0, \ldots, 0) \in B^k(r)$. *Then there exists a holomorphic mapping* $g : B^k(r) \to M$ *satisfying the following conditions:*

1) $H(g'(0)) = H\left(g_* \dfrac{\partial}{\partial z^1} \wedge \ldots \wedge g_* \dfrac{\partial}{\partial z^k}(0) \right) = \dfrac{c}{2}$;

2) $\dfrac{H(g'(z))}{\eta_r(z)} \leq \dfrac{c}{2}$ *for all* $z \in B^k(r)$, *where* $\eta_r(z) = \left(\dfrac{r^2}{r^2 - \| z \|^2} \right)^{(k+1)/2}$, $\| z \|$ *is a Euclidean norm on* $\mathbb{C}^k$;

3) $g(B^k(r)) \subset f(B^k(r))$.

Proof. For $0 \leq t \leq 1$, define holomorphic mappings $f_t \colon B^k(r) \to M$ by $f_t(z) = f(tz)$. Since $f_{t*} \dfrac{\partial}{\partial z^j}(z) = t f_* \dfrac{\partial}{\partial z^j}(tz)$, then

$$H(f_t'(z)) = t^k H(f'(tz))$$

and

$$\frac{H(f_t'(z))}{\eta_r(z)} = t^k \left(\frac{r^2 - \|z\|^2}{r^2 - \|tz\|^2} \right)^{(k+1)/2} \frac{H(f'(tz))}{\eta_r(tz)}. \tag{3}$$

Now we put $\mu(t) = \sup_{z \in B^k(r)} \frac{H(f_t'(z))}{\eta_r(z)}$. The function $\mu(t)$ has the following properties for $t \in [0, 1)$:

(a) $0 \le \mu(t) < +\infty$;

(b) $\mu(t)$ is continuous on $[0, 1)$;

(c) $\mu(t)$ is an increasing function;

(d) $\mu(0) = 0$ and $\mu(t) \ge \frac{H(f_t'(0))}{\eta_r(0)} > tc$.

Now we shall prove these properties.

For (a), $\mu(t) \ge 0$ is evident. Furthermore, by (3) for fixed $t \in [0, 1)$, $\frac{H(f_t'(z))}{\eta_r(z)}$ is continuous on $\overline{B^k(r)}$ (a closure of $B^k(r)$), so

$$\mu(t) = \sup_{t \in B^k(r)} \frac{H(f_t'(z))}{\eta_r(z)} \le \sup_{t \in \overline{B^k(r)}} \frac{H(f_t'(z))}{\eta_r(z)} < +\infty.$$

Thus, (a) is proved.

For (b), since $\frac{H(f_t'(z))}{\eta_r(z)}$ is continuous on $B^k(r) \times [0, 1)$ by (3), $\mu(t)$ is continuous on $[0, 1)$ and (b) is proved.

In order to prove (c), we assume that $0 \le t_1 < t_2 < 1$, if there exists $z_1 \in \overline{B^k(r)}$ such that $\mu(t_1) = \frac{H\left(f_{t_1}'(z_1)\right)}{\eta_r(z_1)}$. Put $z_2 = \frac{t_1}{t_2} z_1$. We obtain

$$\mu(t_2) \ge \frac{H\left(f_{t_2}'(z_2)\right)}{\eta_r(z_2)} = t_2^k \left(\frac{r^2 - \|z_2\|^2}{r^2 - \|t_2 z_2\|^2} \right)^{(k+1)/2} \frac{H(f'(t_2 z_2))}{\eta_r(t_2 z_2)} =$$

$$= t_2^k \left(\frac{r^2 - t_1 \|z_2\|^2 / t_2}{r^2 - \|t_1 z_1\|^2} \right)^{(k+1)/2} \frac{H(f'(t_1 z_1))}{\eta_r(t z_1)} \ge$$

$$\ge t_1^k \left(\frac{r^2 - \|z_1\|^2}{r^2 - \|t_1 z_1\|^2} \right)^{(k+1)/2} \frac{H(f'(t_1 z_1))}{\eta_r(t_1 z_1)} = \frac{H\left(f_{t_1}'(z_1)\right)}{\eta_r(z_1)} = \mu(t_1).$$

Thus, $\mu(t)$ is an increasing function and (c) is proved.

Furthermore, $\mu(0) = 0$ is evident by (3),

$$\mu(t) = \sup_{t \in B^k(r)} \frac{H(f_t'(z))}{\eta_r(z)} \ge \frac{H(f_t'(0))}{\eta_r(0)} = H(f_t'(0)) \ge c > tc, \qquad 0 \le t < 1.$$

Then (d) is proved.

Therefore, we have that $\lim_{t \to 1} \mu(t) \ge c$, $\mu(0) = 0$ and $\mu(t)$ is continuous on $[0, 1)$. By the intermediate-value theorem, there is a number $0 < t_0 < 1$ such that $\mu(t_0) = c/2$. Since $\lim_{\|z\| \to r} \eta_r(z) = +\infty$ and $H\left(f_{t_0}'(z)\right)$ is continuous for z on $\overline{B^k(r)}$, there exists a number $M>0$ such that $H\left(f'_{t_0}(z)\right)\le M\ \ \forall z\in B^k(r)$ and there is a point $z_0\in B^k(r)$ such that

$$\frac{c}{2}=\mu(t_0)=\frac{H\left(f'_{t_0}(z_0)\right)}{\eta_r(z_0)}.$$

Since the group $\operatorname{Aut}(B^k(r))$ acts transitively on $B^k(r)$ (see 3.2), there is a holomorphic transformation $h\in\operatorname{Aut}(B^k(r))$ with $h(0)=z_0$. Put $g=f_{t_0}h$. We prove that g satisfies the properties claimed in theorem.

By Proposition 3.13, $h^*ds^2=ds^2$, so for all $u,v\in T_zB^k(r)$, we have

$$\langle h_*(u),h_*(v)\rangle_{h(z)}=\langle u,v\rangle_z\Rightarrow$$

$$\Rightarrow\left\langle g^r_{h(z)_*}h_*(u),g^r_{h(z)_*}h_*(v)\right\rangle_0=\left\langle g^r_{z*}(u),g^r_{z*}(v)\right\rangle_0\ \text{by 3.13}\Rightarrow$$

$$\Rightarrow\left\langle g^r_{h(z)_*}h_*\left(\frac{\partial}{\partial z^i}\right),g^r_{h(z)_*}h_*\left(\frac{\partial}{\partial z^j}\right)\right\rangle_0=\left\langle g^r_{z*}\frac{\partial}{\partial z^i},g^r_{z*}\frac{\partial}{\partial z^j}\right\rangle_0\quad\text{for}\quad i,j=1,\ldots,k$$

$$\Rightarrow\det\left\{\left\langle g^r_{h(z)_*}h_*\left(\frac{\partial}{\partial z^i}\right),g^r_{h(z)_*}h_*\left(\frac{\partial}{\partial z^j}\right)\right\rangle_0\right\}=\det\left\{\left\langle g^r_{z*}\frac{\partial}{\partial z^i},g^r_{z*}\frac{\partial}{\partial z^j}\right\rangle_0\right\}.$$

By 3.14, we have

$$\det\left[d\left(g^r_{h(z)}h\right)_z\overline{{}^td\left(g^r_{h(z)}h\right)_z}\right]=\det\left[\left(dg^r_z\right)_z\overline{{}^t\left(dg^r_z\right)_z}\right],$$

where $\left(dg^r_z\right)_z$, $d\left(g^r_{h(z)}h\right)_z$ are Jacobian matrix at z of the mappings g^r_z, $g^r_{h(z)}h$ respectively. Furthermore,

$$\det d\left(g^r_{h(z)}h\right)_z=\det\left(dg^r_{h(z)}\right)_{h(z)}\det(dh)_z,$$

$$\det\overline{{}^td\left(g^r_{h(z)}h\right)_z}=\det\overline{{}^t(dh)_z}\det\overline{{}^t\left(dg^r_{h(z)}\right)_{h(z)}}.$$

Then

$$\det\left[\left(dg^r_{h(z)}\right)_{h(z)}\overline{{}^t\left(dg^r_{h(z)}\right)_{h(z)}}\right]\det\left[(dh)_z\overline{{}^t(dh)_z}\right]=\det\left[\left(dg^r_z\right)_z\overline{{}^t\left(dg^r_z\right)_z}\right].$$

By 3.11

$$\det\left(\frac{r\Gamma_r(h(z))}{\upsilon_r(h(z))^2}\frac{\overline{{}^tr\Gamma_r(h(z))}}{\upsilon_r(h(z))^2}\right)|\det(dh)_z|^2=\det\left(\frac{r\Gamma_r(z)}{\upsilon_r(z)^2}\frac{\overline{{}^tr\Gamma_r(z)}}{\upsilon_r(z)^2}\right)\Rightarrow$$

$$\Rightarrow\frac{1}{\upsilon_r(h(z))^{4k}}(\det\Gamma_r(h(z)))^2|\det(dh)_z|^2=\frac{1}{\upsilon_r(z)^{4k}}(\det\Gamma_r(z))^2\ \text{by 3.5}\Rightarrow$$

$$\Rightarrow\frac{1}{\left(r^2-\|h(z)\|^2\right)^{k+1}}|\det(dh)_z|^2=\frac{1}{\left(r^2-\|z\|^2\right)^{k+1}}\quad\text{by 3.9.}$$

Then we have

$$\eta_r(h(z))\det|(dh)_z|=\eta_r(z).\tag{4}$$

On the other hand,

$$H(g'(z))^2=H\left(\left(f_{t_0}h\right)'_z\right)^2=\det\left\{\left\langle\left(f_{t_0}h\right)_*\frac{\partial}{\partial z^i}(z),\left(f_{t_0}h\right)_*\frac{\partial}{\partial z^j}(z)\right\rangle\right\}_{i,j=\overline{1,k}}=$$

$$= \det d\left(f_{t_0}h\right)_z {}^t\overline{d\left(f_{t_0}h\right)_z} \quad \text{by } 3.15 =$$

$$= \det\left(\left(d f_{t_0}\right)_{h(z)}(dh)_z {}^t\overline{(dh)_z}\, {}^t\overline{\left(d f_{t_0}\right)_{h(z)}}\right) =$$

$$= |\det(dh)_z|^2 \det\left(\left(d f_{t_0}\right)_{h(z)} {}^t\overline{\left(d f_{t_0}\right)_{h(z)}}\right) = |\det(dh)_z|^2 H\left(f'_{t_0}(h(z))\right)^2.$$

Thus,

$$H(g'(z)) = |\det(dh)_z| H\left(f'_{t_0}(h(z))\right). \tag{5}$$

From (4) and (5), we obtain

$$\frac{H(g'(z))}{\eta_r(z)} = \frac{H\left(f'_{t_0}(h(z))\right)}{\eta_r(h(z))} \leq \mu(t_0) = \frac{c}{2}$$

for all $z \in B^k(r)$.

Hence, property 2 is proved.

Furthermore,

$$H(g'(0)) = \eta_r(0)\frac{H\left(f'_{t_0}(z_0)\right)}{\eta_r(z_0)} = \frac{c}{2},$$

then property 1 is satisfied.

For $z \in B^k(r)$, we have

$$g(z) = f_{t_0}h(z) = f(t_0 h(z)) \in f\left(B^k(r)\right),$$

so that

$$g(B^k(r)) \subset f(B^k(r)).$$

Thus, the proof of the theorem is complete.

Note. When $k = 1$, this theorem is the Brody reparametrization lemma (see [3, p. 27]).

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