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R. Šleževičienė (Šiauliai Univ., Lithuania),
J. Steuding (Goethe-Univ. Frankfurt, Germany)

ON EXPONENTIAL SUMS RELATED TO THE CIRCLE PROBLEM

ПРО ЕКСПОНЕНЦІАЛЬНІ СУМИ, ЩО ПОВ'ЯЗАНІ З ПРОБЛЕМОЮ КОЛА

Let $r(n)$ count the number of representations of a positive integer n as a sum of two integer squares. We prove a truncated Voronoi-type formula for the twisted Möbius transform

$$\sum_{n \le x} r(n) \exp\left(2\pi i \frac{nk}{4l}\right),$$

where k and l are positive integers such that k and $4l$ are coprime, and give some applications (almost periodicity, limit distribution, an asymptotic mean-square formula, O- and Ω-estimates for the error term).

Нехай функція $r(n)$ підраховує кількість зображень додатного цілого числа n у вигляді суми двох цілих квадратів. Доведено відсічену формулу типу Вороного для скрученого перетворення Мьобіуса

$$\sum_{n \le x} r(n) \exp\left(2\pi i \frac{nk}{4l}\right),$$

де k та l — додатні цілі числа, такі, що k та $4l$ є взаємно простими. Наведено деякі застосування (майже періодичність, граничний розподіл, асимптотичну середньоквадратичну формулу, O- та Ω-оцінки для похибки).

1. A little bit history. About hundred years ago, the Ukrainian mathematician Georgy Fedoseevich Voronoi developed a powerful analytical method in the theory of numbers, by which he obtained an explicit expression for the error term in the divisor problem. For a nicely written and detailed survey on Voronoi"s challenging approach, we refer to the papers [1, 2] of Laurinčikas. In this section, we will only shortly present these and related results to motivate our object of study; the method will become clear in a later section.

Let n be a positive integer and denote by $d(n)$ the number of (positive) divisors of n; $d(n)$ is called the divisor function. The value distribution of $d(n)$ is rather complicated. On one side, it takes very small values, $d(n) = 2$ for prime n, and on the other side, one can construct integers n such that $d(n)$ becomes as large as we please. Hardy and Ramanujan [3] proved that

$$d(n) = (\log n)^{\log 2 + o(1)} \quad \text{for almost all } n;$$

this gives the so-called normal order of $d(n)$. Actually, the latter statement is a consequence of a celebrated result of Hardy and Ramanujan on the prime divisor-counting functions $\omega(n)$ (ignoring multiplicities) and $\Omega(n)$ (counting multiplicities),

$$\frac{1}{N} \sum_{n \le N} (f(n) - \log\log N)^2 \le \log\log N + O(1)$$

for $f(n) = \omega(n)$ and $f(n) = \Omega(n)$, and the trivial inequalities

$$2^{\omega(n)} \le d(n) \le 2^{\Omega(n)}.$$

Another less sophisticated order is the ordinary mean value. Dirichlet proved by his simple but ingenious hyperbola method that

$$\sum_{n \le x} d(n) = x \log x + (2\gamma - 1)x + O\left(x^{1/2}\right),$$

where $\gamma = 0{,}577\ldots$ is the Euler – Mascheroni constant. The so-called divisor problem asks for the best possible estimate for the error term

$$\Delta(x) := \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x.$$

The first steps beyond were done by Voronoi, and his approach and his results are still the basics of the currently best known estimates. By a complicated analytic method, Voronoi [4] obtained an explicit expression for the error term in the divisor problem in terms of certain Bessel functions, defined by

$$K_1(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{(z/2)^{2m+1}}{m!(m+1)!} \left\{ \log\frac{z}{2} - \frac{1}{2}(\Psi(m+1) + \Psi(m+2)) \right\}, \qquad (1)$$

$$Y_1(z) = -\frac{2}{\pi z} + \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+1}}{m!(m+1)!} \left\{ 2\log\frac{z}{2} - \Psi(m+1) - \Psi(m+2) \right\}, \qquad (2)$$

where $\Psi(z)$ is the logarithmic derivative of Euler's gamma-function $\Gamma(s)$. Voronoi proved that

$$\Delta(x) = \frac{1}{4} - \frac{2x^{1/2}}{\pi} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/2}} \left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2} Y_1(4\pi\sqrt{nx}) \right).$$

Furthermore, he obtained also for $N << x$ the truncated version

$$\Delta(x) = -\frac{2x^{1/2}}{\pi} \sum_{n \le N} \frac{d(n)}{n^{1/2}} \left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2} Y_1(4\pi\sqrt{nx}) \right) + O\left(x^{\varepsilon} + x^{1/2+\varepsilon} N^{-1/2}\right).$$

This is very useful for applications. For instance, taking into account the asymptotics of the involved Bessel functions (see formula (12) and (13) below) one gets via the choice $N = x^{1/3}$ the estimate

$$\Delta(x) << x^{1/3+\varepsilon},$$

while letting $N \to \infty$. The sharpest known upper estimate is $\Delta(x) << x^{23/73+\varepsilon}$ due to Huxley [5] (found by a different rather complicated method). On the contrary, Hardy was the first who deduced from Voronoi's truncated formula that the error term cannot be too small for all x; more precisely, there are infinitely many x such that $\Delta(x) >> x^{1/4}$ (actually, he showed a little bit more). We write $f(x) = \Omega(g(x))$ with a positive function $g(x)$ if $\limsup_{x\to\infty}(|f(x)|/g(x)) > 0$ (this is the negation of $f(x) = o(g(x))$). The present best Omega-result is due to Soundararajan [6] who recently proved

$$\Delta(x) = \Omega\left((x \log x)^{1/4} (\log\log x)^{3(\sqrt[3]{2^4}-1)/4} (\log\log\log x)^{-5/8} \right).$$

This improves slightly a celebrated bound of Hafner [7] (which still has the advantage that it gives estimates for both signs of the inequality in question what Soundararajan cannot control with his method); all these estimates rely on Voronoi's discovery. It is widely believed that the truth lies more close to the Omega-result.

There exists an extensive literature concerning generalizations of Voronoi's formula to other arithmetical functions (whose generating Dirichlet series satisfy Riemann-type functional equations). For instance, Peter [8] considered values of Dirichlet L-functions and, recently, Miller and Schmid [9] succeeded in the case of

$GL(2)$- and $GL(3)$-cusp forms. We shall focus on Jutila [10] who obtained a Voronoi-type formula for additive twists of the divisor function. More precisely, he proved

$$\sum_{n\le x} d(n)e\left(\frac{nk}{l}\right) = \frac{x}{l}(\log x+2\gamma-1-2\log k) -$$

$$- x^{1/2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/2}}\left\{\frac{2}{\pi}e\left(\frac{nk^*}{l}\right)K_1\left(\frac{4\pi\sqrt{nx}}{l}\right)+e\left(-\frac{nk^*}{l}\right)Y_1\left(\frac{4\pi\sqrt{nx}}{l}\right)\right\}, \qquad (3)$$

and for $N << x$ the truncated version

$$\sum_{n\le x} d(n)e\left(\frac{nk}{l}\right) =$$

$$= \frac{x}{l}(\log x+2\gamma-1-2\log k) - x^{1/2}\sum_{n\le N}\frac{d(n)}{n^{1/2}}\left\{\frac{2}{\pi}e\left(\frac{nk^*}{l}\right)K_1\left(\frac{4\pi\sqrt{nx}}{l}\right)+\right.$$

$$\left.+ e\left(-\frac{nk^*}{l}\right)Y_1\left(\frac{4\pi\sqrt{nx}}{l}\right)\right\} + O\left(lx^{1/2+\varepsilon}N^{-1/2}\right), \qquad (4)$$

where k and l are coprime integers, k^* is defined by $kk^*\equiv 1 \mod l$, and $e(z) = \exp(2\pi iz)$. These investigations were extended to the general divisor function

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

by Kiuchi [11] for real $\alpha\in(-1,0]$, and the second author [12] for complex α satisfying $|\alpha+1/2|<1/2$ and $|\alpha-1/6|<1/6$, respectively. The proofs of the corresponding Voronoi-type formulae rely in the main part on the analytic properties of the Estermann zeta-function (meromorphic continuation, functional equation).

It is our aim to study the same situation in the classic circle problem. Let $r(n)$ count the number of representations of n as a sum of two integer squares. Obviously, $r(n)=0$ if $n\equiv 3 \mod 4$, but as the divisor function $r(n)$ takes also arbitrarily large values. Gauss observed that the number of integer lattice points inside the disc of radius $\sqrt{x}$ centered at the origin is

$$\sum_{n\le x} r(n) = \pi x + O\left(x^{1/2}\right)$$

(by comparing the area of all unit squares centered at lattice points $(a,b)\in\mathbb{Z}^2$ satisfying $r(n)=a^2+b^2\le x$ with the area of the disc in question). The circle problem asks for the best possible estimate for the error term

$$\mathcal{P}(x) := \sum_{n\le x} r(n) - \pi x.$$

Actually, the circle problem is closely related to the divisor problem. More precisely, it can be rewritten into a more general divisor problem using the representation

$$r(n) = 4\sum_{d|n}\mathcal{X}(d), \qquad (5)$$

where

$$\mathcal{X}(d) = \begin{cases}(-1)^{(d-1)/2} & \text{if} \quad d\equiv 1 \bmod 2,\\ 0 & \text{if} \quad d\equiv 0 \bmod 2,\end{cases}$$

$\mathcal{X}$ is the nonprincipal character modulo 4 (and thus completely multiplicative). Hardy proved a Voronoi-type formula for $r(n)$, namely

$$\mathcal{P}(x) = -1 + \frac{x^{1/2}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}} J_1\left(2\pi\sqrt{nx}\right),$$

where J_1 is the Bessel function defined by

$$J_1(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+1}}{m!(m+1)!}.$$

A truncated version may be obtained following Voronoi's argument in the divisor problem and would lead to $\mathcal{P}(x) \ll x^{1/3+\varepsilon}$. But, as in the divisor problem, Huxley's estimate yields here $\mathcal{P}(x) \ll x^{23/73+\varepsilon}$. The current best known Omega-result is

$$\mathcal{P}(x) = \Omega\left((x\log x)^{1/4}(\log\log x)^{3(\sqrt[3]{2}-1)/4}(\log\log\log x)^{-5/8}\right),$$

where C is an absolute positive constant, also due to Soundararajan [6].

Following Jutila [10], we study

$$\sum_{n \le x} r(n) e\left(\frac{nk}{4l}\right),$$

where k and l are integers such that $l \ge 1$, and k and $4l$ are coprime; the occuring factor 4 in the denominator is very useful with regard to the periodicity $\mathcal{X}(n+4) = \mathcal{X}(n)$.

We start with a study of the analytic properties of the generating Dirichlet series. The estimates in all sections with the exception of Section 5 are uniform in l; ε denotes always an arbitrarily small positive number, having not necessarily the same value at each occurence.

2. A new Dirichlet series. Let $s = \sigma + it$ be a complex variable. Define

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s} e\left(\frac{nk}{4l}\right).$$

Since

$$r(n) \le 4d(n) \ll n^{\varepsilon}, \tag{6}$$

the Dirichlet series in question converges absolutely in the half-plane $\sigma > 1$. In this region, we have by (5)

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = 4\sum_{b=1}^{\infty} \sum_{d=1}^{\infty} \frac{\mathcal{X}(d)}{(bd)^s} e\left(\frac{bdk}{4l}\right) =$$

$$= 4 \sum_{a \bmod 4l} \mathcal{X}(a) \sum_{b=1}^{\infty} \frac{1}{b^s} e\left(\frac{abk}{4l}\right) \sum_{\substack{d=1 \\ d \equiv a \bmod 4l}}^{\infty} \frac{1}{d^s}. \tag{7}$$

We define for $\sigma > 1$

$$\zeta\left(s; e\left(\frac{ak}{4l}\right)\right) = \sum_{b=1}^{\infty} \frac{1}{b^s} e\left(\frac{abk}{4l}\right),$$

and

$$\zeta(s; a \bmod 4l) = \sum_{\substack{d=1 \\ d \equiv a \bmod 4l}}^{\infty} \frac{1}{d^s}.$$

In view of (7) we get for $\sigma > 1$ the representation

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = 4 \sum_{a \bmod 4l} \mathcal{X}(a) \zeta\left(s; e\left(\frac{ak}{4l}\right)\right) \zeta(s; a \bmod 4l). \tag{8}$$

Our first aim is to prove the following statement:

Theorem 1. *The function $\mathcal{R}(s; k/(4l))$ has an analytic continuation throughout the complex plane except for a simple pole at $s = 1$, where it has the Laurent expansion*

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = \frac{\pi i \mathcal{X}(k)}{2l(s-1)} + \textit{higher terms.}$$

Moreover, it satisfies the functional equation

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = \frac{\mathcal{X}(k^*)}{\pi} \left(\frac{\pi}{2l}\right)^{2s-1} \Gamma(1-s)^2 \times$$

$$\times \left\{ \mathcal{R}\left(1 - s; \frac{k^*}{4l}\right) - \cos(\pi s)\, \mathcal{R}\left(1 - s; \frac{-k^*}{4l}\right) \right\},$$

where k^* is given by $kk^* = 1 \bmod 4l$.

Proof. The functions appearing on the right-hand side of (7) are special Lerch zeta-functions. The function $\zeta(s; e(ak/(4l)))$ is entire whenever $\mathcal{X}(a) \neq 0$, and every $\zeta(s; a \bmod 4l)$ is analytic except for a simple pole at $s = 1$ with residue $1/(4l)$ (see [13]). Thus, representation (8) is valid for all complex s and gives the desired analytic continuation for $s \neq 1$. In a neighbourhood of $s = 1$ we have There its main part equals

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = \frac{1}{l(s-1)} \sum_{a \bmod 4l} \mathcal{X}(a) \zeta\left(1; e\left(\frac{ak}{4l}\right)\right) + O(1). \tag{9}$$

Since

$$\sum_{a \bmod 4l} \mathcal{X}(a) e\left(\frac{abk}{4l}\right) = \begin{cases} -2i \sin\left(\frac{\pi bk}{2l}\right) e\left(\frac{bk}{2l}\right) l & \text{if} \quad b \equiv 0 \bmod l, \\ 0 & \text{if} \quad b \not\equiv 0 \bmod l, \end{cases}$$

we get for the sum in (9)

$$-2il \sum_{\substack{b=1 \\ b \equiv 0 \bmod l}}^{\infty} \frac{1}{b} \sin\left(\frac{\pi bk}{2l}\right) e\left(\frac{bk}{2l}\right) = 2i\mathcal{X}(k) \sum_{d=1}^{\infty} \frac{\mathcal{X}(d)}{d}.$$

Thus, by Leibniz' formula,

$$1 - \frac{1}{3} + \frac{1}{5} \mp \ldots = \frac{\pi}{4}, \tag{10}$$

the residue of $\mathcal{R}(s; k/(4l))$ at $s = 1$ equals $\pi i \mathcal{X}(k)/(2l)$ which proves the first assertion of the theorem.

In view of the functional equations (see [13])

$$\zeta\left(s; e\left(\frac{ak}{4l}\right)\right) = \frac{1}{i}\left(\frac{\pi}{2l}\right)^{s-1}\Gamma(1-s) \times$$
$$\times \left\{e\left(\frac{s}{4}\right)\zeta(1-s; -ak \bmod 4l) - e\left(-\frac{s}{4}\right)\zeta(1-s; ak \bmod 4l)\right\},$$

and

$$\zeta(s; a \bmod 4l) = \frac{1}{2\pi i}\left(\frac{\pi}{2l}\right)^{s}\Gamma(1-s) \times$$
$$\times \left\{e\left(\frac{s}{4}\right)\zeta\left(1-s; e\left(\frac{a}{4l}\right)\right) - e\left(-\frac{s}{4}\right)\zeta\left(1-s; e\left(-\frac{a}{4l}\right)\right)\right\},$$

we deduce from (7)

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = \frac{2}{\pi}\left(\frac{\pi}{2l}\right)^{2s-1}\Gamma(1-s)^2 \sum_{a \bmod 4l} \mathcal{X}(a) \times$$
$$\times \left\{e\left(\frac{s}{2}\right)\zeta\left(1-s; e\left(\frac{a}{4l}\right)\right)\zeta(1-s; -ak \bmod 4l) + \right.$$
$$+ \zeta\left(1-s; e\left(\frac{a}{4l}\right)\right)\zeta(1-s; ak \bmod 4l) +$$
$$+ \zeta\left(1-s; e\left(\frac{-a}{4l}\right)\right)\zeta(1-s; -ak \bmod 4l) -$$
$$\left. - e\left(\frac{-s}{2}\right)\zeta\left(1-s; e\left(\frac{-a}{4l}\right)\right)\zeta(1-s; ak \bmod 4l)\right\}.$$

Now let $\delta, \varepsilon \in \{\pm 1\}$. By the condition on k and l, we find

$$4 \sum_{a \bmod 4l} \mathcal{X}(a)\zeta\left(s; e\left(\frac{\delta a}{4l}\right)\right)\zeta(s; \varepsilon ak \bmod 4l) =$$
$$= 4\mathcal{X}(\varepsilon k^*) \sum_{b := \varepsilon ak \bmod 4l} \mathcal{X}(b)\zeta\left(s; e\left(\frac{\delta\varepsilon bk^*}{4l}\right)\right)\zeta(s; b \bmod 4l) = \varepsilon\mathcal{X}(k^*)\mathcal{R}\left(s; \frac{\varepsilon\delta k^*}{4l}\right).$$

This leads immediately to the functional equation.

The theorem is proved.

The functional equation for $\mathcal{R}(s; k/(4l))$ should be compared with the similar one for the Estermann zeta-function (see [11, 14] or [12]) which is also an additive twist of a Dirichlet series with multiplicative coefficients. A similar situation was recently studied by Miller and Schmid [9]. They observed that additively twisted $GL(2)$ L-functions associated with cuspidal modular forms (or even Maass wave forms) satisfy functional equations which allow to obtain Voronoi-type formulae. However, beyond $GL(2)$ such twisted L-functions do not satisfy functional equations any longer. Thus they cannot work with Voronoi's method to prove Voronoi-type formulae for $GL(3)$, but, surprisingly, they succeeded with representation-theoretic arguments.

3. Special values. For our later purpose we have a look on the values taken by $\mathcal{R}(s; k/(4l))$ taken at the integers. We follow Ishibashi's approach towards the values of the Estermann zeta-function [15].

We could work with (8) but here it is more convenient to use

$$\mathcal{R}\left(s; \frac{k}{4l}\right) = 4 \sum_{a, b \bmod 4l} \mathcal{X}(a)e\left(\frac{abk}{4l}\right)\zeta(s; a \bmod 4l)\,\zeta(s; b \bmod 4l),$$

which can be proved similarly. Let $n \in \mathbb{N}$. It is well-known that

$$\zeta(1-n; a \bmod 4l) = -\frac{(4l)^{n-1}}{n} B_n\left(\frac{a}{4l}\right),$$

where the Bernoulli polynomials are defined by

$$\frac{z\exp(xz)}{\exp(z)-1} = \sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!}.$$

This leads to

$$\mathcal{R}\left(1-n; \frac{k}{4l}\right) = \frac{4(4l)^{2n-2}}{n^2} \sum_{a,\, b \bmod 4l} \mathcal{X}(a)\, e\left(\frac{abk}{4l}\right) B_n\left(\frac{a}{4l}\right) B_n\left(\frac{b}{4l}\right). \tag{11}$$

Taking into account the identity

$$\sum_{b \bmod 4l} e\left(\frac{abk}{4l}\right) B_n\left(\frac{b}{4l}\right) = \frac{n}{(4l)^{n-1}}\left(\frac{i}{2}\right)^2 \cot^{(n-1)}\left(\frac{-\pi ak}{4l}\right),$$

valid for coprime ak and $4l$, due to Girstmair [16], we get

$$\mathcal{R}\left(1-n; \frac{k}{4l}\right) = \frac{(2i)^n l^{n-1}}{n} \sum_{a \bmod 4l} \mathcal{X}(a) \cot^{(n-1)}\left(\frac{-\pi ak}{4l}\right) B_n\left(\frac{a}{4l}\right)$$

(since the factor $\mathcal{X}(a)$ equals zero for values of a for which ak is not coprime with $4l$). Putting the terms according to a and $4l - a \bmod 4l$ together the inner sum above can be rewritten as

$$\sum_{a \bmod 2l} \mathcal{X}(a) \left\{\cot^{(n-1)}\left(\frac{-\pi ak}{4l}\right) B_n\left(\frac{a}{4l}\right) - \cot^{(n-1)}\left(\frac{\pi ak}{4l}\right) B_n\left(1-\frac{a}{4l}\right)\right\}.$$

Taking into account the symmetries

$$\cot^{(n-1)}\left(\frac{-\pi ak}{4l}\right) = (-1)^n \cot^{(n-1)}\left(\frac{\pi ak}{4l}\right),$$

and

$$B_n\left(1-\frac{a}{4l}\right) = (-1)^n B_n\left(\frac{a}{4l}\right),$$

we get the following statement:

Theorem 2. *For* $n \in \mathbb{N}$,

$$\mathcal{R}\left(1-n; \frac{k}{4l}\right) = 0.$$

Obviously,

$$\mathcal{R}\left(s; \frac{-k}{4l}\right) = \overline{\mathcal{R}\left(\bar{s}; \frac{k}{4l}\right)}.$$

In view of the functional equation of Theorem 1, we get

$$\mathcal{R}\left(1-n; \frac{k^*}{4l}\right) = \frac{\mathcal{X}(k)(2l)^{2n-1}(n-1)!^2}{\pi^{2n}}\left\{\mathcal{R}\left(n; \frac{k}{4l}\right) - (-1)^{n-1}\overline{\mathcal{R}\left(n; \frac{k}{4l}\right)}\right\}.$$

Thus, we obtain by the previous theorem the following corollary:

***Corollary* 1.** *For* $n \in \mathbb{N}$,

$$\operatorname{Re}\mathcal{R}\left(2n; \frac{k}{4l}\right) = 0 \quad \text{and} \quad \operatorname{Im}\mathcal{R}\left(2n+1; \frac{k}{4l}\right) = 0.$$

4. A truncated Voronoi-type formula. Now we are ready to present the related truncated Voronoi-type formula.

Theorem 3. *Let* $1 \le N << x, l \le \sqrt{Nx}$. *Then*

$$\sum_{n \le x} r(n) e\left(\frac{nk}{4l}\right) = \frac{\pi i \mathcal{X}(k)}{2l} x + \mathcal{P}\left(x; \frac{k}{4l}\right),$$

where

$$\mathcal{P}\left(x; \frac{k}{4l}\right) = x^{1/4} \frac{\mathcal{X}(k^*)}{\pi} \sqrt{\frac{l}{2}} \sum_{n \le N} \frac{r(n)}{n^{3/4}} e\left(-\frac{nk^*}{4l}\right) \cos\left(\frac{\pi\sqrt{nx}}{l} - \frac{\pi}{4}\right) + \\ + O\left(lN^{-1/2} x^{1/2+\varepsilon}\right).$$

The proof is very similar to the one of Julita for (4). Therefore we give only a sketch of the proof.

Proof. Let $N \in \mathbb{N}$ and the parameter $T \ge 1$ be given by

$$(4lT)^2 = 4\pi^2 x \left(N + \frac{1}{2}\right).$$

By Perron's formula we find

$$\sum_{n \le x} r(n) e\left(\frac{nk}{4l}\right) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \mathcal{R}\left(s; \frac{k}{4l}\right) \frac{x^s}{s}\, ds + O\left(lN^{-1/2} x^{1/2+\varepsilon}\right).$$

Now we evaluate the integral above by integrating on the rectangular contour with vertices $1 + \varepsilon \pm iT$, $-\varepsilon \pm iT$. Using the Phragmén – Lindelöf principle we deduce from the functional equation the estimate

$$\mathcal{R}\left(s; \frac{k}{4l}\right) << (l|t|)^{1+\varepsilon-\sigma}$$

for $-\varepsilon \le \sigma \le 1 + \varepsilon$ as $|t| \to \infty$. Consequently, we have for the integrals along the horizontal paths

$$\int_{-\varepsilon \pm iT}^{1+\varepsilon \pm iT} \mathcal{R}\left(s; \frac{k}{4l}\right) \frac{x^s}{s}\, ds << lN^{-1/2} x^{1/2+\varepsilon}.$$

Applying the calculus of residues we find by Theorem 1

$$\sum_{n \le x} r(n) e\left(\frac{kn}{4l}\right) = \frac{\pi i \mathcal{X}(k)}{2l} x + \mathcal{R}\left(0; \frac{k}{4l}\right) + \\ + \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} \mathcal{R}\left(s; \frac{k}{4l}\right) \frac{x^s}{s}\, ds + O\left(lN^{-1/2} x^{1/2+\varepsilon}\right).$$

In view of Theorem 2 the constant $\mathcal{R}(0; k/(4l))$ is equal to zero. Consequently,

$$\mathcal{P}\left(x; \frac{k}{4l}\right) = \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} \mathcal{R}\left(s; \frac{k}{4l}\right) \frac{x^s}{s}\, ds + O\left(lN^{-1/2} x^{1/2+\varepsilon}\right).$$

Using once more the functional equation the integral above equals

$$\frac{\mathcal{X}(k^*)}{2\pi^2 i}\int\limits_{-\varepsilon-iT}^{-\varepsilon+iT}\left(\frac{\pi}{2l}\right)^{2s-1}\Gamma(1-s)^2\left\{\mathcal{R}\left(1-s;\frac{k^*}{4l}\right)-\right.$$

$$\left.-\cos(\pi s)\mathcal{R}\left(1-s;\frac{-k^*}{4l}\right)\right\}\frac{x^s}{s}\,ds+O\left(lN^{-1/2}x^{1/2+\varepsilon}\right)=$$

$$=\frac{4\mathcal{X}(k^*)l}{\pi i}\sum_{n=1}^{\infty}\frac{r(n)}{n}\int\limits_{-\varepsilon-iT}^{-\varepsilon+iT}(2\pi)^{2s-2}\left(\frac{nx}{16l^2}\right)^s\times$$

$$\times\left\{\left(e\left(\frac{nk^*}{4l}\right)-e\left(\frac{-nk^*}{4l}\right)\right)+(1-\cos(\pi s))\,e\left(\frac{-nk^*}{4l}\right)\right\}\frac{ds}{s}+O\left(lN^{-1/2}x^{1/2+\varepsilon}\right).$$

The n-th integral here is the sum of the two integrals corresponding to the two terms in the curly brackets. By Stirling's formula it turns out that the first one is bounded by $\ll n^{-\varepsilon}x^{\varepsilon}$, and so the contribution of these to $\mathcal{P}(x;k/(4l))$ is $\ll lx^{\varepsilon}$. Thus

$$\mathcal{P}\left(x;\frac{k}{4l}\right)=\frac{2\mathcal{X}(k^*)l}{\pi i}\sum_{n=1}^{\infty}\frac{r(n)}{n}\,e\left(\frac{-nk^*}{4l}\right)\times$$

$$\times\int\limits_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}2^{2s}\pi^{2s-2}\sin\left(\frac{\pi s}{2}\right)^2\Gamma(1-s)^2\left(\frac{nx}{16l^2}\right)^s\frac{ds}{s}+O\left(lN^{-1/2}x^{1/2+\varepsilon}\right).$$

The contribution of the terms with $n>N$ in the latter expression is $\ll lx^{\varepsilon}$ by Stirling's formula. For $n\le N$ we may replace the line of integration by $(-i\infty,+i\infty)$ at the expanse of an error $O(lx^{\varepsilon})$. Finally, the formula

$$\frac{1}{n}\int\limits_{-\varepsilon-iT}^{-\varepsilon+iT}2^{2s}\pi^{2s-2}\sin\left(\frac{\pi s}{2}\right)^2\Gamma(1-s)^2\left(\frac{nx}{16l^2}\right)^s\frac{ds}{s}=$$

$$=-\frac{i}{l}\left(\frac{x}{n}\right)^{1/2}\left\{K_1\left(\frac{\pi\sqrt{nx}}{l}\right)+\frac{\pi}{2}Y_1\left(\frac{\pi\sqrt{nx}}{l}\right)\right\},$$

a proof can be found in [17], Section 3.2, leads to

$$\mathcal{P}\left(x;\frac{k}{4l}\right)=-\mathcal{X}(k^*)x^{1/2}\sum_{n\le N}\frac{r(n)}{n^{1/2}}\,e\left(-\frac{nk^*}{4l}\right)\times$$

$$\times\left\{\frac{2}{\pi}K_1\left(\frac{\pi(nx)^{1/2}}{l}\right)+Y_1\left(\frac{\pi(nx)^{1/2}}{l}\right)\right\}+O\left(lN^{-1/2}x^{1/2+\varepsilon}\right).$$

Once more with Stirling's formula one can deduce from (1) and (2)

$$K_1(z)=\sqrt{\frac{2}{\pi z}}\exp(-z)\left(1+O\left(|z|^{-1}\right)\right),\tag{12}$$

$$Y_1(z)=\sqrt{\frac{2}{\pi z}}\exp\left(z-\frac{3\pi}{4}\right)+O\left(|z|^{-3/2}\right).\tag{13}$$

Obviously the K_1-term gives only a small contribution to the error term. Substituting these asymptotics in the last expression for $\mathcal{P}(x;k/(4l))$ completes the proof of the theorem.

In view of (6) Theorem 3 gives

$$\mathcal{P}\left(x; \frac{k}{4l}\right) \ll l^{1/2} x^{1/4} N^{1/4+\varepsilon} + l x^{1/2+\varepsilon} N^{-1/2}.$$

The choice $N = (l^2 x)^{1/3}$ leads to the following corollary:

***Corollary* 2.** *For $l \leq x$,*

$$\mathcal{P}\left(x; \frac{k}{4l}\right) \ll l^{2/3} x^{1/3+\varepsilon}.$$

Following Jutila's argument [10] one can prove by hard analysis, similarly to (4), the exact Voronoi formula

$$\mathcal{P}\left(x; \frac{k}{4l}\right) = -\mathcal{X}(k^*) x^{1/2} \sum_{n=1}^{\infty} \frac{r(n)}{n^{1/2}} \times$$

$$\times \left\{ e\left(-\frac{nk^*}{4l}\right) Y_1\left(\frac{\pi (nx)^{1/2}}{l}\right) + \frac{2}{\pi} e\left(\frac{nk^*}{4l}\right) K_1\left(\frac{\pi (nx)^{1/2}}{l}\right) \right\}.$$

5. Almost periodicity. Heath-Brown [18] observed that one can use Voronoi-type formulae to prove almost periodicity of error terms. This is of special interest since almost periodicity of a function expresses, roughly speaking, a certain regularity of this function. Let $1 \leq q < \infty$. A measurable function $f\colon [1, \infty) \to \mathbb{C}$ is said to be $\mathcal{B}^q$-almost periodic if for every $\varepsilon > 0$ there exists a trigonometric polynomial $p(t) = \sum_{j=1}^{J} c_j e(\alpha_j t)$ with complex coefficients c_j and real exponents α_j such that

$$\| f - p \|_q := \left(\limsup_{X \to \infty} \frac{1}{X} \int_1^X | f(t) - p(t) |^q \, dt \right)^{1/q} < \varepsilon.$$

(For the theory of almost periodic function we refer the reader to [19].) Following the arguments of Peter [8], we will show the validity of the below result.

Theorem 4. *The function $P\colon [1, \infty) \to \mathbb{C}$, defined by*

$$P(t) = t^{-1/2} \mathcal{P}\left(t^2; \frac{k}{4l}\right),$$

is $\mathcal{B}^2$-almost periodic, and $\| P \|_2^2$ is given by (15) *below.*

Proof. With view to our truncated Voronoi-type formula we define

$$p_J(t) = \frac{\mathcal{X}(k^*)}{\pi} \sqrt{\frac{l}{2}} \sum_{n \leq J} \frac{r(n)}{n^{3/4}} e\left(-\frac{nk^*}{4l}\right) \cos\left(\frac{\pi \sqrt{n} t}{l} - \frac{\pi}{4}\right).$$

Now let $M > \sqrt{J}$ and $M \leq t \leq 2M$. Theorem 3 with $x = t^2$ and $N = M^2$ gives

$$P(t) - p_J(t) = S(t) + O(M^{\varepsilon - 1/2}),$$

where

$$S(t) := \frac{\mathcal{X}(k^*)}{\pi} \sqrt{\frac{l}{2}} \sum_{J < n \leq M^2} \frac{r(n)}{n^{3/4}} e\left(-\frac{nk^*}{4l}\right) \cos\left(\frac{\pi \sqrt{n} t}{l} - \frac{\pi}{4}\right).$$

Next we expand $|S(t)|^2$ into a double sum and integrate term by term. By the estimates

$$\int_M^{2M} \cos(\alpha_i t + \delta)\cos(\alpha_j t + \delta)\, dt \ll \min\left\{\left||\alpha_i| - |\alpha_j|\right|^{-1}, M\right\},$$

valid for real $\alpha_i, \alpha_j, \delta,$ and (6) we obtain

$$\int_M^{2M} |S(t)|^2\, dt \ll M \sum_{J < n \le M^2} n^{\varepsilon - 3/2} + \sum_{J < n_1 < n_2 \le M^2} n_1^{\varepsilon - 3/2}\left(\sqrt{n_2} - \sqrt{n_1}\right)^{-1}.$$

The first sum is easily bounded by $J^{\varepsilon - 1/2}$ whereas the second sum is

$$\ll \sum_{J < n_1 < n_2 \le M^2} n_1^{\varepsilon - 3/2} \frac{\sqrt{n_2} + \sqrt{n_1}}{n_2 - n_1} \le$$

$$\ll \sum_{J < n_1 \le M^2} n_1^{\varepsilon - 3/2}\left\{\sum_{n_1 < n_2 \le 2n_1} \frac{M}{n_2 - n_1} + \sum_{2n_1 < n_2 \le M^2} n_1^{-1/2}\right\} \le$$

$$\ll M \sum_{n_1 > J} n_1^{\varepsilon - 3/2} \log(n_1 + 2) \ll MJ^{\varepsilon - 1/2}.$$

These estimates lead to

$$\int_M^{2M} |P(t) - p_J(t)|^2\, dt \ll M^{\varepsilon} + MJ^{\varepsilon - 1/2} \ll MJ^{\varepsilon - 1/2}.$$

Using this with $M = 2^{-j}X$ summation over all $j \in \mathbb{N}$ gives

$$\limsup_{X \to \infty} \frac{1}{X} \int_1^X |P(t) - p_J(t)|^2\, dt \ll J^{\varepsilon - 1/2}.$$

Consequently, $\| P - p_J \|_2$ tends to zero with $J \to \infty$. Hence, P is $\mathcal{B}^2$-almost periodic. From the definition of p_J follows that

$$\| P \|_2^2 = \lim_{J \to \infty} \| p_J \|_2^2 = \frac{l}{2\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}}.$$

It follows from (6) that the function $r(n)/4$ is multiplicative (as a convolution of multiplicative functions). This observation leads for $\sigma > 1$ to

$$\frac{1}{16} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^s} = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{1}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^{-3} \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

A short computation gives

$$\sum_{n=1}^{\infty} \frac{r(n)^2}{n^s} = \frac{16\zeta(s)^2 L(s, \mathcal{X})^2}{(1 + 2^{-s})\zeta(2s)}, \tag{14}$$

where $L(s, \mathcal{X})$ is given by

$$L(s, \mathcal{X}) = \sum_{n=1}^{\infty} \frac{\mathcal{X}(n)}{n^s} = \prod_p \left(1 - \frac{\mathcal{X}(p)}{p^s}\right)^{-1},$$

and the Riemann zeta-function $\zeta(s)$ is defined analogously by replacing $\mathcal{X}$ with 1;

notice that the value (10) for $L(1, \mathcal{X})$ gave a contribution to the main term in the asymptotic formula for our additive twists. Consequently,

$$\| P \|_2^2 = \frac{8l\zeta\left(\frac{3}{2}\right)^2 L\left(\frac{3}{2}; \mathcal{X}\right)^2}{\pi^2(1+2^{-3/2})\zeta(3)}. \tag{15}$$

This proves the theorem.

A nice consequence from the theory of almost periodic functions is the existence of a limit distribution (for a proof see Bleher [20]).

***Corollary* 3.** *The function $P(t)$ possesses a limit distribution; more precisely, there exists a probability distribution function ν such that, for any rectangle $\mathcal{R}$ in the complex plane whose edges are parallel to the axes,*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ T \le t \le 2T : \frac{\mathcal{P}\left(t^2; \frac{k}{4l}\right)}{\sqrt{t}} \in \mathcal{R} \right\} = \iint_{\mathcal{R}} \nu(x, y)\, dx\, dy,$$

where the measure is the ordinary Lebesgue measure.

A little bit more is known in the nontwisted case. Then the corresponding limit distribution is not Gaussian. However, recently Hughes and Rudnick [21] proved that the limit distribution corresponding to the number of lattice points lying in a thin annulus is Gaussian if the width of the annulus tends sufficiently slowly to zero by increasing inner radius.

6. The mean-square. We conclude with another application of almost periodicily, the existence of moments. The mean-square of a $\mathcal{B}^2$-almost periodic function exists. This gives in our case

$$\frac{1}{X} \int_1^X | P(t) |^2 \, dt \sim \| P \|_2^2 .$$

With view to

$$\int_1^X \left| \mathcal{P}\left(x^2; \frac{k}{4l}\right) \right|^2 dx = \frac{2}{3} \int_1^{\sqrt{X}} | P(t) |^2 \, d(t^3) \sim \frac{2}{3} X^{3/2} \| P \|_2^2$$

we get an asymptotic formula, but unfortunately, without error term and not uniform in l. With a bit more effort we can get an asymptotic formula with error term which is uniform in l.

Theorem 5. *For $l \le X$,*

$$\int_1^X \left| \mathcal{P}\left(x; \frac{k}{4l}\right) \right|^2 dx = l X^{3/2} \frac{16\zeta\left(\frac{3}{2}\right)^2 L\left(\frac{3}{2}, \mathcal{X}\right)^2}{3\pi^2\left(1+2^{-3/2}\right)\zeta(3)} + O\left(l^{3/2} X^{5/4+\varepsilon} + l^3 X^{1+\varepsilon}\right).$$

Proof. Theorem 3 with $N = X$ gives

$$\int_X^{2X} \left| \mathcal{P}\left(x; \frac{k}{4l}\right) \right|^2 dx = \frac{l}{2\pi^2} \sum_{m, n < X} \frac{r(m) r(n)}{(mn)^{3/4}} e\left(\frac{(m-n)k^*}{4l} \right) \times$$

$$\times \int_X^{2X} x^{1/2} \cos\left(\frac{\pi\sqrt{mx}}{l} - \frac{\pi}{4} \right) \cos\left(\frac{\pi\sqrt{nx}}{l} - \frac{\pi}{4} \right) dx \; +$$

$$+ \; O\left(l^2 X^{1+\varepsilon} + l^{3/2} X^{\varepsilon} \int_X^{2X} \left| \sum_{n \le X} \frac{r(n)}{n^{3/4}} e\left(\frac{-nk^*}{4l}\right) \cos\left(\frac{\pi\sqrt{nx}}{l} - \frac{\pi}{4}\right) \right| dx \right). \quad (16)$$

The main term comes from the diagonal terms $m = n$. Since

$$\int x^{1/2} \cos\left(\frac{\pi\sqrt{mx}}{l} - \frac{\pi}{4}\right)^2 dx = \frac{2}{3} x^{3/2} + O(x),$$

the diagonal terms $m = n$ in the sum appearing in (16) contribute

$$\frac{2\left(2^{3/2} - 1\right)}{3} X^{3/2} \sum_{n \le X} \frac{r(n)^2}{n^{3/2}} = \frac{2\left(2^{3/2} - 1\right)}{3} X^{3/2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} + O(X^{1+\varepsilon}).$$

The nondiagonal terms $m \ne n$ equal

$$\sum_{m, n \le X} \frac{r(m) r(n)}{(mn)^{3/4}} e\left(\frac{(m-n)k^*}{4l}\right) \left\{ \int_X^{2X} x^{1/2} \cos\left(\frac{\pi\sqrt{mx} - \sqrt{nx}}{l}\right) dx + \right.$$

$$\left. + \int_X^{2X} x^{1/2} \sin\left(\frac{\pi\sqrt{mx} + \sqrt{nx}}{l}\right) dx \right\}.$$

The integrals are bounded by $lX / \left(\sqrt{m} \mp \sqrt{n}\right)$ according to the appearance of cos or sin. Next, splitting the range of summation according to $2n < m$ or not, as in the proof of Theorem 4, it turns out that the nondiagonal terms in (16) contribute $\ll l^2 X^{1+\varepsilon}$. It remains to consider the integral appearing in the error term of (16). By the Cauchy – Schwarz inequality,

$$\int_X^{2X} \left| \sum_{n \le X} \frac{r(n)}{n^{3/4}} e\left(\frac{-nk^*}{4l}\right) \cos\left(\frac{\pi\sqrt{nx}}{l} - \frac{\pi}{4}\right) \right| dx \le$$

$$\le X^{1/2} \left(\int_X^{2X} \left| \sum_{n \le X} \frac{r(n)}{n^{3/4}} e\left(\frac{-nk^*}{4l}\right) \cos\left(\frac{\pi\sqrt{nx}}{l} - \frac{\pi}{4}\right) \right|^2 dx \right)^{1/2}.$$

Taking into account what we have already proved, the latter expression is $\ll X^{5/4} + lX^{1+\varepsilon}$. Hence,

$$\int_X^{2X} \left| \mathcal{P}\left(x; \frac{k}{4l}\right) \right|^2 dx = lX^{3/2} \frac{\left(2^{3/2} - 1\right)}{3\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} + O\left(l^{3/2} X^{5/4+\varepsilon} + l^3 X^{1+\varepsilon}\right).$$

The appearing series may be rewritten by (14) in terms of $L(s, \mathcal{X})$ and Reimann's zeta-function. Using the latter expression with $X2^{-j}$ instead of X and summing up over all $j \in \mathbb{N}$ we obtain the asymptotic formula of the theorem.

As an immediate consequence we deduce the following corollary:

***Corollary* 4.** *For $l \le x$,*

$$\mathcal{P}\left(x; \frac{k}{4l}\right) = \Omega\left(l^{1/2} x^{1/4}\right).$$

Very likely this estimate is more close to the true order of the error term than the one of Corollary 2.

7. Final remarks. It is possible to extend (and, partially, to improve) the obtained results for additive twists of $r(n)$ into several directions (sharpening of the error estimate of Corollary 5, Ω-estimates for the error term in the mean-square formula for $\mathcal{P}$, existence of higher moments). Furthermore, in view of (6) the object of our investigations $r(n)$ is the Dirichlet convolution of $\mathcal{X}$ with the arithmetical function constant 4. The function $r(n)$ occurs as coefficients in the Dirichlet series expansion of the Dedekind zeta-function of the number field $\mathbb{Q}(i)$. Actually, one can replace $r(n)$ by any convolution of a character with a constant function, which includes the class of arithmetical functions $r_{\mathbb{K}}(n)$ that count the number of integral ideals with norm n of the ring of integers associated with a quadratic number field $\mathbb{K}$.

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