UDC 517.5

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ON GENERALIZED HARDY'S SUMS $s_5(h,k)$

ПРО УЗАГАЛЬНЕНІ СУМИ ХАРДІ $s_5(h,k)$

The aim of this paper is to establish generalized Hardy's sums $s_5(h,k)$. By using mediants and adjacent difference of Farey fractions, connections between $s_5(h,k)$ and Farey fractions are obtained. Apllying generalized Dedekind sums and generalized periodic Bernoulli function, generalized Hardy's sums $s_{5,p}(h,k)$ are defined. A connection between $s_{5,p}(h,k)$ and Hurwitz zeta function is established. By using definitions of Lambert series and cot πz, relation between $s_5(h,k)$ and Lambert series is found.

Метою даної статті є вивчення узагальнених сум Харді $s_5(h,k)$. На основі використання медіант та суміжної різниці дробів Фейрі (Farey) встановлено зв'язки між сумами $s_5(h,k)$ і дробами Фейрі. Узагальнені суми Харді $s_5(h,k)$ визначено із застосуванням узагальнених дедекіндових сум та узагальненої періодичної функції Бернуллі. Встановлено зв'язок між сумами $s_5(h,k)$ та дзета-функцією Гурвіца. На основі визначень рядів Ламберта і cot πz знайдено співвідношення між $s_5(h,k)$ та рядами Ламберта.

1. Introduction. The aim of this paper is to study Hardy sums $s_5(h,k)$. These sums are well-known in analytic number theory and Theta functions. We investigate properties of these sums and related the others well-known functions. We now summarize our study in detail as follows:

In Section 1, we give some definitions and notations. In Section 2, arithmetic properties of the Hardy's sums $s_5(h,k)$ are given. By using mediant and adjacent difference of adjacent Farey fractions, a generalized sum $s_5(h,k)$ is given. In Section 3, we establish generalized sum $s_{5,p}(h,k)$. A representation of $s_5(h,k)$ as infinite series is proved. In particular, a connection between $s_{5,p}(h,k)$ and Hurwitz zeta function is established. In Section 4, by using Lambert series and cot πz function, relation between Lambert series and $s_5(h,k)$ is found.

In the twentieth century, the greater integer function has played an important role, often in connection with other functions such as $((x))$, where

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2}, & x \text{ is not an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Dedekind sum $s(h,k)$, arising in the theory of the Dedekind-eta function, is defined by

$$s(h,k) = \sum_{j \bmod k} \left(\left(\frac{j}{k} \right) \right) \left(\left(\frac{hj}{k} \right) \right),$$

where h is an integer and k is a positive integer [1].

The most important property of Dedekind sums is the following reciprocity theorem. If h and k are coprime positive integers, then

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12} \left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk} \right). \tag{1}$$

The proof of (1) was given by Apostol [1].

The higher-order Dedekind sums used in the definition of higher-order Hardy's sums $s_5(h,k)$ were introduced by Apostol. They are defined as

$$s_p(h,k) = \sum_{a \bmod k} \frac{a}{k}\overline{B}_p\left(\frac{ha}{k}\right), \tag{2}$$

where p, h and k are positive integers, $\overline{B}_p(x)$ is the p th periodic Bernoulli function which is defined as follows:

$$\overline{B}_p(x) = -p!\,(2\pi i)^{-p} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} m^{-p}e^{2\pi imx}. \tag{3}$$

Apostol proved reciprocity law of $s_p(h,k)$. When $p = 1$, the sums

$$s_1(h,k) = \sum_{a \bmod k} \frac{a}{k}\overline{B}_1\left(\frac{ha}{k}\right)$$

are known as Dedekind sums.

In this paper, we express Hardy's sums $s_5(h,k)$ explicitly in terms of $s(h,k)$ Sitaramachandraro's sense [2]. In 1905, Hardy [3] was the first to give a proof of the reciprocity theorem for Dedekind sum which does not depend on the theory of the Dedekind-eta function. In fact, by using contour integration, Hardy proved some reciprocity theorems in detail and started eleven more reciprocity theorems for some similar arithmetical sums. Moreover, Hardy clearly indicated the necessary modifications in his foregoing proof to obtain these reciprocity theorems.

In recent years, five of Hardy's reciprocity theorems were studied by Berndt [4] and Goldberg [5]. By using the logarithms of the classical Theta functions $\theta_2(0,q)$, $\theta_3(0,q)$, and $\theta_4(0,q)$ (see for detail [6], Chapter 21), Berndt deduced these reciprocity theorems from his transformation formula [4].

Goldberg [5] showed that these sums also arise in theory of $r_n(n)$, the number of representations of n as a sum of s integral squares and in the study of the Fourier coefficients of the reciprocals of $\theta_n(0,q)$, $n = 2, 3, 4$, while Berndt and Goldberg [7] evaluated certain nonabsolutely convergent-double series in terms of these sums.

Hardy was the first to encounter these sums and to formulate these reciprocity theorems with clear indications of proof. At the end of this paper, Hardy states "... I hope on some other occasion to return to these formulae from an arithmetical point of view..." but it appears that Hardy never returned to the subject. In recent years, the proof of Hardy's reciprocity theorems which do not depend on Berndt's transformations formula was given by Apostol and Vu [8], Berndt and Goldberg [7], Sitaramachandraro [2] and by the author [9, 10]. There are six sums in Hardy's sums, which can be found in detail in [7] and [2], but we will give only $s_5(h,k)$ sum, which is defined as follows.

Hardy's sums $s_5(h,k)$ in terms of Dedekind sums: In studying Hardy's sums $s_5(h,k)$ and its reciprocity theorem, we will use the notation of Berndt and Goldberg [7] and Sitaramachandraro [2]. If h and k are integers with $k > 0$, $s_5(h,k)$ is defined by

$$s_5(h,k) = \sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}\left(\left(\frac{j}{k}\right)\right). \tag{4}$$

Let us now give the reciprocity relations for $s_5(h,k)$ as follows (this relation is the most important in this work):

Theorem 1 [2, 7]. *Let h and k be coprime positive integers. If h and k are odd, then*

$$s_5(h,k) + s_5(k,h) = \frac{1}{2} - \frac{1}{2hk}. \tag{5}$$

Remark 1. A lot of proofs of this theorem have been given by many mathematicians. We will give some information about the proofs of this reciprocity relation and the others. The reciprocity theorems appeared in Hardy [3]. Berndt [4] deduced the other reciprocity relations. Goldberg [5] deduced (5) from Berndt's transformation formulae. Apostol and Vu [8] proved reciprocity law of Hardy sums. By using three-term relations of Carlitz's polynomial, the author [10] and Pettet and Sitaramachandraro [11] established Hardy reciprocity theorems.

It may be noted that Sitaramachandraro [2] expressed, by using elementary arguments, each of the Hardy sums explicitly in terms of Dedekind sums, which he deduced to Theorem 1 and (1). In the theorem below, we will give only the relation between $s(h, k)$ and $s_5(h, k)$. This theorem also contains the other relations between Hardy sums and Dedekind sum, which will not be given and used in this work (see for detail [2, 9, 10].

Theorem 2. *Let h and k be coprime positive integers. If $h + k$ is even, then*

$$s_5(h, k) = -10s(h, k) + 4s(2h, k) + 4s(h, 2k) \tag{6}$$

and if $h + k$ is odd, then

$$s_5(h, k) = 0. \tag{7}$$

In this study, we will give generalized Hardy's sums $s_5(h, k)$ by adjacent Farey fractions. Therefore, we will need the following properties of Farey fractions:

The set of Farey fractions of order n denoted by F_n is the set of reduced fractions in the closed interval $[0, 1]$ with denominators $\leq n$ listed in increasing order of magnitude.

If $\dfrac{h}{k} < \dfrac{H}{K}$ are adjacent Farey fractions, then $hK - kH = -1$. The mediant of adjacent Farey fractions $\dfrac{h}{k} < \dfrac{H}{K}$ is $\dfrac{h+H}{k+K}$. It satisfies the inequality $\dfrac{h}{k} < \dfrac{h+H}{k+K} < \dfrac{H}{K}$. The following inequality can be obtained by repeating the calculation of mediants n-times successively:

$$\frac{h}{k} < \frac{h+H}{k+K} < \ldots < \frac{h+nH}{k+nK} < \frac{H}{K}. \tag{8}$$

The adjacent difference of adjacent Farey fractions $\dfrac{h}{k} < \dfrac{H}{K}$ is $\dfrac{h-H}{k-K}$. It satisfies the inequality $\dfrac{H}{K} < \dfrac{h}{k} < \dfrac{h-H}{k-K}$. The following inequality can be obtained by repeating the calculation of adjacent difference n-times successively

$$\frac{H}{K} < \frac{h}{k} < \frac{h-H}{k-K} < \ldots < \frac{h-nH}{k-nK}. \tag{9}$$

By using (8) and (9), the author proved generalized Dedekind sums in the sense of Rademacher (see for detail [12]). We give generalized Hardy's sums $s_5(h, k)$ along the same line as the generalized Dedekind sums.

2. Arithmetic properties of the Dedekind sums, $s(h, k)$ and Hardy's sums $s_5(h, k)$. The reciprocity law of the Dedekind sums always contains two (and in some generalizations three and even more) Dedekind sums. We focus our attention now on a single Dedekind sum, its properties and its connections with other mathematical topics.

Since $((-x)) = -((x))$, it is clear that

$$s(-h, k) = -s(h, k), s(h, -k) = s(h, k).$$

Let $hh' \equiv 1 \pmod{k}$, then we have

$$s(h', k) = s(h, k).$$

If $h/k < H/K$ are two consecutive fractions of Farey sequence, then $hK - kH = -1$. This implies that

$$hK \equiv -1 \pmod{k}, \qquad Hk \equiv 1 \pmod{K}. \tag{10}$$

Thus, by (10), we have

$$s(h, k) = -s(K, k), \qquad s(H, K) = s(k, K).$$

By using (8) and above relations, the author obtained generalized Dedekind sums [12].

By using arithmetic properties of $s(h, k)$, which were mentioned above, we obtain similar relations for $s_5(h, k)$ as follows: Applying (10) and Theorem 2, we obtain

$$s_5(h, k) = -s_5(K, k), \qquad s_5(H, K) = s_5(k, K).$$

After substituting the above relations into (5), we arrive of the equality

$$s_5(h, k) - s_5(H, K) = -\frac{1}{2} + \frac{1}{2kK}. \tag{11}$$

By using above relations, the following theorem can be obtained:

Theorem 3. *If $h/k < H/K$ are two adjacent Farey fractions, then*

$$s_5(h + H, k + K) = \frac{s_5(h, k) + s_5(H, K)}{2} + \frac{k - K}{4kK(k + K)}.$$

Proof. Let $\dfrac{h}{k} < \dfrac{h + H}{k + K} < \dfrac{H}{K}$ be adjacent Farey fractions. By substituting $\dfrac{h}{k} < \dfrac{h + H}{k + K}$ into (11), we obtain

$$s_5(h, K) - s_5(h + H, k + K) = -\frac{1}{2} + \frac{1}{2k(k + K)}. \tag{12}$$

By substituting $\dfrac{h + H}{k + K} < \dfrac{H}{K}$ into (11), we obtain

$$s_5(h + H, k + K) - s_5(H, K) = -\frac{1}{2} + \frac{1}{2K(k + K)}. \tag{13}$$

By subtracting (12) from (13), we obtain the desired result.

By using (8) and Theorem 3, we can generalize the above theorem as follows.

Theorem 4. *If $\dfrac{h}{k} < \dfrac{nh + H}{nk + K} < \dfrac{(n-1)h + H}{(n-1)k + K}$ are adjacent Farey fractions, then*

$$s_5(nh + H, nk + K) = \frac{s_5(h, k) + s_5((n-1)h + H, (n-1)k + K)}{2} +$$

$$+ \frac{(2 - n)k - K}{4k((n-1)k + K)(k + K)}.$$

Remark 2. By using mathematical induction, the proof of Theorem 4 follows precisely along the same lines as the proof of Theorem 3, and so we omit it.

Corollary 1. *If $\dfrac{h}{k} < \dfrac{H}{K} < \dfrac{H-h}{K-k}$ are adjacent Farey fractions, then*

$$s_5(H - h, K - k) = \frac{1}{2} + \frac{s_5(h, k) + s_5(H, K)}{2} - \frac{k + K}{4kK(K - k)}.$$

Corollary 2. *If* $\dfrac{H}{K} < \dfrac{h(n-1)H}{k-(n-1)K} < \dfrac{h-nH}{k-nK}$ *are adjacent Farey fractions, then*

$$s_5(h-nH, k-nk) = \frac{1}{2} + \frac{s_5(h-(n-1)H, k-(n-1)K) + s_5(H,K)}{2} -$$

$$- \frac{k-(n-2)K}{4K(k-(n-1)K)(k-nK)}.$$

The proofs of the above corollaries follow precisely along the same lines as the proof of Theorem 3, and so we omit them.

3. Representation of $s_5(h, k)$ as infinite series. The aim of this section is to give a generalized sum $s_5(h, k)$. This sum can be represented as infinite series. In particular, we establish a connection between generalized $s_5(h, k)$ and certain finite sums involving Hurwitz zeta function $\zeta(s, a)$.

Under the same hypotheses as Theorem 2 and (2), we define $s_{5,p}(h, k)$ as follows:

$$s_{5,p}(h,k) = \frac{2}{k}\left(\sum_{a \bmod k} a\left(2\overline{B}_p\left(\frac{2ah}{k}\right) - 5\overline{B}_p\left(\frac{ah}{k}\right)\right) + 2 \sum_{b \bmod (2k)} b\overline{B}_p\left(\frac{bh}{2k}\right)\right). \tag{14}$$

For $p = 1$, $s_{5,p}(h, k)$ reduces to $s_5(h, k)$ in Theorem 2.

Theorem 5. *Let* $(h, k) = 1$. *Let p be an odd integer with $p \geq 1$. If $h + k$ is even, then*

$$s_{5,p}(h,k) = \frac{2i(p!)}{(2\pi i)^p} \sum_{\substack{n=1 \\ n \not\equiv 0 \bmod(k)}}^{\infty} n^{-p} f(h, k, n),$$

where

$$f(h, k, n) = -5\cot\left(\frac{\pi hn}{k}\right) + 2\cot\left(\frac{2\pi nh}{k}\right) + 2\cot\left(\frac{\pi hn}{2k}\right).$$

Remark 3. For $p = 1$, Theorem 5 reduces to Theorem 2.

Theorem 6. *Let* $(h, k) = 1$. *Let p be an odd integer with $p > 1$. If $h + k$ is even, then*

$$s_{5,p}(h,k) = \frac{2i(p!)}{(2\pi i k)^p} \sum_{y=1}^{k-1} f(h, k, 1)\zeta\left(p, \frac{y}{k}\right),$$

where $\zeta(s, a)$ is the Hurwitz zeta function.

We will need the following definition and theorems.

Apostol [1] studied the sums $s_p(h, k)$ by means of the theory of the finite sum $\sum_{j=1}^{k-1} jx^j$ which reduces to $\dfrac{k}{x-1}$ if x is any k-th root of unity $\neq 1$ and arrived at the following expression for these sums:

Theorem 7 [1]. *Let* $(h, k) = 1$. *For odd $p \geq 1$, we have*

$$s_p(h,k) = \frac{p!}{(2\pi i)^p} \sum_{\substack{n=1 \\ n \not\equiv 0 \bmod(k)}}^{\infty} n^{-p}\left(\frac{e^{\frac{2\pi i n h}{k}}}{1 - e^{\frac{2\pi i n h}{k}}} - \frac{e^{-\frac{2\pi i n h}{k}}}{1 - e^{-\frac{2\pi i n h}{k}}}\right).$$

Hardy sums were proved and representations as finite trigonometric sums were given by Berndt and Goldberg [7] (here we use the notation of Berndt and Goldberg [7] and Apostol [1]): if h and k are odd (see Eq. (17) and Eq. (7) [7],

$$s_5(h,k) = \frac{1}{2k} \sum_{\substack{j=1 \\ j \neq (k+1)/2}}^{k} \tan\left(\frac{\pi h(2j-1)}{2k}\right) \cot\left(\frac{\pi(2j-1)}{2k}\right),$$

and, if h and k are odd, then

$$s_5(h,k) = \frac{2}{\pi} \sum_{\substack{j=1 \\ 2j-1 \not\equiv 0 \ (\mathrm{mod}\ k)}}^{\infty} \frac{1}{2j-1} \tan\left(\frac{\pi h(2j-1)}{2k}\right).$$

The proof of Theorem 5 is different from the proof of Theorem 7. It can be obtained without any knowledge of the Dedekind eta function, η and the finite sum:

Proof of Theorem 5. Substituting (3) into (14), we obtain

$$s_{5,p}(h,k) = \frac{-(p!)(2\pi i)^{-p}2i}{k}\left(\left(\sum_{a \bmod k} a\left(-10\sum_{m=1}^{\infty} m^{-p}\sin\left(\frac{2\pi ha}{k}\right)+\right.\right.\right.$$

$$+4\sum_{m=1}^{\infty} m^{-p}\sin\left(\frac{4\pi ha}{k}\right)\bigg)\bigg)+2\sum_{b \bmod 2k} b\sum_{m=1}^{\infty} m^{-p}\sin\left(\frac{\pi ha}{k}\right)\bigg).$$

We now define the well-known identity

$$\sum_{a \bmod k} a\sin\left(\frac{2\pi xa}{k}\right) = \frac{-k}{2}\cot\left(\frac{\pi x}{k}\right),$$

where x is an integer, and $x \nmid k$. By using this identity in the above and performing some calculations, we obtain the desired result.

Proof of Theorem 6. Writing $n = gk + y$ with $g = 0, 1, 2, \ldots, \infty$, and, $y = 1, 2, \ldots, k-1$ in Theorem 5, we get

$$s_{5,p}(h,k) = (p!)(2\pi i)^{-p}2i\sum_{y=1}^{k-1}\sum_{g=0}^{\infty}(gk+y)^{-p}f(h,k;gk+y) =$$

$$= (p!)(2\pi ik)^{-p}2i\sum_{y=1}^{k-1} f(h,k,1)\sum_{g=0}^{\infty}\left(g+\frac{y}{k}\right)^{-p},$$

where we must assume $p > 1$ in order to ensure that the series involved should be absolutely convergent and the rearrangements valid. By using the definition of Hurwitz zeta function in the above, we obtain the desired result.

Remark 4. Like the series for $\zeta(s)$, the Riemann zeta function, the Hurwitz zeta function is analytically continued to the whole complex plane except for a simple pole. By using analytic continuation of the Hurwitz zeta function, connection between $s_{5,p}(h,k)$, Bernoulli numbers, and Euler Gamma function $\Gamma(x)$ may be obtained.

4. Relation between Lambert series Theta functions and Hardy sums $s_5(h,k)$.
In [13], the author gave the relations between Theta functions, Hardy sums, Eisenstein and Lambert series. By applying connection between Lambert series and generalized Dedekind sums, the relation between Theta functions and Lambert series were given in [13]. For detail about Lambert series and Hardy sums see [13, 1]. In this section, we give new definition of $s_5(h,k)$, which is related to Lambert series and Theta functions.

By using Theorems 2, 5, and 7, we give relation between Lambert series and $s_5(h,k)$.

Theorem 8. Let $(h,k) = 1$. Let p be an odd integer with $p \geq 1$. If $h+k$ is even, then

$$s_{5,p}(h,k) = \frac{2(p!)}{(2\pi i)^p}(Y_1 + Y_2),$$

where

$$Y_1 = -5G_p(e^{\frac{2\pi h}{k}}) + 2G_p(e^{\frac{4\pi h}{k}}) + 2G_p(e^{\frac{\pi h}{k}}),$$

$$Y_2 = -5G_p\left(e^{\frac{-2\pi h}{k}}\right) + 2G_p\left(e^{\frac{-4\pi h}{k}}\right) + 2G_p\left(e^{\frac{-\pi h}{k}}\right),$$

and $G_p(x)$ is the Lambert series.

Proof. The Lambert series $G_p(x)$ is defined as follows:

$$G_p(x) = \sum_{m,n=1}^{\infty} n^{-p} x^{mn}. \tag{15}$$

By using the well-known relation [13]

$$\frac{e^{2\pi i z}}{1 - e^{2\pi i z}} = i \cot \pi z + \frac{e^{-2\pi i z}}{1 - e^{-2\pi i z}},$$

(15) in Theorem 5, and $f(h, k, n)$ and performing some calculations, we obtain the desired result.

Remark 5. The relation between Theta functions's ϑ_3 and $s_5(h,k)$ was given: If $h + k$ is even, then

$$s_5(h, k) = \frac{Y_{2-} \log \vartheta_3 \left(\dfrac{h}{k}\right)}{\pi}$$

and this relation was generalized in [13].

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